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Presented by

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SUBJECT OF THE MEMOIR :

Compactness method por a class evolution equation Semi Linear

Supported the Monday 02/11/2020 In front of the Jury Composed of :

Mr : K.MAAZOUZ M.C.(B) Tiaret University	President
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DEDICATIONS

♡ I dedicate my work to ♡

My mother and father for their love, their encouragement and their sacrifices.

To my brothers and my sisters.

To my friend Attallah amine and study colleagues.

And to all those who have contributed from near or far to make this project possible, I say thank you.

Soualam Abdelhak

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Chiheb karim

GENERAL RATING

$\mathit{\Omega}$	Bounded open set of $\mathbb{R}^n$.
$\partial\Omega$	The border of Ω
X	Banach space.
$H_0^1(\Omega),\ H^m(\Omega)$	Sobolev spaces.
$B(X)$	Banach algebra of bounded linear operators in X.
I	Identity operator.
$\pounds(X)$	Set of linear maps from X to X.
$D(A)$	Domain of A.
A	bounded linear operator.
$\rho(A)$	Solving set of operator A.
$\sigma(A)$	Spectre set of operator A.
$D^\alpha u$	Weak derivative of u of order α.
$Im(T(t))$	Range of $T(t)$.
X^*	Topological dual of X.
Δ	Laplacien operator.
$\alpha(B)$	Non compacity mesure of Kuratowski.
IVP	Initial value problem.

CONTENTS

CHAPTER 1

SEMI GROUP OF BOUNDED LINEAR OPERATORS

1.1 Preliminaries

Let Ω be a bounded open of $\mathbb{R}^n$

Definition 1.1.1. *Let X a Banach space and*

$$\begin{aligned} PC([0,b],X) \;\; &= \Big\{x : [0,b] \to X : x_k \in C(t_k, t_{k+1}], k = 0, \dots, m \quad \text{and there exists} \\ & x(t_k^-), x(t_k^+), k = 1, \dots, m \text{ with } x(t_k) = x(t_k^-)\Big\}, \end{aligned}$$

which is a Banach space with the norm

$$\|y\|_{PC} = \max_{k=1,\dots m.} \|x_k\|,$$

Definition 1.1.2. *The Sobolev Space is defined by :*

$$H^m(\Omega) = \big\{u \in L^2(\Omega) : \forall \alpha \in N^n, |\alpha| \leq m, D^\alpha u \in L^2(\Omega)\big\}$$

Definition 1.1.3. *The Sobolev Space is defined by :*

$$H_0^1(\Omega) = \big\{u \in H^1(\Omega), u\backslash_{\partial\Omega} = 0\big\}$$

Definition 1.1.4. *An unbounded linear operator A in X is a couple $(A, D(A))$ such that D is a subspace of X representing the domaine of A, and A is the linear map from X into X defined by*

$$A : D(A) \subset X \longrightarrow X$$

Definition 1.1.5. *Let X be a Banach space , A one parameter family $(T(t))_{t\geq 0}$, $t \geq 0$, of bounded linear operators from X into X is semi-group of bounded linear operators on X if* *i)* *T(0)=I,*

ii) *T(s+t)=T(s)·T(t)* , *for every* $t, s \geq 0$.

The Linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} exists \right\}$$

and

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} = \left. \frac{d^+T(t)x}{dt} \right|_{t=0}, for\ x \in D(A)$$

is the infinitesimal generator of the semi-group $T(t)$
$D(A)$ *is the domain of* A

Remark. *Let X a Banach space:*

1. *We note by* $B(X)$ *the Banach algebra of bounded linear operators in* X *and by* I *identity of* $B(X)$.

2. *For a linear operator* $A : D(A) \subset X \longrightarrow X$. *We note by:*

$$\rho(A) = \{\lambda \in \mathbb{C}, (\lambda I - A) \text{ is inversible in } B(X)\},$$

the resolving set of $A \in B(X)$ *and by :*

$$\begin{aligned} R(\cdot; A) : \rho(A) &\longrightarrow B(X) \\ \lambda &\longmapsto (\lambda I - A)^{-1}, \end{aligned}$$

the resolver of A.

Definition 1.1.6. *Let* $\{T(t)\}_{t \geq 0}$ *a semi-group defined on a Banach X space.*

i) *T(t) is uniformly continuous if:*

$$\lim_{t \longrightarrow 0^+} ||T(t) - I||_{B(X)} = 0$$

ii) *T(t) is strongly continuous if:*

$$\forall x, x \in X : \lim_{t \longrightarrow 0^+} ||T(t)x - x||_X = 0$$

Lemma 1.1.1. *Let* $f : [a, b] \longrightarrow X$ *a continuous function then,*

$$\lim_{t \longrightarrow 0} \frac{1}{t} \int_a^{a+t} f(s)ds = f(a) \tag{1.1}$$

Proof. *For everything $t \neq 0$, We have :*

$$\left\| \frac{1}{t} \int_a^{a+t} f(s)ds - f(a) \right\| = \left\| \frac{1}{t} \int_a^{a+t} (f(s) - f(a))\, ds \right\|$$
$$\leq \frac{1}{t} \times \sup_{s \in [a, a+t[} ||f(s) - f(a)|| \times t$$
$$\leq \sup_{s \in [a, a+t[} ||f(s) - f(a)||.$$

The continuity of f allows us to conclude .

Lemma 1.1.2. *Let $\{T(t)\}_{t\geq 0}$ a C_0-semi-group, then,*

$$\lim_{h \longrightarrow 0} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x, \forall t \geq 0, \forall x \in X \tag{1.2}$$

Proof.

$$||\frac{1}{h} \int_t^{t+h} T(s)xds - T(t)x|| \leq \frac{1}{h} \int_t^{t+h} ||T(s)x - T(t)x||ds$$

according to the Lemma 1.1.1 , then :

$$\lim_{h \longrightarrow 0} \frac{1}{h} \int_t^{t+h} ||T(s)x - T(t)x||ds = 0$$

Proposition 1.1.1. *Let $\{T(t)\}_{t\geq 0}$ is a C_0-semi-group and A its infinitesimal generator;*

if $x \in D(A)$, so $T(t)x \in D(A)$

Proof. *Let $x \in D(A)$, so for everything $t \geq 0$, we have :*

$$T(t)Ax = T(t) \lim_{h \longrightarrow 0} \frac{T(h)x - x}{h}$$
$$= \lim_{h \longrightarrow 0} \frac{T(h)T(t)x - T(t)x}{h}.$$

Therefore $T(t)x \in D(A)$.

Proposition 1.1.2. *Let $\{T(t)\}_{t\geq 0}$ and A its infinitesimal generator, then,*

$$\int_0^t T(s)xds \in D(A), \forall x \in D(A) \tag{1.3}$$

Example. *Let*

$$C = \{f : [0, +\infty[\longrightarrow \mathbb{R}, f \text{ is uniformly continuous and bounded}\}.$$

and

$$||f|| = \sup_{\alpha \in [0, +\infty[} |f(\alpha)|, (T(t)f)\alpha = f(t + \alpha) \quad , \forall\, t \geq 0 \text{ and } \alpha \in [0, +\infty[$$

1. $\forall\ f \in\ C, \forall\ \alpha \geq 0,$

$$\begin{aligned}(T(0)f)\alpha &= f(0+\alpha)\\ &= f(\alpha)\end{aligned}$$

Therefore T(0)=I.

2. $\forall\ f \in\ C, \forall\ t,s \geq\ 0$

$$\begin{aligned}(T(t+s)f)\alpha &= f(t+s+\alpha)\\ &= (T(t)f)(s+\alpha)\\ &= (T(t)\cdot T(s)f)\alpha\end{aligned}$$

Therefore T(t+s)=T(t)·T(s).

3. $\forall\ f \in\ C,$

$$\begin{aligned}\lim_{t\longrightarrow 0}||T(t)f-f||_C &= \lim_{t\longrightarrow 0}\{\sup_{\alpha\in[0,+\infty[}|f(t+\alpha)-f(\alpha)|\}\\ &= 0.\end{aligned}$$

Similarly, we have:

$$\begin{aligned}||T(t)f||_C &= \sup_{\alpha\in[0,+\infty[}|(T(t)f)\alpha|\\ &= \sup_{\alpha\in[0,+\infty[}|f(t+\alpha)|\\ &= \sup_{\beta\in[t,+\infty[}|f(\beta)|\\ &\leq \sup_{\beta\in[0,+\infty[}|f(\beta)| = ||f||_C, \forall t\geq 0.\end{aligned}$$

Therefore $||T(t)|| = 1, \forall t \geq 0,$

so $\{T(t)\}_{t\geq 0}$ *is a* C_0*-semi-group of linear operators bounded on C.*
Let $A : D(A) \subset C \longrightarrow C$ *the infinitesimal generator of the semi-group* $\{T(t)\}_{t\geq 0}$
If $f \in D(A)$*, so we have:*

$$\begin{aligned}Af(\alpha) &= \lim_{t\longrightarrow 0}\frac{T(t)f(\alpha)-f(\alpha)}{t}\\ &= \lim_{t\longrightarrow 0}\frac{f(\alpha+t)-f(\alpha)}{t}\\ &= f'(\alpha)\end{aligned}$$

uniformly compared to α*. Therefore*

$$D(A) \subset \{f \in C, f' \in C\}. \tag{1.4}$$

If $f\in C$ *such that* $f'\in C$, *Therefore:*

$$\left\| \frac{T(t)f-f}{t} - f' \right\|_C = \sup_{\alpha\in[0,+\infty[} \left| \frac{(T(t)f)\alpha - f(\alpha)}{t} - f'(\alpha) \right|$$

But

$$\begin{aligned}\left| \frac{(T(t)f)\alpha - f(\alpha)}{t} - f'(\alpha) \right| &= \left| \frac{f(\alpha+t)-f(\alpha)}{t} - f'(\alpha) \right| \\ &= \left| \frac{1}{t}[f(\tau)]_{\alpha}{}^{\alpha+t} - f'(\alpha) \right| \\ &= \frac{1}{t}\left| \int_{\alpha}^{\alpha+t} (f'(\tau)-f'(\alpha))d\tau \right| \\ &\leq \frac{1}{t}\int_{\alpha}^{\alpha+t} |f'(\tau)-f'(\alpha)|d\tau \longrightarrow 0.\end{aligned}$$

uniformly compared to α *for* $t\longrightarrow 0$. *Therefore:*

$$\left\| \frac{T(t)f-f}{t} - f' \right\|_C \longrightarrow 0, If\ t\longrightarrow 0,$$

Therefore $f\in D(A)$ *and :*

$$\{f\in C, f'\in C\}\subset D(A) \tag{1.5}$$

From (1.4) and (1.5) we conclude $D(A)=\{f\in C, f'\in C\}$
Let $f\in C$ *and* $t_n>0$, $n\in\mathbb{N}$ *such that* $\lim_{n\longrightarrow 0} t_n = 0$, *therefore:*
According to (1.3) we obtain

$$f_n = \frac{1}{t_n}\int_0^{t_n} T(s)fds \in D(A), \forall n\in\mathbb{N}$$

so

$$\begin{aligned}\lim_{n\longrightarrow 0} f_n &= \lim_{n\longrightarrow 0} \frac{1}{t_n}\int_0^{t_n} T(s)fds \\ &= T(0)f \\ &= f.\end{aligned}$$

Therefore $\overline{D(A)}=C$
finally we deduce $D(A)$ *is dense in* C .

