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Sur le thème

THE MELLIN TRANSFORMATION AND ITS APPLICATIONS IN CLASSICAL AND FRACTIONAL CALCULUS

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Abstrac

In this work we present the definition of Mellin transformation,its properties,its inverse transform and the concept of convolution in Mellin sense.Also including some examples, then we explore the relation between the Mellin transform and the other transformations as Laplace, Fourier transform. Finally we set some applications of Mellin transform in classical and fractional culculus .

ملخص

نعرض في هذه المذكرة تعريف تحويل ميلين وخواصه والتحويل العكسي لميلين مع ذكر مفهوم الالتفاف له مع ذكر بعض الامثلة والعلاقة بين ملين وتحويلات الاخرى و في الاخير نحدد بعض التطبيقات التحويلية لميلين في الحساب الكلاسيكي والكسري.

Résumer

Dans ce travaille, nous présentons la transformation de Mellin, ses propriétés, sa transformation inverse et le concept de convolution au sens de Mellin. On incluant également quelques exemples. Nous explorons la relation entre la transformation de Mellin et les autres transformations telle que la transformation de Laplace et de Fourier. Enfin nous donnons quelques application de la transformation de Mellin en calcule classique et en calcule fractionnaire.

Contents

Notations

We give some notations that help us in our work.

1. sets

   ✓$\mathbb{C}$: the set of complex numbers.

   ✓$\mathbb{R}$: the set of real numbers.

   ✓$\mathbb{N}$: the set of natural integer numbers.

2. $s = a + ib$ is a complex number, we note that the real part of s is noted by $a = Re(s)$.

3. Transformations

   ✓$\mathcal{L}$: Laplace transform.

   ✓F,$\widehat{f}$: Fourier transform.

   ✓M: Mellin transform.

4. Operators

   ✓J: Riemann-Liouville integral operator.

   ✓W: Weyl integral operator .

   ✓$f * g$: classical convolution.

   ✓$f \;^{*}\; g$: Mellin convolution.

INTRODUCTION

In mathematics, the Mellin transform is an integral transform that may be regarded as the multiplicative version of the two-sided Laplace transform. This integral transform is closely connected to the theory of Dirichlet series, and is often used in number theory, mathematical statistics, and the theory of asymptotic expansions; it is closely related to the Laplace transform and the Fourier transform, and the theory of the gamma function and allied special functions,and it is extremely useful for certain applications including solving Laplaces equation in polar coordinates, as well as for estimating integrals, and occurs in many areas of engineering and applied mathematics.This transformation bears the name of its creator Hjalmar Mellin (18541933).
Our these is divided into three chapters . The first chapter constitute some concepts and a preliminary parts (definition , propositions,theorems). The second chapter is devoted to the study of the transformation of Mellin, of which we give five sections:
In the first and the second section we set out the definition of Mellin transformation, some examples, as well as some properties related to it. Next, the third section contains the inverse formula of the Mellin transformation. Then, the fourth section contains the convolution product of this transformation, and properties related to this product . Then we finish this chapter by setting the relation between Fourier ,Laplace transform and Mellin transform.
The last chapter is based on the applications of the Mellin transformation.Finally, a bibliography at the end of this document.

Chapter 1

Preliminaries

This Chapter constitutes a preliminary part, in which some concepts are recalled and some fundamental results, which are indispensable tools in our work.In the first section of this chapter, we start with definitions of some function spaces, and in the second section we recall Laplace and Fourier transform. The third and the last parts we define some integral inequalities,and some special functions and Zeta function.

1.1 spaces Some functions

1.1.1 The space of integrable functions

Definition 1.1. *(see [15])*
Let Ω be a measurable set of $\mathbb{R}^n, (n \geq 1)$ and let $1 \leq p \leq \infty$. We denote by $L_p(\Omega)$[1] the class of all measurable functions, f defined on Ω , for which

$$\int_\Omega |f(x)|^p dx < \infty \;, if \;\; 1 \leq p < \infty$$

and for $p = \infty$

$$ess\sup|f(x)| = \inf\{M \geq 0 : |f(x)| \leq M < \infty \quad (a.e.)\} < \infty.$$

Theorem 1.1. *Let $\Omega = [a, b]$ a finite or infinite interval from $\mathbb{R}$*

1. *For $1 \leq p < +\infty$ $L_p(\Omega)$ is a Banach space, so the norm is given by :*

$$\|f(x)\|_{Lp(\Omega)} = \left(\int_\Omega |\ f(x)\ |^p dx\right)^{\frac{1}{p}}$$

[1](More generally) let $1 \leq p \leq \infty$ and (Ω, τ, m) be a measure space. we denote by $L_p(\Omega, \tau, m)$ or simply $L_p(\Omega)$ the class of measurable function for which $\int_\Omega |f(x)^p|dm(x) < \infty$

2. *For* $p=\infty$,$L_\infty(\Omega)$ *is a Banach space with the norm:*

$$\|f(x)\|_\infty = ess\sup|f(x)|.$$

3. *For* $p=2$, $L_2(\Omega)$ *is a Hilbert space*[2] $\{L_2(\Omega) : \int_\Omega |f(x)|^2dx < \infty\}$.

Definition 1.2. *(see [11]) The space* A_s

$$A_s = \{f|f:\mathbb{R}_+ \to \mathbb{C} : f(x)x^{s-1} \in L_1(\mathbb{R}_+) < \infty\} \tag{1.1}$$

with it's associated norm $\|f\|_{A_s}$ *for some* $s\in\mathbb{C}$ *is defined by*

$$\|f\|_{A_s} = \int_0^\infty |f(x)x^{s-1}|dx < \infty,$$

1.1.2 The space of continuous and absolutely continuous functions

Definition 1.3. *(see[10])*
Let $\Omega=[a,b]$ *;* $(-\infty < a \le b < +\infty)$ *and* $n\in\mathbb{N}=0,1,2,....$ *we note* $C^n(\Omega)$ *the space of continuous functions* f *that have their derivatives of* n *order or less which are also continuous over* Ω *and include this norm:*

$$\|f\|_{C^n} := \sum_{k=0}^{n} \|f^{(k)}\|_C \quad , n\in\mathbb{N},$$

in particular if $n=0$, $C^0(\Omega)=C(\Omega)$ *the continuous functions space over* Ω *included this norm :*

$$\|f\|_C := \max_{a\le x\le b}|f(x)|.$$

Definition 1.4. *(see[10])*
For $n\in\mathbb{N}=0,1,2.....$ *,we set* $AC^n([a,b])$ *the space of functions that have derivatives (n-1) order and they are continuous over* $[a,b]$ *with* $f^{n-1}\in AC([a,b])$ *it means :*

$$AC^n([a,b]) = \{f:[a,b]\to\mathbb{C}; \quad and \quad f^{n-1}\in AC([a,b])\}$$

In particular $AC^1([a,b]) = AC([a,b]) = \left\{f/\exists\varphi\in L([a,b]) : f(x) = c + \int_a^x \varphi(t)dt\right\}$

[2] $L_2(\Omega)$ is a Hilbert space with respect to the inner product $(f,g) = \int_\Omega f(x)\overline{g(x)}dx$. Holder's inequality for $L_2(\Omega)$ is actually just the Well-known Schwartz inequality $|(f,g)| \le \|f\|_2\|g\|_2$.

1.1.3 The weighted space of continuous functions

Definition 1.5. *(see[10])*
Let $\Omega = [a, b]$ *a finite interval and* $\lambda \in \mathbb{C}$ *;* $(0 \leq Re(\lambda) < 1)$*. we design by* $C_\lambda([a, b])$ *the space of functions* f *defined over* $]a, b]$ *and the function* $(x - a)^\lambda f(x) \in C[a, b]$ *it means :*

$$C_\lambda([a, b]) = \{f :]a, b] \to \mathbb{C}, (. - a^\lambda)f(.) \in C([a, b])\} \tag{1.2}$$

The space $C_\lambda([a, b])$ *called the space of continuous functions with weight , provided with the next norm*

$$\|f\|_{C_\lambda} = \|(x - a^\lambda)f(x)\|_C = max|(x - a)^\lambda f(x)|. \tag{1.3}$$

In particular $C_0([a, b]) = C([a, b])$.