Theorem 1.1.1. *a linear operator* A *is the infinitesimal generator of a uniformly continuous semi-group on* X *, if and only if* A *is a bounded linear operator .*

Proof. ($\Longleftarrow$) *Let* $A\in\mathcal{L}(X)$ *, for* $t\geq 0$ *, Let's pose :*

$$T(t)=e^{tA}=\sum_{n=0}^{+\infty}\frac{t^nA^n}{n!}.$$

This series is convergent and defines a bounded linear operator.
It's clear that $T(0) = I$ *and for all* $t, s \geq 0$

$$T(t+s) = e^{(t+s)A} = e^{tA}.e^{sA} = T(t) \cdot T(s).$$

On the other hand , $\forall t \geq 0$ *:*

$$\begin{aligned} ||T(t) - I|| &= \left\| \sum_{n=0}^{+\infty} \frac{t^n A^n}{n!} - I \right\| \\ &= \left\| \sum_{n=1}^{+\infty} \frac{t^n A^n}{n!} \right\| \\ &\leq \sum_{n=1}^{+\infty} \frac{t^n ||A||^n}{n!} \end{aligned}$$

$$\Longrightarrow \lim_{t \longrightarrow 0} ||T(t) - I||_{B(X)} = 0$$

$\forall\ t > 0$ *we have :*

$$\begin{aligned} \left\| \frac{T(t) - I}{t} - A \right\| &= \left\| \frac{e^{tA} - I}{t} - A \right\| \\ &= \left\| \frac{1}{t} \sum_{n=2}^{+\infty} \frac{t^n A^n}{n!} \right\| \\ &\leq \frac{1}{t} \sum_{n=2}^{+\infty} \frac{t^n ||A||^n}{n!} \\ &\leq \frac{1}{t} \left(e^{t||A||} - 1 - t||A|| \right) \longrightarrow_{t \longrightarrow 0} 0 \end{aligned}$$

Therefore

$$\lim_{t \longrightarrow 0} \frac{T(t) - I}{t} = A.$$

So $(T(t))_{t \geq 0}$ *is a uniformly continuous semi-group .*
$(\Longrightarrow)$ *Let* $(T(t))_{t \geq 0}$ *a uniformly continuous semi-group of infinitesimal generator A .*
By (Lemma 1.1.1) we have :

$$\frac{1}{t} \int_a^{a+t} f(s) ds = f(a)$$

$\forall\ s\ ,\ s' \in [a, a+t]\ ,\ ||f(s) - f(s')|| \leq \epsilon$
The continuity of f allows us to conclude , the application

$$\begin{aligned} \mathbb{R}^+ &\longrightarrow B(X) \\ t\ \ &\longmapsto T(t) \end{aligned}$$

is continuous .
as $\int_0^t T(s)ds$ *is well defined , from the Lemma 1.1.1*

$$\frac{1}{t}\int_0^t T(s)ds \quad converges(in\ norm)\ to\ I.$$

it exists $\Phi > 0$ *such that*

$$\left\| \frac{1}{\Phi}\int_0^{\Phi} T(s) - I \right\| < 1$$

which implies

$$\frac{1}{\Phi}\int_0^{\Phi} T(s)ds \quad is\ invertible$$

and

$$\int_0^{\Phi} T(s)ds. \quad is\ also\ invertible$$

For $h > 0,$ *we have :*

$$\begin{aligned}\left(\frac{T(h)-I}{h}\right)\left(\int_0^{\Phi} T(s)ds\right) &= \frac{1}{h}\left[\int_0^{\Phi}[T(h+s)-T(s)]ds\right]\\ &= \frac{1}{h}\left(\int_{\Phi}^{\Phi+h} T(s)ds - \int_0^h T(s)ds\right).\end{aligned}$$

Therefore

$$\frac{T(h)-I}{h} = \left[\frac{1}{h}\int_{\Phi}^{\Phi+h} T(s)ds - \frac{1}{h}\int_0^h T(s)ds\right]\left[\int_0^{\Phi} T(s)ds\right]^{-1}.$$

From the Lemma 1.1.1 , we obtain by passing the limit $h \longrightarrow 0^+$

$$\lim_{h\longrightarrow 0^+}\frac{T(h)-I}{h} = (T(\Phi)-I)\left(\int_0^{\Phi} T(s)ds\right)^{-1}.$$

Thus the infinitesimal generator of the semi-group $(T(t))_{t\geq 0}$ *is the bounded linear operator .*

$$A = (T(\Phi)-I)\left(\int_0^{\Phi} T(s)ds\right)^{-1}.$$

Corollary 1.1.2. *Let* $T(t)$ *be a uniformly continuous semi-group of bounded linear operators. Then*

a) *There exists a constant* $w \geq 0$ *,* $M \geq 0$ *such that* $\|T(t)\| \leq Me^{wt}$*.*

b) *There exists a unique bounded linear operator* A *such that* $T(t) = e^{tA}$*.*

c) *The operator* A *in part (b) is the infinitesimal generator of* $T(t)$*.*

d) $t \mapsto T(t)$ *is differentiable in norm and*

$$\frac{dT(t)}{dt} = AT(t) = T(t)A.$$

Proof. *See [1].*

Theorem 1.1.3. *Let $T(t)_{t\geq 0}$ be a C_0 semi-group and let A be its infinitesimal generator. Then a*

a) For $x \in X$

$$\lim_{h\longrightarrow 0}\frac{1}{h}\int_t^{t+h} T(s)xds = T(t)x$$

b) For $x \in X$,$\int_0^t T(s)xds \in D(A)$ and

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x$$

c) For $x \in D(A)$,$T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$$

d) For $x \in D(A)$

$$T(t)x - T(s)x = \int_s^t AT(\tau)xd\tau = \int_s^t T(\tau)Axd\tau.$$

Proof. *Let $T(t)$ be a C_0 semi-group and let A be its infinitesimal generator.*
a) one has deduce easily from Lemma 1.1.2
b) Let $x \in X$ and $h > 0$.one has ,

$$\begin{aligned}
\frac{T(h)-I}{h}\int_0^t T(s)xds &= \frac{1}{h}\int_0^t T(s+h)xds - \frac{1}{h}\int_0^t T(s)xds \\
&= \frac{1}{h}\int_h^{t+h} T(\mu)xd\mu - \frac{1}{h}\int_0^t T(s)xds \\
&= \frac{1}{h}\int_0^{t+h} T(\mu)xd\mu - \frac{1}{h}\int_0^h T(\mu)xd\mu - \frac{1}{h}\int_0^t T(\mu)xd\mu \\
&= \frac{1}{h}\int_t^{t+h} T(\mu)xd\mu - \frac{1}{h}\int_0^t T(\mu)xd\mu
\end{aligned}$$

by crossing limit for $h \longrightarrow 0$ and according to Lemma1.1.1,we obtain

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x, \forall x \in X\ .$$

c) Let $x \in D(A)$,$t \geq 0$ and $h>0$ so:

$$\begin{aligned}
\left\|\frac{T(t+h)x - T(t)x}{h} - T(t)Ax\right\| &\leq ||T(t)||\left\|\frac{T(h)x-x}{h} - Ax\right\| \\
&\leq Me^{wt}\left\|\frac{T(h)x-x}{h} - Ax\right\|.
\end{aligned}$$

Therefore,

$$\lim_{h \longrightarrow 0} \frac{T(t+h)x - T(t)x}{h} = T(t)Ax$$

so :

$$\frac{d^+}{dt} T(t)x = T(t)Ax, \forall t \geq 0.$$

if t-h >0 ,we have :

$$\begin{aligned} \left\| \frac{T(t-h)x - T(t)x}{-h} - T(t)Ax \right\| &\leq ||T(t-h)|| \left\| \frac{T(h)x - x}{h} - Ax + Ax - T(h)Ax \right\| \\ &\leq Me^{w(t-h)} \left(\left\| \frac{T(h)x - x}{h} - Ax \right\| + ||T(h)Ax - Ax|| \right). \end{aligned}$$

Therefore,

$$\lim_{h \longrightarrow 0} \frac{T(t-h)x - T(t)x}{-h} = T(t)Ax.$$

So ,

$$\frac{d^-}{dt} T(t)x = T(t)Ax, \forall t \geq 0.$$

We conclude: $T(t)x$ *is differentiable on [0,+∞[regardless of* $x \in D(A)$ *.*
and we have equality :

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax$$

d) The result is obtained by integrating the identity c) between s and t .

Corollary 1.1.4. *If* A *is the infinitesimal generator of a* C_0 *semi-group* $T(t)$ *then* $D(A)$*, the domain of* A*, is dense in* X *and* A *is a closed linear operator .*

Theorem 1.1.5. *Let* $T(t)$ *and* $S(t)$ *be* C_0 *semi-groups of bounded linear operators with infinitesimal generators* A *and* B *respectively. If* $A = B$ *then* $T(t) = S(t)$ *for* $t \longrightarrow 0$ *.*

Proof. *Let* $x \in D(A)$ *,* $t > 0$ *and let* $f : [0, t] \longrightarrow X$ *be given by*

$$f(s) = S(t-s)T(s)x$$

Fo each $s \in [0, t]$ *, By ((c) in Theorem 1.1.3) , it follows f is differentiable on* $[0, t]$ *, and that :*

$$\begin{aligned} f'(s) &= -AS(t-s)T(s)x + S(t-s)AT(s)x \\ &= -AS(t-s)T(s)x + AS(t-s)T(s)x \\ &= 0. \end{aligned}$$

For each $s \in [0, t]$ *, thus f is constant , hence we have f(0)=f(t) , or equivalently S(t)x=T(t)x for each* $x \in D(A)$ *. Since D(A) is dense in X and S(t) , T(t) are linear bounded operators , we easily conclude that S(t)x=T(t)x for each* $x \in X$ *, which completes the proof .*

Theorem 1.1.6. *(Hille-Yosida). A linear (unbounded) operator A is the infinitesimal generator of a C_0 semi-group of contractions $T(t)$, $t \longrightarrow 0$ if and only If*

i) A is closed and $\overline{D(A)} = X$.

ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$ we have :

$$||R(\lambda, A)|| \leq \frac{1}{\lambda}$$

Proof. *See [1]*

Corollary 1.1.7. *If $(A, D(A))$ is the infinitesimal generator of a strongly continuous semi-group on X , $(S(t))_{t\geq 0}$, then $D(A)$ is dense in X, and A is closed .*

Definition 1.1.7. *A strongly continuous semi-group $(S(t))_{t\geq 0}$ on X is a semi-group of contractions if*

$$\|S(t)\| \leq 1$$

for all $t \geq 0$.

Theorem 1.1.8. *(To Schauder)*
Let Y a Banach space .
Let C_0 a closed convex bounded of Y .
and $T : C_0 \longrightarrow C_0$ a continuous application such as $\overline{T(C_0)}$ compact.then T admits at least one fixed point in C_0

Theorem 1.1.9. *(Sadovski)*
Be Y a Banach space .
C_0 a closed convex bounded ,and $T : C_0 \longrightarrow C_0$ continuous
$\alpha(\cdot)$ non-compactness measurement .
$\forall$ B bounded and $\alpha(B) > 0 \qquad \alpha(TB) < \alpha(B)$
So T admits at least one fixed point in C_0.

CHAPTER 2

SOME PARTICULAR C_0-SEMI-GROUPS

2.1 Differentiable C_0-semi-groups

Definition 2.1.1. *A C_0-semi-group $(T(t))_{t\geq 0}$ on X is said differentiable if for every $x \in X$, the application $t \longmapsto T(t)x$ is differentiable for $t > t_0$.*

Theorem 2.1.1. *Let $(T(t))_{t\geq 0}$ a C_0-semi-group and A its infinitesimal generator , the following statements are equivalent :*

i) $(T(t))_{t\geq 0}$ *is a differentiable C_0-semi-group .*

ii) $Im(T(t)) \subset D(A)$, $\forall\ t > 0$.

Proof. $(i) \Longrightarrow (ii)$ *: Let $x \in X$ and $t, h > 0$.*
Since the application $]0, +\infty) \ni t \longmapsto T(t)x \in X$ is differentiable , so

$$\lim_{h\longrightarrow 0} \frac{T(t+h)x - T(t)x}{h} exists$$

so $T(t)x \in D(A)$, from where $Im(T(t)) \subset D(A)$.