1.2 Integral transformations

The integral transformed family maps a function from its original function space into another function space via integration, where some of the properties of the original function might be more easily characterized and manipulated than in the original function space.
The integral transformation is symbolized by the equation

$$F(s) = \int_I K(s, x)f(x)dx.$$

With K(s,x) is called the kernel of the transform.By changing the kernel we can have several different integral transforms.
The most used transformations are : The one of Laplace

$$\begin{cases} I = \mathbb{R}^+, \\ K(s, x) = exp(-sx), s \in \mathbb{C}. \end{cases}$$

The one of Fourier

$$\begin{cases} I = \mathbb{R}, \\ K(s, x) = exp(-2i\pi sx), s \in \mathbb{R}. \end{cases}$$

And the Mellin

$$\begin{cases} I = \mathbb{R}^+, \\ K(s,x) = x^{s-1}, s \in \mathbb{C}. \end{cases}$$

1.2.1 Laplace Transform

Definition 1.6. *(see [4])*
Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a continuous function, Laplace transform of f is defined by :

$$L(f)(s) = \mathcal{L}(s) = \int_0^{+\infty} e^{-sx} f(x)dx = \lim_{\tau \to \infty} \int_0^{\tau} e^{-sx} f(x)dx \tag{1.4}$$

This integral converge for $\alpha < Re(s) < \beta$, $-\infty \leq \alpha \leq \beta \leq +\infty$
The inverse formula of Laplace transformation noted L^{-1} is given by:

$$f(x) = (L^{-1}(\mathcal{L}))(s) = \frac{1}{2\pi i} \int_{\delta - it}^{\delta + it} e^{sx} \mathcal{L}(s) ds. \tag{1.5}$$

Remarque 1.1. *the interval $]\delta - it; \delta + it[$ designates a parallel right to the imaginary axis and a real coordinate δ in the plan.*

1.2.2 Fourier Transform

Definition 1.7. *(see [4])*
Let f be an absolutely integrable function over $\mathbb{R}$. The Fourier transformation of f denoted $F[f(t)](\beta) = F(\beta)$ is defined by:

$$F[f(t)] = F(\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\beta t} f(t)dt, \quad \beta \in \mathbb{R}. \tag{1.6}$$

And it can also be defined with $F[f, \beta] = \widehat{f}(\beta)$

$$F[f, \beta] = \widehat{f}(\beta) = \int_{-\infty}^{+\infty} f(t) e^{-2i\pi\beta t} dt, \quad \beta \in \mathbb{R}. \tag{1.7}$$

The inverse of Fourier transform is defined by:

$$f(t) = F^{-1}[Ff(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\beta t} F(\beta) d\beta \tag{1.8}$$

[3]Where the limit exists (as finite number). when its does , the integral (1.4)is said to converge. If the limit doesn't exist , the integral is said to diverge and there is no Laplace transform defined for f .

And it can be written in other formula :

$$F^{-1}[\widehat{f},\beta] = f(t) = \int_{-\infty}^{+\infty} \widehat{f}(\beta)e^{2i\pi\beta t}d\beta. \tag{1.9}$$

With $F[f,\beta] = \widehat{f}(\beta)$ *is the Fourier transform of* f.

1.3 Some integrals inequalities

1.3.1 Holder inequality

If $1 < p < \infty$, we denote by p' the number $\frac{p}{p-1}$ so that $1 < p' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, p' is called the exponent conjugate to p. If $1 < p < \infty$ and $f \in L_p(\Omega), g \in L_{p'}(\Omega)$, then $fg \in L_1(\Omega)$(see [15])

$$\left(\int_\Omega |fg|^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}} \left(\int_\Omega |g|^{p'} d\mu\right)^{\frac{1}{p'}} \tag{1.10}$$

1.3.2 Minkowski inequality

Let f and g be measurable $(f,g \in L_p(\Omega))$. For $1 \leq p < \infty$ (see [15])

$$\left(\int_\Omega |f+g|^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}} + \left(\int_\Omega |g|^p d\mu\right)^{\frac{1}{p}} \tag{1.11}$$

1.3.3 Fubini's theorem

Theorem 1.2. *(see[13])*
Let f *be a continuous function over a rectangle* $D\times[b;c]$ *.Then :*

$$\begin{aligned}\int\int_D f(x,y)dxdy &= \int_D \left(\int_c^d f(x,y)dy\right)dx \\ &= \int_c^d \left(\int_D f(x,y)dx\right)dy.\end{aligned}$$

In particular ***Dirichlet formula***

$$\int_D dx \int_a^x f(x,y)dy = \int_D f(x,y)dx.$$

1.4 Theorem of Residus

Theorem 1.3. *(see[1]) Let f be an holomorphic [3] function and Ω be an open, F is a finite set of points from Ω, c is a simple curve so*

$$\int_c f(s)ds = 2\pi i \sum_{s_0 \in F} Res(f, s_0).$$

With $f(s) = \sum_{n\in z} a_n(s-s_0)^n$ is Laurent series ,and if s_0 is a pole of order k of f then $Res(f, s_0) = \lim_{s\to s_0} \frac{1}{(k-1)!}[(s-s_0)^k f(s)]^{(k-1)}$.

1.5 Some concepts in fractional calculus

1.5.1 Special functions

One of the basic tools of fractional calculus is the Gamma function which extend the factorial to a real positive number (also to a complex number with real positive part).

Definition 1.8. ***(Gamma function of Euler)*** *(see [9])*
Let $x \in \mathbb{R}_*^+$ .The Euler Gamma function is defined by the integral representation :

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \tag{1.12}$$

(this integral converge for all $x > 0$)

Properties 1.1. *(see [7])*
For all $x > 0$, and for all $n \in \mathbb{N}^*$ we have :

1. *The gamma function generalize the factorial function*

$$\Gamma(n) = (n-1)! \ , \tag{1.13}$$

2. $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

3. $\Gamma(x+1) = x\Gamma(x)$.

[3] **Holomorphic function** is a complex-valued function of one or more complex variables that is, at every point of its domain, complex differentiable in a neighborhood of the point. The existence of a complex derivative in a neighborhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal, locally, to its own Taylor series (analytic).

Definition 1.9. ***(Beta function of Euler)*** *(see [9])*
Let $x, y > 0$, the Beta function is defined the integral representation :

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, (Re(x) > 0, Re(y) > 0) \tag{1.14}$$

Properties 1.2. *For all $(x, y) \in \mathbb{C}$ with $Re(x) > 0$ and $Re(y) > 0$, we have :*

1. *The Beta function is symmetric (i.e) :*
   $B(x, y) = B(y, x)$,
2. $B(x + 1, y) = xB(x, y + 1)$,
3. *If $n = y + 1$ is an integer ,that gives a recurrence relation*
   $B(x, y) = \frac{n-1}{x} B(x + 1, n - 1)$
4. $B(x, 1) = \frac{1}{x}$
5. *If $x = m$ and $y = n$,we get* $B(m, n) = \frac{(m-1)(n-1)!}{(m+n-1)!}$

Proposition 1.1. *(see [9])*
The Beta function is related with the Gamma function with the relation :
$\forall x, y > 0$, we have

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.15}$$

Proof*(see[9])*

Definition 1.10. *(**Zeta Riemann function**) (see [16]) The zeta Reimann function is defined over $]1, +\infty[$ with:*

$$(\forall x > 1), \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \tag{1.16}$$

For a given real x, the series of general term $\dfrac{1}{n^x}$, $n \geq 1$, converge if only if $x > 1$.

Remarque 1.2. *For a complex z and $n \in \mathbb{N}^*$ the series of general term $\dfrac{1}{n^z}$ absolutely converge if only if $Re(z) > 1$*

Proposition 1.2. *$\forall s \in \mathbb{C}$, tq $\Re(s) > 1$, we have :*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

Proof

For $Re(s) \geq 1$

$$
\begin{aligned}
\zeta(s)\Gamma(s) &= \sum_{n\geq 1} \frac{\Gamma(s)}{n^s} \\
&= \sum_{n\geq 1} \int_0^{\infty} \left(\frac{u}{n}\right)^{s-1} e^{-u} \frac{du}{n} \\
&= \sum_{n\geq 1} \int_0^{\infty} e^{-nt} t^{s-1} dt, \qquad with\, the\, variable\, change\, (u = nt) \\
&= \int_0^{\infty} \frac{e^{-t}}{1-e^{-t}} t^{s-1} dt \\
&= \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt.
\end{aligned}
$$

Also with, for $\Re > 1$:

$$
\zeta(s)\Gamma(s) = M\left[\frac{1}{e^t - 1}\right](s)
$$

.