$(ii) \Longrightarrow (i)$ *: Let $x \in X$ and $t, h > 0$.*
That $T(t)x \in D(A)$, so :

$$\frac{d^+}{dt}T(t)x = \lim_{h\longrightarrow 0^+} \frac{T(t+h)x - T(t)x}{h} = AT(t)x.$$

On the other hand , for $h \in]0,t[$ and $\delta \in]0,t-h[$ we have :

$$\begin{aligned}\left\|\frac{T(t-h)x-T(t)x}{-h}-AT(t)x\right\| &= \left\|\frac{T(t-\delta)T(\delta)x-T(t-h-\delta)T(\delta)x}{h}-AT(\delta)T(t-\delta)x\right\| \\ &= \left\|\frac{1}{h}\left[\int_{t-h-\delta}^{t-\delta}\frac{d^+}{ds}T(s)T(\delta)xds-\int_{t-h-\delta}^{t-\delta}AT(\delta)T(t-\delta)xds\right]\right\| \\ &= \left\|\frac{1}{h}\int_{t-h-\delta}^{t-\delta}\left[AT(\delta)T(s)-AT(\delta)T(t-\delta)\right]xds\right\| \\ &\leq \frac{1}{h}||AT(\delta)||\int_{t-h-\delta}^{t-\delta}||T(s)x-T(t-\delta)x||ds \\ &= \frac{1}{h}||AT(\delta)||h||T(c)x-T(t-\delta)x|| \\ &= ||AT(\delta)||||T(c)x-T(t-\delta)x||.\end{aligned}$$

where $c \in [t-h-\delta, t-\delta]$. Therefore

$$\begin{aligned}\frac{d^-}{dt}T(t)x &= \lim_{h\longrightarrow 0^+}\frac{T(t-h)x-T(t)x}{-h} \\ &= AT(t)x.\end{aligned}$$

so $(T(t))_{t\geq 0}$ is a differentiable C_0-semi-group .

Proposition 2.1.1. *Let $(T(t))_{t\geq 0}$ is a differentiable C_0-semi-group so the application :*

$$]0,+\infty) \ni t \longmapsto T(t) \in B(X)$$

is continuous for the topology of uniform convergence .

Proof. *Let $x \in X$ and $t_1, t_2 \in]0,+\infty)$ such as $t_1 < t_2$.given the Theorem 2.1.1 , we obtain :*

$$\begin{aligned}||T(t_1)x-T(t_2)x|| &= \left\|\int_{t_1}^{t_2}\frac{d}{ds}T(s)xds\right\| \\ &= \left\|\int_{t_1}^{t_2}AT(t_1)T(s-t_1)xds\right\| \\ &\leq ||AT(t_1)||\int_{t_1}^{t_2}Me^{(s-t_1)w}||x||ds.\end{aligned}$$

Therefore,we have :

$$||T(t_1)x-T(t_2)x|| \leq ||AT(t_1)||M\int_{t_1}^{t_2}e^{(s-t_1)w}||x||ds \ .$$

it follows the uniform continuity of the application considered in the statement .

Theorem 2.1.2. *Let $T(t)$ a C_0-semi-group and A its infinitesimal generator so :*

i) $\forall\ n \in \mathbb{N}^*$, $\forall\ x \in X$ *:*

$$T(x) \in D(A^n)$$

and

$$A^n T(t)x = \left[AT\left(\frac{t}{n}\right)\right]^n x, \quad \forall t > 0.$$

ii) For every $n \in \mathbb{N}^*$ *, the application :*

$$]0, +\infty) \ni t \longmapsto T(t) : X \longrightarrow D(A^n)$$

is n times differentiable for the topology of uniform convergence and :

$$T^{(n)}(t) = \frac{d^n}{dt^n} T(t) = A^n T(t)$$

iii) For every $n \in \mathbb{N}^*$ *, the application :* $[0, +\infty) \ni t \longmapsto T^{(n)}(t) \in B(X)$ *is continuous for the topology of uniform convergence.*

Proof. *Prove the statement of the statement by induction.*
i) with the Theorem 2.1.1 , we see that for every $x \in X$ *we have T(t)x* $\in$ *D(A) and :*

$$AT(t)x = \left[AT\left(\frac{t}{1}\right)\right]^1 x, \quad \forall t > 0.$$

Suppose for every $x \in X$ *we are* $T(t)x \in D(A^k)$ *and :*

$$A^k T(t)x = \left[AT\left(\frac{t}{k}\right)\right]^k x, \quad \forall t > 0.$$

Let $x \in X$ *and* $\delta \in]0, t[$*. we see that*

$$T(t-\delta)T(\delta)x \in D(A)$$

and :

$$\begin{aligned} AT(t)x &= AT(t+\delta)T(\delta)x \\ &= T(t-\delta)AT(\delta)x \in D(A^k) \end{aligned}$$

Therefore $T(t)x \in D(A^{k+1})$, $\forall\ t > 0$*. Furthermore :*

$$\begin{aligned} A^{k+1}T(t)x &= A[A^k T(t-\delta)T(\delta)]x \\ &= A[T(t-\delta)A^k T(\delta)]x \\ &= AT(t-\delta)\left[AT\left(\frac{\delta}{k}\right)\right]^k x. \end{aligned}$$

if $\delta = \frac{kt}{k+1}$*, he comes :*

$$A^{k+1}T(t)x = \left[AT\left(\frac{t}{k+1}\right)\right]^{k+1} x.$$

Finally , we obtain i)
ii) For n=1 , given the Theorem 2.1.1 and of the Proposition 2.1.1 , it follows that the application :

$$]0,+\infty) \ni t \longmapsto T(t) : X \longrightarrow D(A)$$

is differentiable for the topology of uniform convergence and :

$$T'(t) = AT(t) \quad ,\forall\, t > 0.$$

as A is a closed operator and T(t) $\in$ B(X),it follows that AT(t) is closed operator defined on X.With the closed graph theorem,we see that AT(t) $\in$ B(X) , $\forall$ t>0 , suppose the application :

$$]0,+\infty) \ni t \longmapsto T(t) : X \longrightarrow D(A^k)$$

is k times differentiable for the topology of uniform convergence and :

$$T^{(k)}(t) = A^kT(t) \;\in\; B(X) \quad ,\forall\, t > 0.$$

Moreover, With the previous proof , we see that T(t)x $\in D(A^{k+1})$, for every $t > 0$.
Be $x \in X$,$||x||$ $\leq$1 and $t > 0$, if $h > 0$ and $\delta \in$]0,t[,we have :

$$\begin{aligned}
\left\| \frac{T^{(k)}(t+h) - T^{(k)}(t)}{h} - A^{k+1}T(t)x \right\| &= \left\| \frac{A^kT(\delta)}{h}\left[T(t+h-\delta) - T(t-\delta)\right]x - A^{k+1}T(\delta)T(t-\delta)x \right\| \\
&= \left\| \frac{A^kT(\delta)}{h}\int_{t-\delta}^{t+h-\delta} \frac{d}{d\tau}T(\tau)xd\tau - A^{k+1}T(\delta)\frac{1}{h}\int_{t-\delta}^{t+h-\delta} T(t-\delta)xd\tau \right\| \\
&= \left\| \frac{A^kT(\delta)}{h}\int_{t-\delta}^{t+h-\delta} AT(\tau)xd\tau - A^{k+1}T(\delta)\frac{1}{h}\int_{t-\delta}^{t+h-\delta} T(t-\delta)xd\tau \right\| \\
&= \left\| \frac{1}{h}A^{k+1}T(\delta)\int_{t-\delta}^{t+h-\delta} \left[T(\tau) - T(t-\delta)\right]xd\tau \right\| \\
&\leq \frac{||A^{k+1}T(\delta)||}{h}\int_{t-\delta}^{t+h-\delta} ||T(\tau) - T(t-\delta)||||x||d\tau \\
&= ||A^{k+1}T(\delta)||||T(c) - T(t-\delta)||||x||,
\end{aligned}$$

where $c \in [t-\delta, t+h-\delta]$. it follows that :

$$\left\| \frac{T^{(k)}(t+h) - T^{(k)}(t)}{h} - A^{k+1}T(t) \right\| \leq ||A^{k+1}T(\delta)||||T(c) - T(t-\delta)||,$$

where $c \in [t-\delta, t+h-\delta]$.Therefore :

$$\lim_{h\longrightarrow 0} \frac{T^{(k)}(t+h) - T^{(k)}(t)}{h} = A^{k+1}T(t) \;,\forall\, t > 0.$$

if h>0 such that t-h>0 and $\delta \in]0, t-h[$, *so :*

$$\begin{aligned}
\left\| \frac{T^{(k)}(t-h) - T^{(k)}(t)}{-h} - A^{k+1}T(t)x \right\| &= \left\| A^k T(\delta) \frac{1}{h} \left[T(t-\delta) - T(t-h-\delta) \right] x - A^{k+1} T(\delta) T(t-\delta) x \right\| \\
&= \left\| A^k T(\delta) \frac{1}{h} \int_{t-h-\delta}^{t-\delta} \frac{d}{d\tau} T(\tau) x d\tau - A^{k+1} T(\delta) \frac{1}{h} \int_{t-h-\delta}^{t-\delta} T(t-\delta) x d\tau \right\| \\
&= \left\| A^k T(\delta) \frac{1}{h} \int_{t-h-\delta}^{t-\delta} A T(\tau) x d\tau - A^{k+1} T(\delta) \frac{1}{h} \int_{t-h-\delta}^{t-\delta} T(t-\delta) x d\tau \right\| \\
&= \left\| \frac{1}{h} A^{k+1} T(\delta) \int_{t-h-\delta}^{t-\delta} \left[T(\tau) - T(t-\delta) \right] x d\tau \right\| \\
&\leq \frac{||A^{k+1}T(\delta)||}{h} \int_{t-h-\delta}^{t-\delta} ||T(\tau) - T(t-\delta)|| ||x|| d\tau \\
&= ||A^{k+1}T(\delta)|| ||T(c) - T(t-\delta)|| ||x||,
\end{aligned}$$

where $c \in [t-h-\delta, t-\delta]$. *it follows that :*

$$\left\| \frac{T^{(k)}(t-h) - T^{(k)}(t)}{-h} - A^{k+1}T(t) \right\| \leq ||A^{k+1}T(\delta)|| ||T(c) - T(t-\delta)||,$$

where $c \in [t-h-\delta, t-\delta]$. *Therefore :*

$$\lim_{h \longrightarrow 0} \frac{T^{(k)}(t-h) - T^{(k)}(t)}{-h} = A^{k+1}T(t) \ , \forall \ t > 0.$$

it follows that $T^{(k)}(t)$ *is differentiable for the topology of uniform convergence and*

$$\left(T^{(k)}(t) \right)' = T^{(k+1)}(t) = A^{k+1} T(t) \ \forall \ t > 0.$$

as A *is a closed operator and* $A^k T(t) \in B(X)$,*it follows that* $A\left(A^k T(t) \right)$ *is a closed operator defined on* X . *with the closed graph theorem , we see that :*

$$T^{(k+1)}(t) = A^{k+1}T(t) \ \in \ B(X) \ \forall \, t > 0.$$

Finally , we obtain ii).
iii) Let $x \in X$ *with* $||x|| \leq 1$ *and t>0.for h>0 and* $\delta \in]0, t[$ *we obtain :*

$$\begin{aligned}
||T'(t+h)x - T'(t)x|| &= ||AT(t+h)x - AT(t)x|| \\
&\leq ||AT(\delta)|| ||T(t+h-\delta) - T(t-\delta)|| ||x||,
\end{aligned}$$

it follows

$$||T'(t+h) - T'(t)|| \leq ||AT(\delta)|| ||T(t+h-\delta) - T(t-\delta)||.$$

Similarly, for h>0 and $\delta \in]0, t-h[$ *we obtain :*

$$\begin{aligned}
||T'(t-h)x - T'(t)x|| &= ||AT(t-h)x - AT(t)x|| \\
&\leq ||AT(\delta)|| ||T(t-h-\delta) - T(t-\delta)|| ||x||,
\end{aligned}$$