1.5.2 Riemann-Liouville fractional integral

Riemanns Liouvilles fractional integral operator is a direct generalization of Cauchys formula for a n-fold integral

$$
(J_a^n f)(x) = \int_a^x dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_n} f(t_n) dt_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt \ \ ,(n \in \mathbb{N}^*)
$$

Definition 1.11. *The fractional integral of order $\alpha(\alpha > 0)$ of Riemann Liouville of a function $f \in \mathbb{C}[a,b]$ is given by*

$$
(J_a^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds. \tag{1.17}
$$

Where $\Gamma(\alpha)$ is the gamma function.

Proposition 1.3. *if $\alpha > 0$ and $\beta > 0$, then*

$$
\begin{aligned}
(J_a^{\alpha} J_a^{\beta} f)(t) &= (J_a^{\alpha+\beta} f)(t) \\
&= (J_a^{\beta} J_a^{\alpha} f)(t).
\end{aligned} \tag{1.18}
$$

Example 1.1. *Let $\alpha > 0$, $\beta > (-1)$ and $f(t) = (t-a)^{\beta}$,then*

$$(J_a^{\alpha} f)(t) = \frac{\Gamma(\beta+1)}{\beta+\alpha+1}(t-a)^{\beta+\alpha}.$$

Indeed,

$$(J_a^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(s-a)^{\beta}ds.$$

By changing the variable $s = a + u(t-a), 0 \leq u \leq 1$, then

$$(J_a^{\alpha} f)(t) = \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}u^{\beta}du.$$

Using the properties of the Beta function, we find

$$\begin{aligned}(J_a^{\alpha} f)(t) &= \frac{B(\beta+1,\alpha)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha} \\ &= \frac{B(\beta+1,\alpha)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}.\end{aligned}$$

1.5.3 The Weyl fractional integral

Definition 1.12. *(see[4]) The Weyl fractional integral of $f(x)$ is defined by*

$$W^{-\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} (t-x)^{\alpha-1}f(t)dt,\ 0 < Re(\alpha) < 1,\ x > 0. \tag{1.19}$$

Often ${}_xW_{\infty}^{-\alpha}$ is used instead of $W^{-\alpha}$ to indicate the limits to integration. Result (1) can be interpreted as the Weyl transform of $f(t)$, defined by

$$W^{-\alpha}[f(x)] = F[x,\alpha]\frac{1}{\Gamma(\alpha)}\int_x^{\infty} (t-x)^{x-\alpha}f(t)dt. \tag{1.20}$$

1.5.4 Riemann-Liouville derivative

Definition 1.13. *The fractional derivate of order $\alpha(\alpha > 0)$ of Riemann Liouville of a function defined on an interval $[a,b]$ of $\mathbb{R}$ is given by*

$$\begin{aligned}(D_a^{\alpha} f)(t) =&(D_a^n J_a^{n-\alpha} f)(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d}{dt}^n\left(\int_a^t (t-s)^{n-\alpha-1}f(s)ds\right).\end{aligned} \tag{1.21}$$

where $n = [\alpha] + 1$, *and* $[\alpha]$ *is the integer part of* α.
In particular if $\alpha = 0$, *then*

$$(D_a^0 f)(t) = (J_a^0 f)(t) = f(t)$$

If $\alpha = n\,(n \in \mathbb{N})$, *then*

$$(D_a^n f)(t) = f^{(n)}(t).$$

If $0 < \alpha < 1$, *then* $n = 1$, *hence*

$$(D_a^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dt} \left(\int_a^t (t-s)^{-\alpha} f(s) ds \right).$$

Proposition 1.4. *Let* $\alpha > 0$ *and* $f \in \mathbb{C}[a,b]$, *then*

$$(D_a^\alpha)(J_a^\alpha f)(t) = f(t)$$

$$(J_a^\alpha)(D_a^\alpha f)(t) \neq f(t)$$

1.5.5 Caputo fractional derivative

Definition 1.14. *(Caputo 1969) (see [12])*
The Caputo fractional derivative of $f \in C_{-1}^m$ *,*$m \in \mathbb{N}$ *is defined*

$$D_c^\alpha f(x) = J^{m-\alpha} D^n f(x)$$
$$D_c^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t) dt \ , m-1 < \alpha \leq m.$$

Theorem 1.4. *If* $m-1 < \alpha \leq m \in \mathbb{N}$,$f \in C_\mu^m$, $\mu > -1$, *then the following properties holds*

(1) $\quad D_c^\alpha [J_c^\alpha f(x)] = f(x)$

(2) $\quad J^\alpha [D_c^\alpha f(x)] = f(x) - \sum_{k=0}^{m-1} f^k(0) \left(\frac{x^k}{k!} \right)$.

Chapter 2

The Mellin transformation

This chapter constitutes four parts. In the first part (of this chapter), we introduce the definition of the Mellin transform and its properties and some examples. The second part the definition of Mellins inversion formula Thus, in the third section, gives the definition of the Mellin convolution product and some properties related to this product.In the last section we give the relation between Mellin, Laplace and Fourier transformations .

Definition 2.1. *(see [2])*
Let f(x) be locally Lebesgue[1] *integrable over* $(0, +\infty)$ *,* $s \in \mathbb{C}$*. The Mellin transformation of f(x) is defined by:*

$$M[f(x), s] = \int_0^{+\infty} x^{s-1} f(x) dx \tag{2.1}$$

This integral converges only for values which are located in the strip $< \alpha, \beta >$ *(i.e) the integral does exist only for complex values of* s*. The strip in some cases may extend to a half-plane or to the whole complex s-plane.*

Examples

Example 2.1. *Consider:*

$$f(x) = H(x - x_0)x^a$$

Where H *is the Heaviside's step function defined by:* $\forall x \in \mathbb{R}$

$$H(x) = \begin{cases} 0, & if x < 0 \\ 1, & if x \geq 0 \end{cases}$$

[1]A function is called locally integrable if,around every point in the domain ,there is a neighborhood in which the function is integrable. The space of locally Lebesgue integrable function is denoted L^1_{loc} any continuous function is locally Lebesgue integrable on $\mathbb{R}$

And let x_0 be a positive number and a is complex. The Mellin transform of f is given by:

$$M[f;s] = \int_{x_0}^{\infty} x^{a+s-1} dx = -\frac{x_0^{a+s}}{a+s} \; using\{if\, t \in \mathbb{R}\,, s \in \mathbb{C}\,, Re(s) < 0, then \lim_{t \to \infty} t^s = 0\}.$$

Example 2.2. *The most simple example of The Mellin transform is provided by the Legendre integral representation of the Gamma function. Let f be the function defined by*

$$f(x) = e^{-x},$$

right away we find :

$$M[f(x)](s) = \int_0^{\infty} e^{-x} x^{s-1} dx = \Gamma(s).$$

Is holomorphic for $Re(s) > 0$.

Example 2.3. *Consider the function:*

$$f(x) = \frac{1}{1+x},$$

By changing the variables $x = \frac{v}{1-v}$,$1 - v = \frac{1}{1+x}$, $dx = \frac{dv}{(1-v)^2}$ So the transform is expressed by:

$$M[f;s] = \int_0^1 v^{s-1}(1-v)^{(1-s)-1} dv$$

This integral is known, it can be written by the beta function like

$$M[\frac{1}{1+x}] = B(s, 1-s) = \Gamma(s).\Gamma(1-s)$$

Example 2.4. *Consider the function:*

$$f(x) = \frac{1}{e^x - 1}$$

The Mellin transform of f is given by

$$M[f;s] = \int_0^{\infty} x^{s-1} \frac{1}{e^x - 1} dx$$

Using

$$\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}}$$
$$\sum_{n=1}^{\infty} e^{-nx} = \frac{1}{e^{x}-1}$$

$$\begin{aligned} M[\frac{1}{e^x-1}] &= \sum_{n=1}^{\infty}\int_0^{\infty} x^{s-1}e^{-nx}dx \\ &= \sum_{n=1}^{\infty}\frac{\Gamma(s)}{n^s} \\ &= \Gamma(s)\zeta(s). \end{aligned}$$

Where the function ζ is the zeta function.

Proposition 2.1. *(see [15]) Let f be a function satisfied*
1.f defined and continuous for $x > 0$.
2.The integral (2.1) is absolutely continuous for $Re(s) = \alpha$ or $Re(s) = \beta$
There for it absolutely converges for $\alpha \leq Re(s) \leq \beta$.Plus the function $s \to M[f](s)$ is continuous and limited in the closed strip and holomophic in the inside .

2.1 Mellin Transform properties

We give some of elementary properties of the Mellin integral (see[4] , [9],[2]) :
Let f,g : $\mathbb{R} \to \mathbb{C}$, $M[f(s)]$ and $M[g(s)]$ are the Mellin transform of the functions f and g respectively .

$P1$) "The Mellin transform is a linear transformation" $\alpha, \beta \in \mathbb{C}$:

$$M[\alpha f(x) + \beta g(x)](s) = \alpha M[f](s) + \beta M[g](s).$$

$P2$) " Scaling property " For $a > 0$, $\alpha, \beta \in \mathbb{C}$ we have

$$M[f(ax)](s) = a^{-s}M[f(x)](s).$$

$P3$) "Multiply by x^a " For all $a \in \mathbb{R}^+$ and $\Re(s + a) \in]\alpha, \beta[$,(i.e) $s \in]\alpha - a, \beta - a[$ we have

$$M[x^a f(x)] = M[f(x)](s + a).$$

$P4$)"Raising the Independent variable to a real Power " For $a \in \mathbb{R}^+$,$s \in < a\alpha, a\beta >$, so

$$M[f(x^a)](s) = \frac{1}{a}M[f(x)](\frac{s}{a}).$$

$P5$) For $s \in < 1-\beta, 1-\alpha >$,so

$$M[\frac{1}{x}f(\frac{1}{x})](s) = M[f(x)](1-s).$$

$P6$) "Mellin Transforms of Derivatives ".

The Mellin transformed of the derivative of the function f is given by the relation:

$$M[f'(x)](s) = (-1)(s-1)M[f(x)](s-1)$$

Provided with $[x^{s-1}f(x)]$ vanishes as $x \to 0$ and $x \to \infty$.

In general

$$M[f^{(k)}(x)](s) = (-1)^k(s-k)_k M[f(x)](s-k).$$

Where

$$(s-k)_k = \frac{\Gamma(s)}{\Gamma(s-k)}$$

Provided with $x^{s-r-1}f^{(r)} \to 0$ as $x \to 0, \infty$ for $r = 1, 2, 3, ..., (k-1)$ vanishes as $x \to 0$ and $x \to \infty$.

$P7$) Mellin transform and derivative

Let f be Mellin transformable function defined on $\mathbb{R}_+$ .Then if differentiation under the integral sign is allowed :

$$\frac{d}{ds}M(f)(s) = M[\log x f(x)].$$

$P8$) $M[xf'(x)] = -sM[f(x)](s)$ provided $x^s f(x) \to 0$ as $x \to 0$ and $x \to \infty$

$$In\ general.\ \ M[x^n f^{(n)}(x)] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} M[f(x)](s).$$

Proof

$P1)$ For all α,$\beta \in \mathbb{C}$, we have

$$\begin{aligned} M[\alpha f(x)+\beta g(x)](s) &= \int_0^\infty (\alpha f+\beta g)(x)x^{s-1}dx \\ &= \int_0^\infty \alpha f(x)x^{s-1}dx + \int_0^\infty \beta f(x)x^{s-1}dx \\ &= \alpha \int_0^\infty f(x)x^{s-1}dx + \beta \int_0^\infty g(x)x^{s-1}dx \\ &= \alpha M[f](s)+\beta M[g](s). \end{aligned}$$

$P2)$ since $a>0$ and for $s \in <\alpha,\beta>$

$$\begin{aligned} M[f(ax)](s) &= \int_0^\infty f(ax)x^{s-1}dx \\ &= \int_0^\infty \left(\frac{t}{a}\right)^{s-1} f(t)\frac{dt}{a} \\ &= a^{-s}M[f(x)](s). \end{aligned}$$

$P3)$ we have

$$\begin{aligned} M[f(x)x^a](s) &= \int_0^\infty x^{s-1}x^a f(x)dx \\ &= \int_0^\infty x^{s+a-1}f(x)dx \\ &= M[f(x)](s+a). \end{aligned}$$

$P4)$ since

$$\begin{aligned} M[f(x^a)](s) &= \int_0^\infty f(x^a)x^{s-1}dx \\ &= \int_0^\infty f(t)(t^{\frac{1}{a}})^{s-1}\frac{dt}{a} \quad (\textit{changing variables}) \\ &= \frac{1}{a}\int_0^\infty f(t)(t^{\frac{1}{a}})^{s-1}dt \\ &= \frac{1}{a}M[f(x)](\frac{s}{a}). \end{aligned}$$

$P5)$we set $g(x)=\frac{1}{x}f(\frac{1}{x})$

$$[M(g(x))](s) = \int_0^\infty \left[\frac{1}{x}f\left(\frac{1}{x}\right)\right]x^{s-1}dx.$$

Then

$$\begin{aligned}M[\frac{1}{x}f(\frac{1}{x})](s) &= -\int_{\infty}^{0}(t)^{1-s}tf(t)\frac{dt}{t^2}\\ &= \int_0^{\infty}f(t)(t)^{(1-s)-1)}dt\\ &= M[f(x)](1-s).\end{aligned}$$

$P6$) since

$$M[f'(x)](s) = \int_0^{\infty}x^{s-1}f'(x)dx,$$

We use the integration by parts

$$M[f'(x)](s) = [[x^{s-1}f(x)] - (s-1)] - \int_0^{\infty}x^{s-2}f(x)dx,$$

Since limit of f will be 0 when $x \to 0$ and $x \to \infty$.
so we find

$$\begin{aligned}M[f'(x)](s) &= \int_0^{\infty}x^{s-1}f'(x)dx\\ &= (-1)(s-1)M[f(x)](s-1)\end{aligned}$$

$$\begin{aligned}M[f''(x)](s) &= \int_0^{\infty}x^{s-1}f''(x)dx\\ &= (s-1)(s-2)M[f(x)](s-2)\end{aligned}$$

We continue with recurrence method we find the general form

$$M[f^{(k)}(x)](s) = (-1)^k(s-k)_kM[f(x)](s-k).$$

Where

$$\begin{aligned}(s-k)_k &= (s-k)(s-k+1)(s-k+2)........(s-1)\\ &= \frac{(s-1)!}{(s-k-1)!}\\ &= \frac{\Gamma(s)}{\Gamma(s-k)}.\end{aligned}$$

$p7$) We have by definition

$$M[(logx)f(x)] = \int_0^{\infty}x^{s-1}(\log x)f(x)dx,$$

In the another part we have

$$\begin{aligned} M[f(x)](s) &= \int_0^\infty x^{s-1} f(x)dx, \\ \frac{d}{ds}M[f(x)] &= \int_0^\infty \frac{d}{ds}(x)^{s-1} f(x)dx, \end{aligned}$$

Using $\frac{d}{ds}(a^x) = (\log a).a^x$ we get

$$\begin{aligned} \frac{d}{ds}M[f(x)] &= \int_0^\infty x^{s-1}(\log x) f(x)dx, \\ &= M[(\log x)f(x)]. \end{aligned}$$

$P8$)

$$\begin{aligned} M[xf'(x)] &= \int_0^\infty x^s f'(x)dx, \\ &= [x^s f(x)]_{x=0}^\infty \ - s\int_0^\infty f(x)dx, \\ &= -sM[f(x)](s). \\ M[x^2 f''(x)] &= (-1)^2 s(s+1))M[f(x)](s), \end{aligned}$$

In general

$$M[x^n f^{(n)}(x)] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} M[f(x)](s).$$