Therefore

$$||T'(t-h) - T'(t)|| \leq ||AT(\delta)|| ||T(t-h-\delta) - T(t-\delta)||.$$

it is clear that the application :

$$]0,+\infty) \ \ni \ t \ \longmapsto \ T'(t) \ \in \ B(X)$$

is continuous for the topology of uniform convergence . Suppose that the application:

$$]0,+\infty) \ \ni \ t \ \longmapsto \ T^{(k)}(t) \ \in \ B(X)$$

is continuous for the topology of uniform convergence.
if h>0 and $\delta \ \in]0,t[$ *, so :*

$$\begin{aligned} ||T^{(k+1)}(t+h)x - T^{(k+1)}(t)x|| &= ||A^{k+1}T(t+h)x - A^{k+1}T(t)x|| \\ &\leq ||A^{k+1}T(\delta)|| ||T(t+h-\delta) - T(t-\delta)|| ||x||, \end{aligned}$$

Therefore :

$$||T^{(k+1)}(t+h) - T^{(k+1)}(t)|| \leq ||A^{k+1}T(\delta)|| ||T(t+h-\delta) - T(t-\delta)||.$$

On the other hand , for h>0 and $\delta \ \in]0,t-h[$ *we obtain :*

$$\begin{aligned} ||T^{(k+1)}(t-h)x - T^{(k+1)}(t)x|| &= ||A^{k+1}T(t-h)x - A^{k+1}T(t)x|| \\ &\leq ||A^{k+1}T(\delta)|| ||T(t-h-\delta) - T(t-\delta)|| ||x||, \end{aligned}$$

and we can see :

$$||T^{(k+1)}(t-h) - T^{(k+1)}(t)|| \leq ||A^{k+1}T(\delta)|| ||T(t-h-\delta) - T(t-\delta)||.$$

so the application :

$$]0,+\infty) \ \ni \ t \ \longmapsto \ T^{(k+1)}(t) \ \in \ B(X)$$

is continuous for the topology of uniform convergence.
Finally , we obtain iii).

Example. *Let*

$$X = \{f \in C([0,1],\mathbb{R}); f(1) = 0\}.$$

Equipped with the standard norm of uniform convergence,space X is a Banach space.
Let $(T(t))_{t\geq 0}$ *the family of linear operators defined on X by :*

$$\begin{cases} (T(t)f)(x) = f(x+t) & si \quad x+t \leq 1. \\ (T(t)f)(x) = 0 & si \quad x+t > 1. \end{cases}$$

for every $f \in X$ *and every* $x \in [0,1]$.
so $(T(t))_{t\geq 0}$ *is a* C_0*-semi-group of contractions on X.its infinitesimal generator A is given by :*

$$D(A) = f \in C^1([0,1],\mathbb{R}) \cap X; f' \in X$$

and

$$Af = f', for f \in D(A)$$

Furthermore $(T(t))_{t\geq 0}$ *is differentiable for* $t > 1$.

Remark. *if $(T(t))_{t\geq 0}$ is a differentiable C_0-semi-group of infinitesimal generator A , so for every $x \in X$:*

i) $\frac{d}{dt}T(t)x = AT(t)x$ *for* $t > 0$

ii) $t \longmapsto AT(t)x$ *is lipschitzienne for* $t > 0$.

iii) Application : $]0, +\infty) \ni t \longmapsto T(t) \in B(X)$ *is classy* $C^{\infty}{}_{]0,+\infty)}$

2.2 Analytical C_0-semi-groups

Definition 2.2.1. *Let*

$$\Delta = \{z \in \mathbb{C} : \varphi_1 < argz < \varphi_2, \varphi_1 < 0 < \varphi_2\},$$

and for z$\in \Delta$ let T(z) be a bounded linear operator . The family $(T(z))_{z\in\Delta}$ is an analytic semi group in Δ if :

i) T(0)=I , $T(z_1 + z_2) = T(z_1) \cdot T(z_2)$ *,* $\forall\ z_1, z_2 \in \Delta$.

ii) $\lim\limits_{z\longrightarrow 0} T(z)x = x$ *,* $\forall\ x \in X$ *,* $\forall\ z \in \Delta$.

iii) The application $z \longmapsto$ *T(z) is analytical on* Δ *for the standard of B(X).*

Theorem 2.2.1. *Let $(T(t))_{t\geq 0}$ be a uniformly bounded C_0-semi-group such that $||T(t)|| \leq M$ and A its infinitesimal generator such That 0 $\in\rho$(A). The assertions are equivalent :*

i) it exists $\delta > 0$ such that $(T(t))_{t\geq 0}$ can be extended into an analytical C_0 semi-group on the sector $\Delta_\delta = \{z \in C : |arg(z)| < \delta\}$ and $(T(z))_{z\in\Delta_{\delta'}}$ is uniformly bounded in all closed subsectors $\overline{\Delta_{\delta'}} \subset \Delta_\delta$,or $\delta' \in]0, \delta[$.

ii) there is a constant $C > 0$ such that for every $\sigma > 0$ and every $\tau \neq 0$ we have:

$$||R(\sigma + i\tau, A)|| \leq \frac{C}{|\tau|}. \tag{2.1}$$

iii) it exists $0 < \delta < \frac{\pi}{2}$ and $K > 0$ why

$$\rho(A) \supset \sum_{\delta} = \left\{\lambda \in C : |arg\lambda| < \frac{\pi}{2} + \delta\right\} \cup \{0\} \tag{2.2}$$

and

$$||R(\lambda, A)|| \leq \frac{K}{|\lambda|}, \forall\lambda \in \sum_{\delta} \setminus \{0\} \tag{2.3}$$

iiii) The application $]0, +\infty[\ni\ t \longmapsto$ T(t) is differentiable and there is a constant $c > 0$ such that :

$$||AT(t)|| \leq \frac{c}{t}, \forall t > 0. \tag{2.4}$$

CHAPTER 3

NON LINEAR EVOLUTION PROBLEMS

3.1 Rappel

Let the homogeneous abstract problem of Cauchy :

$$\begin{cases} \frac{\partial u}{\partial t} = Au(t) \\ u(0) = u_0 \end{cases} \tag{3.1}$$

Let X a Banach space . $A : D(A) \longrightarrow X$ linear closed.
if A is an infinitesimal generator of a C_0-semi-group $(T(t))_{t>0}$ on X.
if $u_0 \in D(A)$ then the solution u of (3.1) is given by :$u(t) = T(t).u_0$ is unique solution of (3.1) .
if $u_0 \notin D(A)$ .t $\longrightarrow$T(t).u_0 is called mild solution

$$\begin{cases} \frac{\partial u}{\partial t} = Au(t) + f(t, u(t)) \quad , \quad t > 0 \\ u(0) = u_0 \end{cases} \tag{3.2}$$

3.2 Non-homogeneous problem

$$\begin{cases} \frac{\partial u}{\partial t} = Au(t) + f(t) \quad , \quad t \geq 0 \\ u(0) = u_0 \end{cases} \tag{3.3}$$

$f : \mathbb{R}^+ \longrightarrow X$.

Definition 3.2.1. *a continuous function $u : \mathbb{R}^+ \longrightarrow X$ is said to be a strict solution of (3.3) if*
-$u \in C^1(\mathbb{R}^+, X)$
-u(t) $\in D(A)$. $\forall$ $t \geq 0$
-u satisfies the equation (3.3) .

Definition 3.2.2. *$u : \mathbb{R}^+ \longrightarrow X$ continuous is said to be a classical solution if :*
-$u \in C^1(\mathbb{R}^+, X)$
-u(t) $\in D(A)$. $\forall$ $t > 0$
-u satisfies the equation (3.3) for t>0.

Definition 3.2.3. *$u : \mathbb{R}^+ \longrightarrow X$ continuous is said to be a strong solution if :*
-u is almost everywhere derivable on $\mathbb{R}^+$
-u(t) $\in$ D(A) almost everywhere
-u satisfies the equation (3.3) almost everywhere.

Theorem 3.2.1. *if u is strict solution of (3.3) . so :*
$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds \qquad , t \geq 0$

Remark. *if $A \in \pounds(X) \Longrightarrow T(t)=e^{tA}$ and the solution given by : $u(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}f(s)ds$.*

Proof. *u being the strict solution . $g : [0,t] \longrightarrow X$ given by : $g(s) = T(t-s)u(s)$, so $g \in C^1([0,t])$, moreover*

$$\begin{aligned} g'(s) &= -AT(t-s)u(s) + T(t-s)u'(s) \\ &= -AT(t-s)u(s) + T(t-s)Au(s) + T(t-s)f(s) \\ &= T(t-s)f(s). \end{aligned}$$

$$\int_0^t g'(s)ds = g(t) - g(0) = \int_0^t T(t-s)f(s)ds$$

$$\Longrightarrow u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds. \tag{3.4}$$

Definition 3.2.4. *If u satisfies (3.4) , so u is called mild solution of (3.3).*

Remark. *strict solution $\Longrightarrow$ classical solution $\Longrightarrow$ strong solution $\Longrightarrow$ mild solution*

Example (*). *Let $y \in X\backslash D(A)$, and consider the next problem :*

$$\begin{cases} X'(t) = AX(t) + T(t)y \quad , \quad t \geq 0 \\ X(0) = X_0 \end{cases} \tag{3.5}$$

Let X the mild solution of (3.5) .

$$\begin{aligned} X(t) &= \int_0^t T(t-s)T(s)yds \\ &= tT(t).y \quad \textit{is not differentiable.} \end{aligned}$$

Theorem 3.2.2. *(regularity)*
if $u_0 \in D(A)$ and $f \in C^1(\mathbb{R}^+, X)$ so the mild solution is a strict solution.

Proof. *since $u_0 \in D(A)$, $t \longrightarrow T(t).u_0$ is C^1 , $\frac{\partial T}{\partial t}(t)u_0$ exist .*
Set $v(t) = \int_0^t T(t-s)f(s)ds$.

$$\begin{aligned} \left(\frac{T(h)-I}{h}\right)v(t) &= \frac{1}{h}\int_0^t T(t+h-s)f(s)ds - \frac{1}{h}\int_0^t T(t-s)f(s)ds \\ &= \frac{1}{h}\int_0^{t+h} T(t+h-s)f(s)ds + \frac{1}{h}\int_{t+h}^t T(t+h-s)f(s)ds - \frac{1}{h}\int_0^t T(t-s)f(s)ds \\ &= \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds \longrightarrow v'(t) - T(0)f(t). \end{aligned}$$

Lemma 3.2.1. *v is class C^1 on $\mathbb{R}^+$ such that :*
v'(t)=f(t)+$\int_0^t$ T(t-s)f'(s)ds.

$$\begin{aligned}\frac{v(t+h)-v(t)}{h} &= \frac{1}{h}\int_0^{t+h} T(s)f(t+h-s)ds - \frac{1}{h}\int_0^t T(s)f(t-s)ds \\ &= \int_0^t T(s)\frac{f(t+h-s)-f(t-s)}{h}ds + \frac{1}{h}\int_t^{t+h} T(s)f(t+h-s)ds\end{aligned}$$

$\Longrightarrow$ *v'(t)=T(t)f(0)+$\int_0^t$ T(t-s)f'(s)ds*
$\Longrightarrow$ *v' $\in$ $C^1(\mathbb{R}^+, X)$.*

Remark. $\lim_{h\longrightarrow 0^+} \frac{T(h)-I}{h}v(t) = Av(t)$ *(because this limit exists)*
Av(t)=v'(t)-f(t) . so v'(t)=Av(t)+f(t).
such that the mild solution u given by u(t)=T(t)u_0+f(t).
so

$$\begin{aligned}u'(t) &= AT(t)u_0 + Av(t) + f(t) \\ &= A[T(t)u_0 + v(t)] + f(t) \\ &= Au(t) + f(t).\end{aligned}$$

3.3 Reflective space

X is said to be reflexive if any sequence $(x_n)_n$ bounded in X admits a sub sequence $(x_{nk})_n$ which converges weakly :

$$\forall x^* \in X^* \quad (dual\ topological) \qquad < x^*, x_{nk} > \longrightarrow_{k\longrightarrow+\infty} < x^*, x > \qquad is\ convergent.$$

Examples of reflective spaces

- Hilbert spaces
- $L^p(\Omega)$ $1 < p < \infty$
- Uniformly convex space.

Theorem 3.3.1. *Let f:[a,b] $\longrightarrow$ H (H Hilbert)be a Lipschitzienne function $\Longrightarrow$ f is almost by everywhere differentiable on [a,b].*

Theorem 3.3.2. *We suppose that f:[a,b] $\longrightarrow$ X is lipschitzienne and $u_0 \in$ D(A) , so the mild solution of (3.3) is a strong solution on [a,b] .*

$$x_{nk} \longrightarrow x \Longrightarrow \forall x^* \in X^*, < x^*, x_{nk} > \longrightarrow < x^*, x > .$$

Proof. *Let u be the mild solution of (3.3)*
so : u(t)=T(t)u_0+v(t) ,such that

$$v(t) = \int_0^t T(s)f(t-s)ds.$$

"the domain of derivation of f $\subset$ the domain of derivation of v"

Or : $\left[\frac{T(h)-I}{h}\right] v(t) = \frac{v(t+h)-v(t)}{h} + \frac{1}{h}\int_{t+h}^{t} T(t+h-s)f(s)ds$
Almost everywhere $t \in [a,b]$

$$Av(t) = \frac{v(t+h)-v(t)}{h} + \frac{1}{h}\int_{t+h}^{t} T(t+h-s)f(s)ds$$

Remark. *$v'(t)=Av(t)+f(t)$ almost everywhere $t \in [0,a]$.*

Example. *Let Ω fairly regular open . Let the following partial differential equation :*

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \theta(t,x) \qquad ,t \geq 0, x \in \Omega \\ u(t,x) = 0 \qquad ,t \geq 0, x \in \partial\Omega \\ u(0,x) = u_0(x) \end{cases} \tag{3.6}$$

or : θ:[0,a]$\times\Omega\longrightarrow\mathbb{R}$ is continuous.
$|\theta(t_1,x)-\theta(t_2,x)| \leq K|t_1-t_2|$
$H=L^2(\Omega)$: $D(\Delta)=H^2(\Omega)\cap {H^1}_0(\Omega)$
• $-\Delta$ is maximal monotonic , so Δ is infinitesimal generator of a C_0-semi-group ;
Let

$$\begin{aligned} f : [0,a] &\longrightarrow H^2(\Omega) \\ t &\longmapsto f(t) = \theta(t.\cdot) \end{aligned}$$

Let's pose v(t)=u(t,0) . $\forall$ t $\in$ [0,a] , $\int_\Omega f^2(t,x)dx < +\infty$.
So

$$\begin{cases} v'(t) = Av(t) + f(t) \qquad , \; t \geq 0 \;\; (\textit{so that } \theta \textit{ is in } L^2(\Omega)) \\ v(0) = v_0 \end{cases} \tag{3.7}$$

Let $t_1,t_2 \in [0,a]$

$$\begin{aligned} ||f(t_1) - f(t_2)||^2{}_{L^2(\Omega)} &= \int_\Omega (\theta(t_1,x) - \theta(t_2,x))^2 dx \\ &\leq K^2 mes(\Omega)|t_1 - t_2|^2 \end{aligned}$$

$\Longrightarrow ||f(t_1) - f(t_2)|| \leq K\sqrt{mes(\Omega)}|t_1 - t_2|$
The mild solution v of (3.7) get strong
u(t,x)=v(t)(x) satisfies the following model C_0-semi-group associated with Δ :

$$\Omega =]0,\pi[, H = L^2(]0,\pi[)$$

T(Δ)=$\{-n^2, n \geq 1\}$
-n^2 eigenvector associated with the eigenvector sin(nx).

3.4 Non linear equation

$$\begin{cases} \frac{\partial x}{\partial t} = Ax(t) + f(t, x(t)) \\ x(0) = x_0 \in X \end{cases} \tag{3.8}$$

A is an infinitesimal generator of a C_0-semi-group on X
f : $\mathbb{R}^+ \times X \longrightarrow X$ a continuous function

strict solution $\Longrightarrow$ *classical solution* $\Longrightarrow$ *strong solution* $\Longrightarrow$ *mild soltion.*

A mild solution of (3.8) on [0,a] (on $\mathbb{R}^+$) is a continuous function on [0,a] satisfies :

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x, x(s))ds \quad t \in [0, a]$$

Theorem 3.4.1. *Suppose that f is lipschitzienne at 2nd variable :* $\exists$ $K{:}\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *continuous such that :*

$$||f(t, x) - f(t, y)||_X \le K(t)||x - y||_X \;\; \forall t \ge 0, \forall x, y \in X.$$

So (3.8) admits a mild solution set to $\mathbb{R}^+$.

Proof. *Let* $a > 0$, $M_a = \sup_{t\in[0,a]} |T(t)|$; $K_a = \sup_{t\in[0,a]} |K(t)|$; $C_a = C([0,a], X)$
For $y \in C_a$

$$L : C_\alpha \longrightarrow C_\alpha$$
$$y \longmapsto (Ly)(t) = T(t)x_0 + \int_0^t T(t-s)f(s, y(s))ds.$$

Let $y_1, y_2 \in C_a$

$$Ly_1(t) - Ly_2(t) = \int_0^t T(t-s)(f(s, y_1(s)) - f(s, y_2(s)))ds$$
$$|Ly_1(t) - Ly_2(t)| \le M_a K_a t |y_1 - y_2| \quad , \forall t \in [0, a]$$

2nd iteration

$$|L^2 y_1(t) - L^2 y_2(t)| \le \frac{(M_a K_a t)^2}{2} |y_1 - y_2| \le \frac{(M_a K_a a)^2}{2} |y_1 - y_2|$$

$\exists$ *n :*

$$\frac{(M_a K_a a)^n}{n!} < 1$$

so $||L^n y_1(t) - L^n y_2(t)| \le \frac{(M_a K_a a)^n}{n!} |y_1 - y_2|$
From where K admits a unique fixed point.

Remark.

$$f \text{ continuous} \not\Rightarrow \text{existance of mild solution}$$

*See the **Example**(*)*

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(0) = x_0 \end{cases} \tag{3.9}$$

Theorem 3.4.2. *If T(t) is compact for* $t > 0$
And $f : \mathbb{R}^+ \times X \longrightarrow X$ *continuous ;*
so $\exists\ b > 0$ *such that : (3.8) admits at least mild solution on* $[0, b]$*.*

Proof. *f is continuous in* $[0, x_0]$
$\exists\ M > 0 \quad |f(t, y)| \le M \quad ; \exists\ r > 0\ , \exists\ \alpha > 0$
Let $\widetilde{C}_\alpha = \{y \in C_\alpha : |y(t) - x_0| \le r \quad , \forall t \in [0, \alpha]\}$
$\widetilde{C}_\alpha$ *is a closed and bounded convex .*
so $LC_\alpha \subseteq \widetilde{C}_\alpha$ *Let* $y \in \widetilde{C}_\alpha$

$$|Ly(t) - x_0| \le |T(t)x_0 - x_0| + \int_0^t |T(t-s)f(s, y(s))|ds$$

for t small enough $|T(t)x_0 - x_0| \le \frac{r}{2}$

$$\begin{aligned} \int_0^t |T(t-s)||f(s, y(s))|ds &\le tM_\alpha.M \\ &\le \frac{r}{2} \quad \text{for } t \text{ pertty small.} \end{aligned}$$

$\exists\ \beta > 0$ *such that if* $t \le \beta$ *,* $|Ly(t) - y_0| \le r$
so

$$LC_\beta \subseteq C_\beta$$

We prove $\overline{LC_\alpha}$ *is compact.*

Equi-boundedness

Let $t \in [0, \beta]$ *:* $\overline{\{Ly(t), y \in C_\beta\}}$ *compact on X*
if t=0 obvious
Let $0 < t < \beta$*,*

$$Ly(t) = T(t)x_0 + \int_0^t T(t-s)f(s, y(s))ds.$$

Let $\epsilon >$ *0 such that* $\epsilon < t$*,*

$$\begin{aligned} \int_0^t T(t-s)f(s, y(s))ds &= \int_0^{t-s} T(t-s)f(s, y(s))ds + \int_{t-s}^t T(t)f(s, y(s))ds \\ &= T(\epsilon)\int_0^t T(t-\epsilon-s)f(s, y(s))ds + O(\epsilon) \end{aligned}$$

Non compactness measure

i) α(B)=0 $\Leftrightarrow$ $\overline{B}$ Compact.

ii) $\alpha(B_1+B_2) \leq \alpha(B_1)\alpha(B_2)$

iii) $\alpha(B(0,\epsilon)) \leq 2\epsilon$

iiii) $\alpha(\lambda B) \leq |\lambda|\ \alpha(B)$

$$\alpha\left(\{Ly(t), y \in C_\beta\}\right) \leq 0 + O(\epsilon) \qquad \forall \epsilon < t$$
$$\epsilon \longrightarrow 0 \qquad \alpha\left(\left\{Ly(t), y \in \widetilde{C}_\beta\right\}\right) = 0$$

so $\qquad \overline{\left\{Ly(t), y \in \widetilde{C}_\beta\right\}}$ is compact.

The continuity

Let $x_n \longrightarrow_{n\longrightarrow+\infty} x$ we demonstrate that :

$$Lx_n(t) \longrightarrow_{n\longrightarrow+\infty} Lx(t)$$

According to the Theorem 3.4.1 we have :

$$|Lx_n(t) - Lx(t)| = |\int_0^t T(t-s)(f(s, x_n(s)) - f(s, x(s)))ds|$$
$$\leq M_a K_a t |x_n - x| \longrightarrow_{n\longrightarrow+\infty} 0$$

with M_a= $\sup_{t\in[0,a]} |T(t)|$; K_a= $\sup_{t\in[0,a]} |K(t)|$
Therefore K is continuous.

Theorem 3.4.3. *Let Y is a Banach space and* $(T_n)_n \subseteq \pounds(Y)$
Such that $T_n \longrightarrow_{n\longrightarrow+\infty} T$
so : $\forall$ *B compact in Y*

$$\sup_{y\in B} |T_n(y) - T(y)| \longrightarrow_{n\longrightarrow+\infty} 0$$

Back to demonstration:

Equi-continuous: $\forall\ t_0 \in [0,\beta]$
$\sup_{y\in\widetilde{C}_\beta} |Ly(t) - Ly(t_0)| \longrightarrow_{t\longrightarrow t_0} 0 \qquad t > t_0$

$$|Ly(t) - Ly(t_0)| \leq |T(t)x_0 - T(t_0)x_0| + |\int_0^{t_0} (T(t-s) - T(t_0 - s))f(s, y(s))ds| + \int_{t_0}^t |T(t-s)f(s, y(s))|ds$$

$|T(t)x_0 - T(t_0)x_0| \longrightarrow 0 \qquad t \longrightarrow t_0$.
$|\int_{t_0}^t (T(t-s)f(s, y(s))ds| \leq k|t - t_0| \qquad \forall y \in \widetilde{C}_\beta$.
$\int_0^{t_0} (T(t-s) - T(t_0 - s))f(s, y(s))ds = (T(t - t_0) - I) \int_0^{t_0} (T(t-s)f(s, y(s))ds$
$\sup_{z\in K_0} |T(t - t_0)z - z| \longrightarrow_{t\longrightarrow t_0} 0$
so $\qquad L\widetilde{C}_\beta$ is compact.

$$Schauder's\ theorem \Longrightarrow admits\ at\ least\ one\ fixed\ point.$$

Theorem 3.4.4. *If moreover f is bounded ;*
so (3.8) admits a maximum solution defined on $[0,t_{max}[$,
with $0 < t_{max} \leq +\infty$.
If $t_{max} < +\infty \Longrightarrow \lim_{t \longrightarrow t_{max}} |x(t)| = +\infty$.