2.2 The inverse of the Mellin transform :

Definition 2.2. *(see[4])*
Let $f : [0, +\infty] \to \mathbb{C}$ be a continuous function, $M[f; s]$ is the Mellin transform of f, so we denote its inverse by M^{-1} :

$$f(x) = M^{-1}[M[f(x); s]] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[f(x); s]x^{-s}ds \tag{2.2}$$

proof
For $s = c + i\beta$ with $c > 0$, we have

$$\begin{aligned} M[f(x);s] &= \int_0^\infty f(x)x^{s-1}dx, \\ &= \int_0^\infty f(x)e^{s\log x}\frac{ds}{x}, \\ &= \int_0^\infty f(x)e^{c\log x}e^{i\beta\log x}\frac{dx}{x}, \end{aligned}$$

By changing the variable $\log x = u$ we get

$$M[f(x);s] = \int_{-\infty}^{+\infty} f(e^u)e^{cu}e^{i\beta u}du,$$

And with another variable change $u = -2\pi x$ we have

$$M[f(x);s] = 2\pi\int_{-\infty}^{+\infty} f(e^{-2\pi x})e^{-2\pi i\beta x}e^{-2\pi cx}dx$$

With $\beta \in \mathbb{R}$ and with the Fourier transform (1.7) of the real function, then , we have:

$$M(s) = 2\pi F[f(e^{-2\pi x})e^{-2\pi cx}, \beta].$$

We use the inverse formula of Fourier transform (1.9), we get

$$e^{-2\pi cx}f(e^{-2\pi x}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} M(s)e^{2i\pi\beta x}d\beta,$$

if we put $e^{-2\pi x} = t$, we find

$$f(t) = t^{-c}\frac{1}{2\pi}\int_{-\infty}^{+\infty} M(s)t^{-i\beta}d\beta,$$

In another part

$$f(t) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} M(s)t^{-(c+i\beta)}id\beta,$$

Finally the inverse formula is given by:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(s)t^{-s}ds.$$

Example 2.5. *We consider the Cahen-Mellin* [2] *integral*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}ds; x > 0$$

[2] Cahen is a surname and/or a first name that refer to the integral in the example

we check out for every $c > 0$ *and* $Re(s) > 0$ *:*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)x^{-s}ds = e^{-x}.$$

plus,the function $x \to \Gamma(s)x^{-s}$ *admits a simple poles and with theorem of residue we get:*

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)x^{-s}ds &= \sum_{k=1}^{\infty} Res(x^{-s}\Gamma(s), s=-k),\\
&= \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}x^k,\\
&= e^{-x},\\
so \quad M^{-1}[\Gamma(s)](x) &= e^{-x}.
\end{aligned}$$

Corollary 2.1. *(see [2],[9]) Let* $M[f;s]$*and* $M[g;s]$ *be the Mellin transforms of the functions* f *and* g *with strips of convergence* s_f *and* s_g *respectively, and suppose that* c *is a real number exist such that* $c \in s_f$ *and* $1-c \in s_g$*, the Parseval formula holds*

$$\int_0^{+\infty} f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[f;1-s]M[g;s]ds. \tag{2.3}$$

Proof : we have

$$\begin{aligned}
J &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[g](s)M[f](1-s)ds,\\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[f](1-s)\left(\int_0^{+\infty} x^{s-1}g(x)dx\right)ds,\\
&= \int_0^{\infty} g(x)\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{s-1}M[f](1-s)ds\right)dx.
\end{aligned}$$

we use the inverse formula for the function f in (2.2), we find

$$\begin{aligned}
\int_0^{\infty} f(x)g(x)dx &= \frac{1}{2\pi i}\int_{c-\infty}^{c+\infty} M[f](1-s)M[g](s)ds\\
&= \frac{1}{2\pi i}\int_{c-\infty}^{c+\infty} M[f;1-s]M[g;s]ds
\end{aligned}$$

2.3 The convolution product

Definition 2.3. *let f, g be two integrable functions, the classical convolution is defined with*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy.$$

Definition 2.4. *(see[3])*
Let f, g be two functions defined in $\mathbb{R}_*^+$.The Mellin Convolution product of f and g is defined as :

$$(f\ ^*\ g)(x) = \int_0^\infty \left(\frac{1}{y}\right)^s f(y)g\left(\frac{x}{y}\right) y^s \frac{dy}{y} = \int_0^\infty f(y)g(\frac{x}{y})\frac{dy}{y}. \tag{2.4}$$

Proposition 2.2. *If F and G are the Mellin transforms of the functions f and g respectively, then $\alpha < Re(s) < \beta$, and for a real number c well defined*
we have :

$$\int_0^\infty f(x)g(x)x^{s-1}dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s-z)G(z)dz. \tag{2.5}$$

In particular for: $s = 1$, we get

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(1-z)g(z)dz.$$

For more

$$M^{-1}[F(s)G(s)](x) = \int_0^\infty f(y)g(\frac{x}{y})\frac{dy}{y}. \tag{2.6}$$

It means

$$M[f\ ^*\ g(x)](s) = F(s)G(s). \tag{2.7}$$

Proof

to prove (2.5) we choose a real number c with $c \in I_c(s) =]Sup(\alpha_f, Re(s)-\beta_g), inf(\beta_f, Re(s)-\alpha_g)[$.And with the inversion theorem, we get

$$\begin{aligned}\int_0^\infty f(x)g(x)x^{s-1}dx &= \int_0^\infty f(x)x^{s-1}dx\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(z)x^{-z}dz,\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(z)dz\int_0^\infty f(x)x^{s-z-1}dx \quad (withFubini)\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s-z)G(z)dz\end{aligned}$$

Its the same for proving (2.6), we use the inverse theorem :

$$\begin{aligned} M^{-1}[F(s)G(s)](x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)G(z)x^{-s}ds, \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(s)x^{-s}\left(\int_0^\infty f(u)u^{s-1}du\right)ds, \\ &= \int_0^\infty f(u)\frac{du}{u}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(s)\left(\frac{x}{u}\right)^{-s}ds, \\ &= \int_0^\infty f(u)g\left(\frac{x}{u}\right)\frac{du}{u}. \end{aligned}$$

And that what we wanted.

2.3.1 Properties of the Mellin convolution:

The Mellin convolution product admit the next properties (see[3]) :

1. **Commutative:**
   Let f,g be two functions defined on $\mathbb{R}_*^+$, then $(f * g)(x)(s) = (g * f)(x)(s)$
2. **Associative:** Let f,g,h be three functions defined on $\mathbb{R}_*^+$, $(f * g) * h = f * (g * h)$
3. **Unit element:**
   Let f be a continuous function, so $(f * \delta(x-1)) = f$. Where δ is the Dirac [3]delta function.
4. **The action of the operator** $x\frac{d}{dx}$

$$\begin{aligned} \left(x\frac{d}{dx}\right)^k (f * g) &= \left(\left(x\frac{d}{dx}\right)^k f\right)^* g \\ &= f * \left(\left(x\frac{d}{dx}\right)^k g\right). \end{aligned}$$

5. **Multiplication by** $\ln x$ **:**
   Let f,g be two continuous functions , so

$$(lnx)(f * g) = [(\ln x)f] * g + f * [(\ln x)g].$$

6. If $f, g \in A_s$(defined in (1.1))then he convolution $(f * g)(x)$ exists almost every where on R_+,and belongs to the space A_s and further

[3]The Dirac delta function is defined as
$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\|(f\ ^{*}\ g)(x)\|_{A_s} \leq \|f\|_{A_s}.\|g\|_{A_s}.$$

proof

1. **Commutative**: we have

$$(f\ ^{*}\ g)(x) = \int_0^{\infty} f(y)g(\frac{x}{y})\frac{dy}{y}$$

By the variable change $u = \frac{x}{y}$

$$\left\{du = -\tfrac{x}{y^2}dy, -\tfrac{du}{u} = \tfrac{dy}{y}\right.$$

So we have

$$\begin{aligned}(f\ ^{*}\ g)(x) &= \int_0^{+\infty} f(y)g(\frac{x}{y})\frac{dy}{y} \\ &= \int_0^{+\infty} f(\frac{x}{y})g(u)\frac{du}{u}\end{aligned}$$

Finally we get $(f\ ^{*}\ g)(x)(s) = (g\ ^{*}\ f)(x)(s)$.