Theorem 3.4.5. *Let* $f :]\alpha, \beta[\longrightarrow Y$ *full , uniformly continuous , so* $\lim_{x \longrightarrow \alpha} f(x)$, $\lim_{x \longrightarrow \beta} f(x)$ *exist in* Y.

Theorem 3.4.6. $(T(t))_{t \geq 0}$ C_0*-semi-group compact operators.*
$f : \mathbb{R}^+ \times X \longrightarrow X$ *bounded continuous .*
so (3.8) admits a maximum solution defined on $[0,t_{max}[$
with $0 < t_{max} \leq +\infty$.
If $t_{max} < \infty$ *so* $\overline{\lim}_{t \longrightarrow t_{max}} |x(t)| = +\infty$.

Proof. *we suppose that* $t_{max} < +\infty$ *and* $\overline{\lim}_{t \longrightarrow t_{max}} |x(t)| < +\infty$
$\exists\ M > 0 : \forall\ t < t_{max} : |x(t)| \leq M$. *Let* $t < t_{max}$ *and* $h > 0$

$$
\begin{aligned}
x(t+h) - x(t) &= T(t)\left(T(h)x_0 - x_0\right) + \int_0^{t+h} T(t+h-s) f(s, x(s)) ds - \int_0^t T(t-s) f(s, x(s)) ds \\
&= T(t)\left(T(h)x_0 - x_0\right) + \int_0^t (T(t+h-s) - T(t-s)) f(s, x(s)) ds \\
&\quad - \int_t^{t+h} T(t+h-s) f(s, x(s)) ds
\end{aligned}
$$

$T(t)\left(T(h)x_0 - x_0\right) \longrightarrow_{h \longrightarrow 0} 0$ *and* $\int_t^{t+h} T(t+h-s) f(s, x(s)) ds = O(h)$
$\int_0^t (T(t+h-s) - T(t-s)) f(s, x(s)) ds = (T(h) - I) \int_0^t T(t-s)) f(s, x(s)) ds$

$$
D = \left\{ \int_0^t T(t-s)) f(s, x(s)) ds \ : \ t \in [0, t_{max}[\right\} \quad \textit{is compact ?}
$$

Let $(t_n)_n \subseteq [0, t_{max}[\ : \exists\ (t_{nk})_k \longrightarrow \bar{t}$
if $\bar{t} < t_{max}$: $\int_0^{t_n} T(t_n - s)) f(s, x(s)) ds$
converge to $\int_0^{\bar{t}} T(\bar{t} - s)) f(s, x(s)) ds$
if $\bar{t} = t_{max}$. $\int_0^{t_{nk}} T(t_{nk} - s)) f(s, x(s)) ds = \int_0^{t_{max}} 1_{[0, t_{nk}]} T(t_{nk} - s)) f(s, x(s)) ds$
Or
$\forall\ s \in [0, t_{max}[$

$$
1_{[0, t_{nk}]} T(t_{nk} - s)) f(s, x(s)) ds \longrightarrow_{k \longrightarrow +\infty} T(t_{max} - s) f(s, x(s)).
$$

By the Lebesgue dominated convergence
$\int_0^{t_{nk}} T(t_{nk} - s)) f(s, x(s)) ds \longrightarrow \int_0^{t_{max}} T(t_{nk} - s)) f(s, x(s)) ds$
$\overline{D}$ *compact* $\Longrightarrow \sup_{z \in D} |T(h)z - z| \longrightarrow_{h \longrightarrow 0} 0 \qquad \forall\ t.$
From where $x(t+h) - x(t) \longrightarrow_{h \longrightarrow 0} 0$

Regularity :

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)) & \forall t \geq 0 \\ x(0) = x_0 \end{cases} \tag{3.10}$$

A infinitesimal generator of a C_0-semi-group $(T(t))_{t\geq 0}$
$|f(t,x) - f(t,y)| \leq k|x - y|$,
(3.10) admits at a mild solution . $t \geq 0$, $x, y \in X$.

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds \qquad t \geq 0$$

Proposition 3.4.1. *if $x_0 \in D(A)$ and if another "$t \longmapsto f(t, x(t))$" is class C^1 , so x is a strict solution of (3.10)*

Proof. *Let $u(t) = f(t, x(t))$.*

$$(3.10) \Longleftrightarrow \begin{cases} x'(t) = Ax(t) + u(t) & \forall t \geq 0 \\ x(0) = x_0 \end{cases}$$

$u \in C^1(\mathbb{R}^+, X)$, $x_0 \in D(A)$

$$x(t) = T(t)x_0 + \int_0^t T(t-s)u(s)ds.$$

According to the regulator for non homogeneous problems.
$X \in C^1(\mathbb{R}^+, X)$ *and*

$$\begin{aligned} x'(t) &= Ax(t) + u(t) \\ &= Ax(t) + f(t, x(t)) \end{aligned}$$

Remark. *Under which conditions $t \longmapsto f(t, x(t))$ is a class C^1 .*

Theorem 3.4.7. *If f a $C^1(\mathbb{R}^+ \times X, X)$ and $D_t f$, $D_x f$ are locally lipschitzian 2^{nd} variable. if $x_0 \in D(A)$, so the mild solution becomes strict.(By the regularity)*

Proof. *It suffices to show that $t \longmapsto f(t, x(t))$ is a class C^1 (See Proposition 3.4.1) it comes down to showing that $x \in C^1(\mathbb{R}^+, X)$*

$$\begin{cases} \forall x_0 \in X, \exists V \ \text{ open }, \exists x_0 \ \text{ such that :} \\ |D_t f(t,x) - D_t f(t,y)| \leq k|x-y| \qquad \forall x, y \in V \\ |D_x f(t,x) - D_x f(t,y)| \leq k|x-y| \end{cases} \Longrightarrow \begin{cases} \forall K \text{ compact of } X, \exists k > 0 \text{ such that :} \\ |D_t f(t,x) - D_t f(t,y)| \leq k|x-y| \qquad \forall x, y \in K \\ |D_x f(t,x) - D_x f(t,y)| \leq k|x-y| \end{cases}$$

$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds \quad t \in [0, a]$
either the following problem :

$$\begin{cases} y'(t) = Ay(t) + D_t f(t, x(t)) + D_x f(t, x(t))(y(t)) \\ y(0) = Ax_0 + f(0, x_0) \end{cases} \tag{3.11}$$

y'(t)=Ay(t)+g(t,y(t))

$$|g(t,y_1)-g(t,y_2)|\leq K(t)|y_1-y_2|$$

Such that : $K(t)=|D_xf(t,x(t))|$,
so (3.11) admits a mild solution in [0,a] .

$$y(t)=T(t)(Ax_0+f(0,x_0))+\int_0^t T(t-s)\left[D_sf(s,x(s))+D_xf(s,x(s))(y(s))\right]$$

Let $z(t)=x_0+\int_0^t y(s)ds$*;*
so $z\in C^1([0,a],X)$ *, such that* $x\equiv z$
Therefore

$$z(t)=x_0+\int_0^t T(s)Ax_0ds+\int_0^t T(s)f(0,x_0)ds+\int_0^t\int_0^s T(s-\tau)\left[D_\tau f(\tau,x(\tau))+D_xf(\tau,x(\tau))(y(\tau))\right]dxd\tau$$

On the other hand

$$\begin{aligned}\int_0^t T(s)Ax_0ds &= A\int_0^t T(s)x_0ds\\ &= T(t)x_0-x_0.\end{aligned}$$

$$t\longmapsto z(t) is\ a\ class C^1\Longrightarrow(t\longmapsto f(t,z(t))\ is\ a\ class\ C^1)$$

$$\Longrightarrow t\longrightarrow\int_0^t T(t-s)f(s,x(s))ds\ is\ a\ class\ C^1$$

We have :

$$\begin{aligned}\frac{d}{dt}\int_0^t T(s)f(t-s,z(t-s))ds &= \int_0^t T(s)\left[D_tf(t-s,z(t-s))+D_xf(t-s,z(t-s))y(t-s)\right]ds\\ &\quad +T(t)f(0,x_0)\\ &= T(t)f(0,x_0)+\int_0^t T(t-s)\left[D_tf(s,z(s))+D_xf(s,z(s))y(s)\right]d\tau ds.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_0^t T(s)f(0,x_0)ds &= \int_0^t T(t-s)f(s,z(s))ds\\ &\quad -\int_0^t\int_0^s T(s-\tau)\left[D_\tau f(\tau,z(\tau))+D_xf(\tau,z(\tau))y(\tau)\right]d\tau ds\end{aligned}$$

So,

$$\begin{aligned}z(t) &= T(t)x_0+\int_0^t T(t-s)f(s,z(s))ds\\ &\quad +\int_0^t\int_0^s T(s-\tau)\left[D_\tau f(\tau,x(\tau))-D_\tau f(\tau,z(\tau))\right]\\ &\quad +\int_0^t\int_0^s T(s-\tau)\left[D_xf(\tau,x(\tau))-D_xf(\tau,z(\tau))\right]y(\tau)\end{aligned}$$

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds$$

so

$$|x(t) - z(t)| \leq k \int_0^t |x(s) - y(s)|ds$$

$$K = \{x(s), \tau(s), s \in [0, a]\} \quad \textit{is compact.}$$

Lemma 3.4.1. *(Gronwall)*
$x \equiv z$ *on* $[0, a]$ *ie* $\quad x \in C^1(\mathbb{R}^+, X)$. *Then* $t \longmapsto f(t, x(t))$ *is a class* C^1

Theorem 3.4.8. *We suppose that* X *is a Hilbert space and* $f : \mathbb{R}^+ \times X \longrightarrow X$ *bi-lipschitzian ie:* $\exists\ k > 0$ *such that :*

$$||f(t, x) - f(s, y)|| \leq k(||t - s||_{\mathbb{R}^+} + ||x - y||_X)$$

If $x_0 \in D(A)$ *, so the mild solution becomes strong.(See the Theorem 3.3.2)*

Proof. *Let* $a > 0$*, it suffices to prove that the mild solution* x *is lipschitzian .*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)e(s)ds$$

this amounts to showing that :

$$s \longmapsto e(s) = f(s, x(s)) \;\; \textit{is lipschitzian.}$$

Let $h > 0$ *,*

$$x(t+h) - x(t) = T(t+h)x_0 - T(t)x_0 + \int_0^t T(s)\left(f(t+h-s, x(t+h-s)) - f(t-s, x(t-s))\right)ds$$
$$+ \int_t^{t+h} T(s)f(t+h-s, x(t+h-s))ds$$

So

$$\begin{aligned}|x(t+h) - x(t)| &\leq |T(t+h)x_0 - T(t)x_0| + M_a k \int_0^t |x(t+h-s) - x(t-s)|ds \\ &\leq o(h) + M_a k \int_0^t |x(h+s) - x(s)|ds \\ &\leq o(h)e^{M_a k a}.\end{aligned}$$

Let $t \to x(t)$ is lipschitzian on $[0, a] \Rightarrow t \to f(t, x(t))$ is lipschitzian on $[0, a]$, $t \to e(t)$ is lipschitzian on $[0, a]$.
Hence the existence of the strong solution .
Stability and asymptotic behavior .
Either the autonomous system :

$$(1) \begin{cases} x'(t) = Ax(t) + f(x(t)), & t > 0 \\ x(0) = 0 \end{cases}$$

A infinitesimal generator of a C_0-semi-group $(T(t))_{t\geq 0}$ and $f : X \to X$ lipschitzian

$$\begin{aligned} x(t) : X &\longrightarrow X \\ x_0 &\longmapsto x(t)x_0 = x(t, x_0) \end{aligned}$$

$x(t, x_0)$exists it is the mild solution of (1),
$(x(t))_{t\geq 0}$ is a C_0-semi-group

We calculated this limit

$$\lim_{t\to+\infty} x(t, x_0)$$

$$|x(t)x_0 - x(t)y_0| \leq \beta e^{\alpha t}|x_0 - y_0|, \qquad \forall t \geq 0,$$

$x_0 \to x(t, x_0)$ is continuous.