2. **Associative**: Let f, g, h be a continuous functions

$$\begin{aligned}(f\ ^{*}\ g)\ ^{*}\ h &= \int_0^{\infty} (f\ ^{*}\ g)(y)h(\frac{x}{y})\frac{dy}{y}, \\ &= \int_0^{\infty}\left(\int_0^{\infty} f(t)g\left(\frac{y}{t}\right)\frac{dt}{t}\right)h\left(\frac{x}{y}\right)\frac{dy}{y}, \\ &= \int_0^{\infty}\int_0^{\infty} f(t)g\left(\frac{y}{t}\right)h\left(\frac{x}{y}\right)\frac{dy}{y}\frac{dt}{t} \qquad ....withFubini \\ &= \int_0^{\infty} f(t)\left(\int_0^{\infty} g\left(\frac{y}{t}\right)h\left(\frac{x}{y}\right)\frac{dy}{y}\right)\frac{dt}{t},\end{aligned}$$

We put $\frac{y}{t} = v$, so we get

$$\begin{aligned}(f\ ^{*}\ g)\ ^{*}\ h &= \int_0^{\infty} f(t)\left(\int_0^{\infty} g(v)h\left(\frac{x}{tv}\right)\frac{dv}{v}\right)\frac{dt}{t} \\ &= \int_0^{\infty} f(t)\left(\int_0^{\infty} g(v)h\left(\frac{\frac{x}{t}}{v}\right)\frac{dv}{v}\right)\frac{dt}{t} \\ &= \int_0^{\infty}\int_0^{\infty} (g\ ^{*}\ h)(\frac{x}{t})\frac{dt}{t} \\ &= f\ ^{*}\ (g\ ^{*}\ h).\end{aligned}$$

2.4 The relation between Mellin transform and other transforms

Mellin transform and Laplace (see [2],)
Let f be a function and $M[f]$ its Mellin transform, then

$$M[f](s) = \int_0^\infty f(x)x^{s-1}dx$$

with a variable change $x = e^{-t}$, the Mellin transform will be

$$\begin{aligned} M[f](s) &= \int_{-\infty}^{+\infty} f(e^{-t})e^{-t(s-1)}e^{-t}dt \\ &= \int_{-\infty}^{+\infty} f(e^{-t})e^{-st}dt. \end{aligned}$$

So

$$M[f](s) = L[f(e^{-t})](s). \tag{2.8}$$

Mellin transform and Fourier (see [4])
We derive the Mellin transform from the complex Fourier transform (1.6).

$$F[f(t)](k) = G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikt}f(t)dt$$

Making the changes of variables $e^t = x$ and $ik = c - s$, where c is a constant, we obtain

$$G(is - ic) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{s-c-1}g(\log x)dx$$

We now write $\frac{1}{\sqrt{2\pi}}x^{-c}g(\log x) \equiv f(x)$ to get the Mellin transform of $f(x)$

$$M[f(x)] = \int_0^\infty x^{s-1}f(x)dx.$$

Chapter 3

Applications

In this chapter there is two sections, in the first one we discuss the Mellin transform in solving problems like integral equations and a method of summation of series. In the second section we apply the Mellin transform in solving problem with fractional integral and derivatives(Riemann-Liouville, Caputo).

3.1 Applications in classical calculus

3.1.1 Integral equation

The integral equation can be easily solved with the Mellin Transform (see [4]):

$$g(x) = \int_0^\infty f(y)k(xy)dy, \quad x > 0. \tag{3.1}$$

Where f is unknown and g, k are given functions.

Theorem 3.1. *(see [4]) Let f, g be two functions and $M[f](s)$ and $M[g](s)$ be their Mellin transform respectively, we defined the next operator by*

$$M[f(x)og(x)] = M\left[\int_0^\infty f(xy)g(y)dy\right] = M[f](s)M[g](1-s) \tag{3.2}$$

Proof *We have*

$$\begin{aligned}
M[f(x) o\, g(x)] &= M\left[\int_0^\infty f(xy)g(y)dy\right] \\
&= \int_0^\infty x^{s-1}dx \int_0^\infty f(xy)g(y)dy\,(xy=\eta), \\
&= \int_0^\infty g(y)dy \int_0^\infty \eta^{s-1}y^{1-s}f(\eta)\frac{d\eta}{y} \\
&= \int_0^\infty y^{1-s-1}g(y)dy \int_0^\infty \eta^{s-1}f(\eta)d\eta = M[f](s)M[g](1-s).
\end{aligned}$$

Note that, the operation o is not commutative

Application of the Mellin transform with respect to x to equation (3.1) combined with (3.2) gives

$$M[f](1-s)M[k](s) = M[g](s).$$

which gives , replacing s by $1-s$,

$$M[f](s) = M[g](1-s)M[h](s).$$

where

$$M[h](s) = \frac{1}{M[k](1-s)}$$

.

The inverse Mellin transform combined with (3.2) leads to the solution

$$\begin{aligned}
f(x) = M^{-1}[M[g](1-s)M[h](s)] &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\int_0^\infty y^{1-s-1}g(y)dy\int_0^\infty u^{s-1}h(u)du\right)ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^\infty g(y)dy\int_0^\infty u^{s-1}y^{1-s}h(u)\frac{du}{u}ds \quad with\,(u=xy) \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_0^\infty y^{s-1}dyds\int_0^\infty h(xy)g(y)dy \\
&= \int_0^\infty g(y)h(xy)dy
\end{aligned}$$

provided $h(x) = M^{-1}[M[h](s)]$exists .Thus, the problem is formally solved. In particular,if $M[h](s) = M[k](s)$, then the previous equality becomes

$$f(x) = \int_0^\infty g(y)k(xy)dy, \quad x > 0. \tag{3.3}$$

Application Solve the integral equation

$$\int_0^\infty f(y)g(\frac{x}{y})\frac{dy}{y} = h(x), \tag{3.4}$$

Where $f(x)$ is unknown and $g(x)$ and $h(x)$ are given functions. Applying theorem (3.1) the solution of the equation (3.4) is

$$M[f(x)](s) = M[h(x)](s)M[k(x)](s)\ ,\ M[k(x)](s) = \frac{1}{M[g(x)](s)}.$$

Inversion, by the convolution proposition (2.7), gives the solution

$$f(x) = M^{-1}\left[M[h(x)](s)M[k(x)](s)\right] = \int_0^\infty h(y)k(\frac{x}{y})\frac{dy}{y}. \tag{3.5}$$

3.1.2 Summation of series

We recall the Zeta Riemann function $\zeta(x) = \sum_{n=1}^{\infty}\frac{1}{n^x}$

Theorem 3.2. *(see [4])*
For all f defined over $\mathbb{R}^+$,$c \in <\alpha_f, \beta_f>$

$$\sum_{n=0}^{\infty} f(n+a) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[f(x)](s)\zeta(s,a)ds, \tag{3.6}$$

with Hurwitz Zeta function $\zeta(s,a)$ is defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}; \qquad 0 \le a \le 1,\ \Re(s) > 1 \tag{3.7}$$

Proof By applying the Inverse Mellin transform , we get

$$f(n+a) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)(n+a)^{-s}ds \tag{3.8}$$

Summing this over all n gives

$$\sum_{n=0}^{\infty} f(n+a) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[f(x)](s)\zeta(s,a)ds. \tag{3.9}$$

Theorem 3.3. *Similarly, On using the Mellin properties (P_2) we get*

$$f(nx) = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{c-i\infty}^{c+i\infty} F(s)n^{-s}x^{-s}ds,$$

Thus

$$\sum_{n=1}^{\infty} f(nx) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\zeta(s)x^{-s}ds. \tag{3.10}$$

Corollary 3.1. *when $x = 1$, result (3.10) reduces to*

$$\sum_{n=1}^{\infty} f(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\zeta(s)ds. \tag{3.11}$$

This can be obtained from (3.9) when $a = 0$.

Example 3.1.

$$f(n) = \sum_{n=1}^{+\infty} \frac{cos\ an}{n^2}$$

.

The Mellin transform of $f(x) = \frac{cos(ax)}{x^2}$ with $s \in< 2, 3 >$ is

$$\begin{aligned}
M[f](s) &= \int_0^{\infty} x^{s-3} cos(ax)dx \\
&= M[cos(ax)](s-2) \quad (\textit{with properties "Multiply by } x^a \textit{ "}) \\
&= \frac{1}{a^{s-2}} M[cosx](s-2) \quad (\textit{with "Scaling properties"}) \\
&= -\frac{1}{a^{s-2}} \Gamma(s-2) cos\left(\frac{\pi s}{2}\right)
\end{aligned}$$

We now that

$$\zeta(1-s) = \frac{2}{2\pi^s} \cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s) \tag{3.12}$$

substituting this result into (3.11) gives and we use the relation (3.12) we find

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{cos\ an}{n^2} &= -\frac{a^2}{2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{2\pi}{a}\right)^s \frac{\zeta(1-s)\Gamma(s-2)}{\Gamma(s)} ds \\
&= -\frac{a^2}{2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{2\pi}{a}\right)^s \frac{\zeta(1-s)}{(s-1)(s-2)} ds,
\end{aligned}$$

The integral has three simple poles at $0, 1$ and 2 with residues $\frac{-1}{2}$ and $\frac{\pi}{a}$, and $-\frac{\pi^2}{3a^2}$ respectively, and the complex integral is evaluated by calculating the residues at these poles. Thus, the sum of the series is

$$\sum_{n=1}^{\infty} \frac{cos\ an}{n^2} = \frac{a^2}{4} - \frac{\pi a}{2} + \frac{\pi^2}{6},$$

if a goes to 0 we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3.2 Application in fractional calculus

Theorem 3.4. *(see[12]) Let $f(x)$ be Mellin transformable function on $(0,\infty)$, where $0 \leq n-1 < \alpha < n$ then*

(1) $M[J^\alpha J^\beta f(x); s] = \frac{\Gamma(1-\alpha-\beta-s)}{\Gamma(1-s)} M[f(t); \alpha+\beta+s]$,

(2) $M[D_c^\alpha J_c^\alpha f(x); s] = M[f(x); s]$.

(3) $M[J^\alpha D_c^\alpha f(x); s] = M[f(x); s] - \sum_{k=0}^{m-1} \frac{f^k(0)}{k!(k+s)}, Re(s) > -Re(k)$,

Proof

(1) Now, we are applying Mellin transform of $J^\alpha J^\beta$

$$\begin{aligned} M[J^\alpha J^\beta f(x); s] &= M[J^{\alpha+\beta} f(x); s] \\ &= \int_0^\infty x^{s-1} J^{\alpha+\beta} f(x) dx \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty x^{s-1} \left(\int_0^x (x-t)^{\alpha+\beta-1} f(t) dt \right) dx \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty f(t) dt \int_t^\infty (x-t)^{\alpha+\beta-1} x^{s-1} dx. \end{aligned}$$

setting $x = \frac{t}{u}$ then the x-integral becomes

$$\begin{aligned} M[J^\alpha J^\beta f(x); s] &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty f(t) dt \int_0^1 t^{\alpha+\beta-1} \left(\frac{1}{u} - 1 \right)^{\alpha+\beta-1} \left(\frac{t}{u} \right)^{s-1} \frac{du}{u^2} \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty t^{\alpha+\beta+s-1} f(t) dt \int_0^1 u^{-\alpha-\beta-s} (1-u)^{\alpha+\beta-1} du. \end{aligned}$$

where $Re(\alpha+\beta) > 0, Re(\alpha+\beta+s) < 1$.
After using beta function which is defined by $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt$ and the fact that $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, hence obtain,

$$\begin{aligned} M[J^\alpha J^\beta f(x); s] &= \frac{B(-\alpha-\beta-s+1, \alpha+\beta)}{\Gamma(\alpha+\beta)} \int_0^\infty t^{\alpha+\beta+s-1} f(t) dt \\ &= \frac{\Gamma(-\alpha-\beta-s+1)}{\Gamma(1-s)} M[f(t); \alpha+\beta+s]. \end{aligned}$$

(2) The result is obtained by applying Mellin transform to both sides of the first property (1) in Theorem 1.15

$M[D_c^\alpha J_c^\alpha f(x); s] = M[f(x); s]$.

(3) We apply Mellin transform on the part (2) in Theorem 1.15 , then we obtain

$$\begin{aligned} M[J^\alpha D_c^\alpha f(x); s] &= M[f(x); s] - M\Big[\sum_{k=0}^{m-1} \frac{f^k(0)x^k}{k!}; s\Big] \\ &= M[f(x); s] - \sum_{k=0}^{m-1} \frac{f^k(0)}{k!} \int_0^\infty x^{k+s-1} dx \\ &= M[f(x); s] - \sum_{k=0}^{m-1} \frac{f^k(0)}{k!(k+s)}, \quad Re(s) > -Re(k) \end{aligned}$$

Theorem 3.5. *(see[12]) Let f be Mellin transformable defined on $\mathbb{R}_+$, $n-1 < \alpha < n, n \in \mathbb{N}$ then*

$$M[f^\alpha f(x); s] = \frac{\Gamma(s)}{\Gamma(s-\alpha)} M[f(x); s-\alpha].$$

Remarque 3.1. *By using the same technique in above theorem, Mellin transform of fractional integral can be yielded as the following formula :*

$$M[J^\alpha f(x); s] = \frac{\Gamma(s)}{\Gamma(s+\alpha)} M[f(x); s+\alpha].$$

Example 3.2. *(1) $M[f^{\frac{1}{2}}(x); s] = \int_0^{+\infty} x^{s-1} f^{\frac{1}{2}}(x) dx$, by using fractional integration by parts and fractional derivative of power function, we obtain*

$$\begin{aligned} M[f^{\frac{1}{2}}(x); s] &= \int_0^{+\infty} x^{s-1} f^{\frac{1}{2}}(x) dx = \int_0^{+\infty} f(x) D^{\frac{1}{2}} x^{s-1} dx \\ &= \frac{\Gamma(s)}{\Gamma(s-\frac{1}{2})} M\big[f(x); s - \frac{1}{2}\big]. \end{aligned}$$

(2) $M[f^{\frac{3}{2}}(x); s] = \int_0^{+\infty} x^{s-1} f^{\frac{3}{2}}(x) dx$, by using fractional integration by parts and fractional derivative of power function, we obtain

$$\begin{aligned} M[f^{\frac{3}{2}}(x); s] &= \int_0^{+\infty} x^{s-1} f^{\frac{3}{2}}(x) dx = \int_0^{+\infty} f(x) D^{\frac{3}{2}} x^{s-1} dx \\ &= \frac{\Gamma(s)}{\Gamma(s-\frac{3}{2})} M\big[f(x); s - \frac{3}{2}\big]. \end{aligned}$$

Theorem 3.6. *(see[12]) let f be Mellin transformable defined on $\mathbb{R}_+$, then*

$$M[x^\alpha f^\alpha(x); s] = \frac{\Gamma(s+\alpha)}{\Gamma(s)} M[f(x); s].$$

Example 3.3.

$$M[x^{\frac{1}{2}}f^{\frac{1}{2}}(x);s] = \int_0^{+\infty} f^{\frac{1}{2}}(x)x^{s-\frac{1}{2}}dx = \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}M[f(x);s].$$

$$M[x^{\frac{3}{2}}f^{\frac{3}{2}}(x);s] = \int_0^{+\infty} f^{\frac{3}{2}}(x)x^{s+\frac{1}{2}}dx = \frac{\Gamma(s+\frac{3}{2})}{\Gamma(s)}M[f(x);s].$$

Example of application
Solve the problem :

$$x^{\frac{1}{2}}f^{\frac{1}{2}}(x) = x^{\frac{3}{2}}f^{\frac{3}{2}}(x) = \Gamma(x-\alpha).$$

Solution : By applying the Mellin transform to both side we have

$$\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}M[f(x);s] + \frac{\Gamma(s+\frac{3}{2})}{\Gamma(s)}M[f(x);s] = a^{s-1}.$$

By solving the equation and applying the inverse Mellin transform by using complex inversion integral in order to cover the $f(x)$ explicitly as the solution

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{\Gamma(s+\frac{1}{2})\Gamma(s+\frac{3}{2})}a^{s-1}x^{-s}ds.$$