Proposition 3.4.2. *If* $\lim_{t\to+\infty} = \bar{x}$. *So* $\bar{x}$ *satisfies* $A\bar{x} + f(\bar{x}) = 0$ *ie* $\bar{x}$ *is equilibrium point .*

Proof. *Let* $\bar{x} = \lim_{t\to+\infty} x(t, x_0) = \lim_{t\to+\infty} x(t)x_0$

$$\begin{aligned} x(s)\bar{x} &= x(s)(\lim_{t\to+\infty} x(t)x_0) \\ &= \lim_{t\to+\infty}(x(s)x(t)x_0) = \lim_{t\to+\infty}(x(s + t)x_0). \end{aligned}$$

$u(s)\bar{x} = \bar{x}, \forall s \geq 0$
ie $\bar{x}$ *is a fixed point for* $u(s)$,$\forall s \geq 0$

$$u(t)\bar{x} = T(t)\bar{x} + \int_0^t T(t - s)f(u(s)\bar{x})ds$$
$$u(t)\bar{x} = T(t)\bar{x} + \int_0^t T(t - s)f(\bar{x})ds$$

$0 = \frac{u(t)\bar{x}-\bar{x}}{t} = \frac{T(t)\bar{x}-\bar{x}}{t} + \frac{1}{t}\int_0^t T(t - s)f(\bar{x})ds \to_{t\longrightarrow 0} A(\bar{x}) + f(\bar{x})$

Equation at equilibrium

$$\begin{cases} A\bar{x} + f(\bar{x}) = 0 \\ \bar{x} \in D(A) \end{cases}$$

Definition 3.4.1. *Let* $\bar{x}$ *is equilibrium point :*
$\bar{x}$ *is stable* $\Longleftrightarrow \forall\varepsilon > 0, \exists\eta > 0$ $|x_0 - \bar{x}| < \eta$
$\Rightarrow sup_{t\geq 0}|x(t)x_0 - \bar{x} \leq \varepsilon$ $\bar{x}$ *is said to be* ***asymptotically stable*** *if* $\bar{x}$ *is stable and another* $\exists U$ *contains* $\bar{x}$ *such that :*

$$\forall x_0 \in U \lim_{t\to+\infty} u(t)x_0 = \bar{x}$$

$\bar{x}$ *is said to be* ***exp-stable*** *if* $\bar{x}$ *is stable another we have* $\exists U$ *contains* $\bar{x}$,$\exists\alpha$, $M > 0$ *Such that*

$$\forall x_0 \in U \ |u(t)x_0 - \bar{x}| \leq Me^{-\alpha t}$$

$\bar{x}$ *is said to be unstable if* $\bar{x}$ *is not stable*

and

$$\exists x_n \longrightarrow x, \exists (t_n), |u(t_n)x_n - \bar{x}| > \varepsilon$$

Linear equation

$$(2) \begin{cases} x'(t) = Ax(t) \ t \geq 0 \\ x(0) = x_0 \in X \end{cases}$$

0 is always a point of equilibrium.
we know $\exists M \geq 1, \exists w \in \mathbb{R}$ such that $|T(t)| \leq Me^{wt}$.
The type of $(T(t)_{t\geq 0}$ is defined by : $w_0(T) = \inf[w \in \mathbb{R} \ \sup_{t\geq 0} (e^{-wt}|T(t)|) < +\infty]$

Proposition 3.4.3. *if* $w_0(T) < 0$*, so:*

$$|T(t)| \longrightarrow 0$$

Theorem 3.4.9. $w_0(T) = \lim_{t\to+\infty} \frac{log|T(t)|}{t}$

Definition 3.4.2. *The spectral terminal* $\zeta(A)$ *of* A *is defined by :*

$$\zeta(A) = sup\{Re(\lambda) : \lambda \in \sigma(A)\}$$

Theorem 3.4.10. *Let* $T(t)$ *est compact for* $t > 0$ *so :*

$$\sigma(A) = \sigma_p(A)$$
$$w_0(T) = sup\{Re(\lambda) : \lambda \in \sigma(T)\} < +\infty.$$

Remark. *In the general case we have :* $\zeta(A) \leq w_0(T)$

General formula :

$w_0(T) = max(\zeta(A), w_{ess}(T))$
On the other hand

$$w_{ess}(T) = sup\left\{w \in \mathbb{R} : \ sup \ \left\{e^{-wt}|T(t)|_{ess}\right\} < +\infty\right\}$$

On the other hand $|\cdot|_{ess}$ is defined by :
for $K \in \pounds(X)$

$$|k|_{ess} = inf\,\{M > 0 \ \ |\alpha(K(B))| \leq M\alpha(B)\} \qquad \forall \ B \ bounded \ of \ X$$

$\alpha(\cdot)$ measure of non-compactness.

Lemma 3.4.2. *If* $T(t)$ *is compact for* $t > 0$ *, so :*

(i) $\zeta(A) < 0 \quad \Longrightarrow T(t) \longrightarrow 0$

(ii) $\zeta(A) = 0 \quad \Longrightarrow \exists x_0 \in X \quad |T(t)x_0| = |x_0| \ , \ \forall t > 0$

(iii) $\zeta(A) > 0 \quad \Longrightarrow \exists x_0 \in X : \quad T(t)x_0 \longrightarrow +\infty$

Proof. *(ii)*
T(t) compact $\Longrightarrow \zeta(A)$ *is reached , so*

$$\exists \lambda_0 \in \sigma_p(A) \text{ such that } \zeta(A) = Re(\lambda_0).$$

$Re(\lambda_0) = 0$. $\exists x_0 \neq 0$ *such as :* $Ax_0 = \lambda_0 x_0 \Longrightarrow T(t)x_0 = e^{\lambda_0 t}x_0$ *, so*

$$|T(t)x_0| = |x_0|.$$

formula :

If the eigenvalues are simple and if $\zeta(A) = 0$ *so :*

$$\exists m \geq 1 \text{ such that } |T(t)| \leq M \qquad (\text{here } T(t) \text{ compact})$$

Theorem 3.4.11. *Let* $\overline{x}$ *equilibrium point.*
We suppose that f is differential in $\overline{x}$ *.Let B=f*$(\overline{x}) \in \pounds(X)$
We suppose that $T(t)$ *is compact ,* $\forall\ t > 0$.
Let $(v(t))_{t\geq 0}$ *the* C_0*-semi-group solution of the linear system .*

$$\begin{cases} y'(t) = (A+B)y(t) & t > 0 \\ y(0) = y_0 \end{cases}$$

so we have :

(i) $\zeta(A+B) < 0 \Longrightarrow \overline{x}$ *is exponentially stable*

(ii) $\zeta(A+B) > 0 \Longrightarrow \overline{x}$ *is unstable*

$$v(t)x = T(t)x + \int_0^t T(t-s)Bv(s)xds \qquad v(t) \text{ is compact} \quad , \quad for\ t > 0$$

$$\zeta(A+B) < 0 \quad \Longrightarrow \quad v(t) \longrightarrow 0.$$

$$\zeta(A+B) > 0 \quad \Longrightarrow \quad v(t) \longrightarrow +\infty.$$

Theorem 3.4.12. *(Desh + Shappache).*
Let Y a Banach space and $(S(t))_{t\geq 0}$ C_0*-semi-group, we suppose* $\exists\ \bar{X} \in X$ *, such that* $S(t)\bar{x} = \bar{x} \quad , t \geq 0$*, be* $(R(t))_{t\geq 0}$ C_0*-semi-group of generator Q.*
if $w_0(Q) < 0$ *so* $\bar{x}$ *is exponentially stable , then* $R(t)$ *is compact for* $t > 0$.
if $w_0(Q) > 0$ *so* $\bar{x}$ *is unstable.*

Proof. *(theorem 3.4.11) we suppose that* $\bar{x} = 0$
it suffices to show that $v(t) = u'(t)(0)$ *and we conclude with the theorem* ***(Desh + Shappache)****.*
ie

$$\forall \epsilon > 0, \exists \delta > 0, |x_0| < \delta \Longrightarrow |u(t)x_0 - v(t)x_0| < \epsilon |x_0|,$$

and

$$|u(t)x_0| < \beta e^{\alpha t} |x_0|.$$

$$u(t)x_0 = T(t)x_0 + \int_0^t T(t-s) f(u(s)x_0) ds$$

$$v(t)x_0 = T(t)x_0 + \int_0^t T(t-s) B(v(s)x_0) ds$$

$u(t)x_0 - v(t)x_0 = \int_0^t T(t-s) \left(f(u(s)x_0) - B(u(s)x_0) \right) ds + \int_0^t T(t-s) \left(B(u(s)x_0) - B(v(s)x_0) \right) ds$
f is differential in $\bar{x}$ *= 0 . Let* $\epsilon > 0$

$$\exists\ \delta > 0 \textit{ such that } : |x| < \delta + |f(x) - B(x)| < \epsilon |x|$$

We have $|u(t)x_0| \leq M e^{\beta t} |x_0|$.

$$|f(u(s)x_0) - B(u(s)x_0)| < \epsilon \cdot |u(s)(x_0)| < \epsilon K_1 |x_0|$$

$$|u(t)x_0 - v(t)x_0| \leq \bar{K_1} \epsilon |x_0| + \bar{K_2} \int_0^t |u(s)x_0 - v(s)x_0| ds \qquad \forall s \leq t$$

By Gronwall's lemma we have :

$$|u(t)x_0 - v(t)x_0| \leq \bar{K_1} \epsilon |x_0| e^{\bar{K_2} t} = o(\epsilon) |x_0|$$

Hence the result.

Application:

Reaction-diffusion model.
Ω is regular open bounded of $\mathbb{R}^n$.

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(u(t,x)) \\ u(t,x) = 0, \forall t \geq 0. \forall x \in \partial \Omega \\ u(0,x) = u_0(x) \end{cases}$$

$f : \mathbb{R} \longrightarrow \mathbb{R}$ is regular .
$X = L^2(\Omega)$; $A = -\Delta$
$D(A) = H^2(\Omega) \cap H^1{}_0(\Omega)$.

Theorem 3.4.13.

$$\begin{aligned} \sigma(\Delta) &= \sigma_p(\Delta) \\ &= \{\lambda_n \ \ , \ \ n \geq 1\} \end{aligned}$$

$$\cdots < \lambda_{n+1} < \lambda_n < \cdots < \lambda_0 < 0.$$

Moreover the eigenvectors normalized in associated with λ_n form a Hilbertian basis in $L^2(\Omega)$, $\forall\, f \in L^2(\Omega)$:

$$f = \sum_{n=1}^{+\infty} < f, e_n > e_n.$$

$$||f|| = \sqrt{\sum_{n=1}^{+\infty} | < f, e_n > |^2}$$

We have :

$$T(t)f = \sum_{n=1}^{+\infty} e^{\lambda_n t} < f, e_n > e_n$$

Theorem 3.4.14. *T(t) is compact for $t > 0$*

$$\begin{aligned} T(t)f &= \lim_{N \longrightarrow +\infty} \sum_{n=1}^{N} e^{\lambda_n t} < f, e_n > e_n \\ &= \lim_{N \longrightarrow +\infty} S_N(t)f. \end{aligned}$$

$$S_N(t)\left(L^2(\Omega)\right) = vect\,\{e_1, \cdots, e_N\} \Longleftrightarrow of\ finished\, rank.$$

so $S_N(t)$ is compact.

$$||T(t) - S_N(t)|| \longrightarrow_{N \longrightarrow +\infty} 0.$$

We suppose that u is solution of the reaction-diffusion module on $[0, a] \times \Omega$
Let

$$\begin{aligned} v : [0, a] &\longrightarrow L^2(\Omega) \\ t &\longmapsto v(t) \end{aligned}$$

$v(t)(x) = u(t, x).$

$$\begin{aligned} F : L^2 &\longrightarrow L^2 \\ v &\longmapsto F(v) \end{aligned}$$

$F(v, x) = F(v(x)).$
So

$$(3) \begin{cases} v(t) = Av(t) + F(v(t)) & , t \geq 0 \\ v(0) = u_0 \end{cases}$$

Lemma 3.4.3. $|f(x)| \leq a|x| + b. \qquad a, b > 0.$
$f : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous*
so $f : L^2 \longrightarrow L^2$ *is continuous .*
(3) *admits a maximal mild solution defined on* $[0, t_{max}[$.