3.2.1 Mellin transforms of the Weyl fractional integral

Theorem 3.7. *(see[4]) The Mellin transform of the Weyl fractional integral of $f(x)$ is defined by*

$$M[W^{-\alpha}f(x),s] = \frac{\Gamma(s)}{\Gamma(s+\alpha)}M[f(x),s+\alpha] \tag{3.13}$$

Proof :
We use The Weyl fractional integral(1.19)

$$\begin{aligned}(W^{-\alpha}f)(x) &= \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}f(t)dt\\ &= \frac{1}{\Gamma(\alpha)}\int_x^{\infty}t^{\alpha-1}\left(1-\frac{x}{t}\right)^{\alpha-1}f(t)dt\\ &= \frac{1}{\Gamma(\alpha)}\int_x^{\infty}t^{\alpha}f(t)\left(1-\frac{x}{t}\right)^{\alpha-1}\frac{dt}{t}.\end{aligned}$$

we calculate the Mellin transform of the Weyl fractional integral by putting
$h(t) = t^{\alpha}f(t)$ *and* $g(\frac{x}{t}) = \frac{1}{\Gamma(\alpha)}(1-\frac{x}{t})^{\alpha-1}H(1-\frac{x}{t})$ *, where* $H(1-\frac{x}{t})$ *is the Heaviside*

function, so

$$W^{\alpha}[f(t)] = \int_0^{\infty} h(t)g(\frac{x}{t})\frac{dt}{t}$$

which is, by the proposition in the equation (2.7)

$$\begin{aligned}
M[W^{\alpha}f(x),s] &= M[x^{\alpha}f(x)]M[\frac{1}{\Gamma(\alpha)}(1-x)^{\alpha-1}H(1-x)] \\
&= M[x^{\alpha}f(x)]\frac{1}{\Gamma(\alpha)}\int_0^1 x^{s-1}(1-x)^{\alpha-1}dx \\
&= M[x^{\alpha}f(x)]\left(\frac{B(s,\alpha)}{\Gamma(\alpha)} - \frac{\Gamma(s)}{\Gamma(s+\alpha)}\right) \\
&= M[x^{\alpha}f(x)]\frac{\Gamma(s)}{\Gamma(s+\alpha)}.
\end{aligned}$$

Tabular of the Mellin transform

Original function Mellin transform

$f(t), t > 0$	$M[f;s] = \int_0^\infty f(x)x^{s-1}dx$	Strip of holomorphy
$e^{-pt}, p > 0$	$p^- s\Gamma(s)$	$Re(s) > 0$
$H(t-a)t^b, a > 0$	$-\frac{a^{b+s}}{b+s}$	$Re(s) < -Re(b)$
$(1+t)^{-1}$	$\frac{\pi}{sin(\pi s)}$	$0 < Re(s) < 1$
$(1+t)^{-1}$	$\frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}$	$Re(s) > 0$
$H(t-1)(t-1)^{-b}Re(b) > 0$	$\frac{\Gamma(s)\Gamma(b)}{\Gamma(s+b)}$	$Re(s) > 0$
$H(t-1)sin(aln(t))$	$\frac{a}{s^2+a^2}$	$Re(s) < - \mid \Im(a) \mid$
$ln(1+t)$	$\frac{\pi}{ssin(\pi s)}$	$-1 < Re(s) < 0$
$t^{-1}ln(1+t)$	$\frac{\pi}{(1-s)sin(\pi s)}$	$0 < Re(s) < 1$
$(e^t - 1)^{-1}$	$\Gamma(s)\zeta(s)$	$Re(s) > 1$
$t^{-1}e^{-t^{-1}}$	$\Gamma(1-s)$	$-\infty < Re(s) < 1$
e^{-x^2}	$\frac{1}{2}\Gamma(\frac{s}{2})$	$0 < Re(s) < +\infty$
$tan^{-1}(t)$	$\frac{-\pi}{2scos(\pi s/2)}$	$-1 < Re(s) < 0$
$cotan^{-1}(t)$	$\frac{\pi}{2scos(\pi s/2)}$	$0 < Re(s) < 1$
$\sigma(t-p), p > 0$	p^{s-1}	whole plane
$\sum_{n=1}^{\infty} \sigma(t-pn), p > 0$	$p^{s-1}\zeta(1-s)$	$Re(s) < 0$

Annex

Robert Hjalmar Mellin, the son of the clergyman Gustaf Robert Mellin (1826-1880) was born in Liminka, northern Ostrobothnia, in Finland in 1854, he grew up and received his schooling in Hmeenlinna then undertook his university studies at the University of Helsinki and he continues the research of his doctorate at the same university, where he met his greatest influence Mittag-Leffler, he introduced Mellin to function theory in the style of Weierstrass.
He studied the transform which now bears his name and established its reciprocal properties.He applied this technique systematically in a long series of papers to the study of the gamma function, hypergeometric functions, Dirichlet series, the Riemann zeta function and related number-theoretic functions. He also extended his transform to several variables and applied it to the solution of partial differential equations. The use of the inverse form of the transform, expressed as an integral parallel to the imaginary axis of the variable of integration. In 1895 Mellin received a prize from the Finnish Society of Sciences and Letters, and also in 1927 he received a major award from the Alfred Kordelin Foundation set up in 1918 to support the sciences, literature, the arts and public education with grants and awards. He spent his last years working with his beliefs, closing his eyes to eternal sleep on 5 April 1933 in the evening.For More biography of R.H.Mellin including a sketch of his works can be found in [14].
However several mathematicians work to develop the Mellin transform,such as Riemann and M.Cahen (a further extension) are given in [5],O.I.Marichev (who has extended the Mellin method and devised a systematic procedure to make it practical)[[8]].

Bibliography

[1] ***Bellassoued Mourad***, Cours et exercices danalyse Complexe, 2008 .

[2] ***Bertrand.J, Bertrand.P, Ovarpez.J***, The Mellin Transform : The Transforms and Applications Handbook, Second Edition,Alexen.

[3] ***Buschaman.R.G , Srivastava,H.M.***, Theory and Applications of Convolution Integral Equations, Springer Science+Business Media B.V., 1992.

[4] ***Debnath.L, and Bhatta.D***, Integral Transforms and Their Applications, Third Edition, Boca Raton : CRC Press LLC, 2015.

[5] ***Ditkin.V.A and Prudnikov.A.P,*** Integral Transforms and Operational Calculus, Pergamon Press, 1965.

[6] ***Yves Driencourt,*** Rsum du cours dAnalyse 3 (L3 Maths). Printemps 2010.

[7] ***Erdelyi. A, Magnus. W, Oberhettinger. F and Tricomi .F***,Higher Transcendental Functions , Voll.III,Krieger Pub ,Melbourne, Florida,(1981).

[8] ***Gradshteyn.I.S and Ryzhik.I.M***, Table of Integrals, Series and Products, A.Jeffrey editor, fifth edition, Academic Press 1994.

[9] ***Paris.R.B, Kaminski.D*** : Asymptotics and Mellin-Barnes Integrals , Encyclopedia Of Mathematics And Its Applications, 85.

[10] ***A.A.A Kilbas,H.M.Sirvastava and JJ.Trujillo***,Theory and application of fractional differential Equation North-Holland Mathematical studies 204, Ed van Mill. Amsterdam, (2006).

[11] ***Kilicman.A***,A Note on Mellin transform and distributions 2004.

[12] ***Kilicman.A and Omran.M***, A Note on fractional Mellin transform and applications 2006, Dol 10.1186/s40064-016-1711-x.

[13] ***Kudryavtsev,L.D.*** ,Fubini theorem, in Hazewinkel, Michiel, Encyclopedia of Mathematics, Springer Science.Business Media B.V., Kluwer Academic Publishers, ISBN 978,,5608,0, 1994.

[14] ***F.Oberhettinger***, Tables of Mellin Transforms, Springer-Verlag, New York 1974.

[15] ***Robert.A.Adams***,SOBOLEV SPACES ,New York San Francisco London 1975.

[16] ***Sandeep Tyagi,*** Christian Holm Frankfurt Institute for Advanced Studies, Max-von-Laue-Strae 1, D-60438 Frankfurt am Main, Germany.