$if\ t_{max} < \infty \Longrightarrow \overline{lim}_{t\longrightarrow t_{max}}|u(t)| = +\infty$

$if\ Lipf < \infty \Longrightarrow LipF < \infty$

(3) *admits a mild solution defined on* $[0, +\infty[$.

stability :

f(0)=0

Equation of equilibrium : $\begin{cases} \Delta v + f(v) = 0 \\ vn_{\partial\Omega} = 0 \end{cases}$ f differentiable in 0 .

The linearised equation around 0 is given by :

$$(4) \begin{cases} \frac{\partial v}{\partial t}(t,x) = \Delta v(t,x) + av(t,x) \quad t \geq 0, x \in \Omega \\ v(t,x) = 0, \forall t \geq 0. \forall x \in \partial\Omega \\ v(0,x) = v_0(x) \end{cases}$$

with $a = f'(0)$

$$(4) \Longleftrightarrow \begin{cases} w' = Bw \\ w(0) = w_0 \end{cases}$$

with $B = \Delta + aI$.

Let $(S_B(t))_{t\geq 0}$ the C_0-semi-group generated by B .

Therefore

$$\begin{aligned} S_B(t)f &= e^{at}T(t)f. \\ &= e^{at}\sum_{n=1}^{\infty} e^{\lambda_n t} < f, e_n > e_n. \end{aligned}$$

$S_B(t)$ is compact for $t \geq 0$ so

$$\begin{aligned} w_0(S_B) &= \zeta(B) \\ &= \sup\{Re(\lambda), \lambda \in \sigma_p(B)\} \end{aligned}$$

λ_n function of Ω

$$\begin{aligned} \sigma_p(B) &= \sigma_p(\Delta + aI) \\ &= \{\lambda_n + a, n \geq 1\} \end{aligned}$$

so : $\zeta(B) = \lambda_0 + a$.

According the Theorem 3.4.11 we have :

$$\lambda_0 + a < 0 \Longrightarrow 0\ is\ exp-stable.$$

$$\lambda_0 + a > 0 \Longrightarrow 0\ is\ unstable.$$

CHAPTER 4

STRUCTURE OF SOLUTION SETS FOR DIFFERENTIAL EVOLUTION EQUATIONS IN BANACH SPACE

4.1 Introduction

In this Chapter, we study structure of solutions sets to an initial value problem (IVP for short) for a non-homogeneous differential evolution equation with non-local initial conditions More precisely we consider the IVP

$$x'(t) = Ax(t) + f(t, x(t)), \qquad t \in [0, b] \tag{4.1}$$

$$x(0) = x_0 - g(x), \tag{4.2}$$

where $f : [0, b] \times X \to X$. $g : C([0, b], X) \to X$ functions that will be specified later, $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, and X a real separable Banach space with norm $|.|$.

4.2 Contractibility for the solution sets for the nonhomogeneous problem

4.2.1 Preliminaries

First, we recall some elementary notions and notations from geometric topology . In what follows (X, d) and $(Y, d^{'})$ stand for two metric spaces.

Definition 4.2.1. *Let $A \in P(X)$. The set $P(A)$ is called a contractible space provided there exists a continuous homotopy $H : A \times [0, 1] \longrightarrow A$ and $x_0 \in A$ such that* *(a)* $H(x, 0) = x$, *for every* $x \in A$,

(b) $H(x, 1) = x_0$, *for every* $x \in A$,

Definition 4.2.2. *$A \in P(X)$ is a retract of X if there existe a continuous map $r : X \longrightarrow A$ such that $r(a) = a$ for every $a \in A$.*

Definition 4.2.3. *A compact nonempty space X is called an $R_\delta -$ set provided there exists a decreasing sequence of compact nonempty contractible spaces $\{X_n\}_{n\in N}$ such that $X = \bigcap_{n=1}^{\infty} X_n$.*

Definition 4.2.4. *A space X is called an absolute retract (in short $X \in AR$) provided that for every space Y, every closed subset $B \subseteq Y$ and any continuous map $\varphi : B \longrightarrow X$, there exists a continuous extension $g : Y \longrightarrow X$ of φ over Y that is*

$$g(x) = \varphi(x) \text{ for every } x \in B.$$

In other words, for every space Y and for any embedding $\varphi : X \longrightarrow Y$, the set $\varphi(X)$ is a retract of Y.

Let us recall the well-known Lasota-Yorke approximation lemma, [8, 7].

Lemma 4.2.1. *Let E be a normed space, X a metric space and $F : X \longrightarrow E$ be a continuous map. Then for each $\varepsilon > 0$ there is a locally Lipschitz map $F_\varepsilon : X \longrightarrow E$ such that*

$$\|F(x) - F_\varepsilon(x)\| < \varepsilon, \text{ for every } x \in X.$$

Next, we present a result about the topological structure of the solution set of some nonlinear functional equations due to N. Aronszajn and developed by Browder and Gupta [5, 3].

Theorem 4.2.1. *Let (X, d) be a metric space, $(E, \|.\|)$ a Banach space and $F : X \longrightarrow E$ a proper map, i.e., F is continuous and for every compact $K \subset E$, the set $F^{-1}(K)$ is compact. Assume further that for each $\varepsilon > 0$, a proper map $F_\varepsilon : X \longrightarrow E$ is given, and the following two conditions are satisfied*

(a) $\|F_\varepsilon(x) - F(x)\| < \varepsilon$, *for every* $x \in X$,

(b) for every $\varepsilon > 0$ and $u \in E$ in a neighborhood of the origin such that $\|u\| \leq \varepsilon$, the equation $F_\varepsilon(x) = u$ has exactly one solution x_ε,

then the set $S = F^{-1}(0)$ is an $R_\delta -$ set.

Lemma 4.2.2. *Let E be a Banach space, $C \subset E$ be a nonempty closed bounded subset of E and $F : C \longrightarrow E$ is an completely continuous map, then $G = Id - F$ is proper.*

4.2.2 Existence result

In this section, we denote by $S(f, x_0, g)$ the set of all solutions of the problem $(4.1) - (4.2)$. We have to prove that solution sets $S(f, x_0, g)$ is contractible.
. Before stating and proving this result, we give the definition of its mild solution.

Definition 4.2.5. *A function $x \in C([0,b], X)$ is said to be a mild solution of problem (4.1)–(4.2) if $x(0) = x_0 - g(x)$, and x is a solution of integral equation*

$$x(t) = T(t)(x_0 - g(x)) + \int_0^t T(t-s)\tilde{f}(s, x(s))ds, \quad t \in [0,b].$$

We consider the following problem

$$x'(t) = Ax(t) + \tilde{f}(t, x(t)), \qquad t \in [0,b] \tag{4.3}$$

$$x(0) = x_0 - g(x), \tag{4.4}$$

Theorem 4.2.2. *Assume that*

(H1) There exists a constant $\eta > 0$ such that $|g(t,u) - g(t,u_1)| \leq \eta|u - u_1|$, for each $t \in [0,b]$, and each $u, u_1 \in X$.

(H2) There exists a constant $\mu > 0$ such that $|\tilde{f}(t,u) - \tilde{f}(t,u_1)| \leq \mu|u - u_1|$, for each $t \in [0,b]$, and each $u, u_1 \in X$.

If

$$M(\eta + \mu b) < 1. \tag{4.5}$$

then (4.3)-(4.4) *has a unique solution $\overline{x}$ on* $[0,b]$.

We transform the problem $(4.3) - (4.4)$ into a fixed point problem. Consider the operator $N : C([0,b], X) \longrightarrow C([0,b], X)$ defined by

$$N(x)(t) = T(t)(x_0 - g(x)) + \int_0^t T(t-s)\tilde{f}(s, x(s))ds, \quad t \in [0,b].$$

We use Banach fixed point theorem to prove the existence solution. Clearly, the fixed points of the operator N are solution of the problem (4.3)-(4.4). We shall use the Banach contraction principle to prove that N has a fixed point. We shall show that N *is a contraction.* Let $x_1, x_2 \in C([0,b], E)$. Then, for each $t \in [0,b]$ we have

Proof.

$$\begin{aligned} |N(x)(t) - N(\bar{x})(t)| &\leq |T(t)(g(\bar{x}) - g(x))| + \int_0^t \left|T(t-s)\left[\tilde{f}(s, x(s)) - \tilde{f}(s, \bar{x}(s))\right]\right| ds \\ &\leq M\eta|x - \bar{x}| + M\int_0^t \mu|x - \bar{x}|ds \\ &\leq M(\eta + \mu b)|x - \bar{x}|, \end{aligned}$$

if $M(\eta + \mu b) < 1$ so N is a contraction , with $M = \sup_{t\in[0,b]} |T(t)|$.
Therefore the fixed point of the operator N is solution of the problem (4.3)-(4.4) .

Claim 1. *Define the homotopy* $H : S(f, x_0, g) \times [0, b] \longrightarrow S(f, x_0, g)$ *by*

$$H(x, \lambda)(t) = \begin{cases} x(t), & 0 < t \leq \lambda b \\ \overline{x}(t) & \lambda b < t \leq b. \end{cases}$$

where $\overline{x} = S(f, x_0, g)$ *is the unique solution of problem* $(4.3) - (4.4)$. *In particular*

$$H(x, \lambda) = \begin{cases} x, & \text{for } \lambda = 1, \\ \overline{x}, & \text{for } \lambda = 0. \end{cases}$$

We prove that H *is a continuous homotopy. Let* $(x_n, \lambda_n) \in S(f, x_0, g) \times [0, b]$ *be such that* $(x_n, \lambda_n) \longrightarrow (x, \lambda)$, *as* $n \longrightarrow +\infty$.
We shall prove that $H(x_n, \lambda_n) \longrightarrow H(x, \lambda)$, *we have*

$$H(x_n, \lambda_n)(t) = \begin{cases} x_n(t), & \text{for } t \in (0, \lambda_n b], \\ \overline{x}(t), & \text{for } t \in (\lambda_n b, t]. \end{cases}$$

We consider several cases,

(**a**) *if* $\lim_{n \longrightarrow +\infty} \lambda_n = 0$,

$$\begin{aligned} &|H(x_n, \lambda_n)(t) - H(x, \lambda)(t)| \leq |H(x_n, \lambda_n)(t) - H(x, \lambda)(t)|_{[0,\lambda\ b]} \\ &+ |H(x_n, \lambda_n)(t) - H(x, \lambda)(t)|_{[\lambda\ b, \lambda_n\ b]} + |H(x_n, \lambda_n)(t) - H(x, \lambda)(t)|_{[\lambda_n\ b, b]} \\ &\leq |x_n(t) - x(t)|_{[0,\lambda\ b]} + |x_n(t) - \overline{x}(t)|_{[\lambda\ b, \lambda_n\ b]} + |\overline{x}(t) - \overline{x}(t)|_{[\lambda_n\ b, b]} \\ &\leq |x_n(t) - x(t)|_{[0,\lambda\ b]} + |x_n(t) - \overline{x}(t)|_{[\lambda\ b, \lambda_n\ b]} \\ &\leq (M\eta + M\mu b)\|x_n - x\| + (M\eta + M\mu b)|x_n(t) - \overline{x}(t)|_{[\lambda\ b, \lambda_n\ b]}, \end{aligned}$$

which tends to 0 as $n \longrightarrow +\infty$.

(**b**) *If* $\lim_{n \longrightarrow \infty} \lambda_n = 1$,

it's treated similarly.
If $\lambda_n \neq 0$ *and* $0 < \lim_{n \longrightarrow \infty} \lambda_n < 1$,
two cases must be treated,

- $t \in (0, \lambda_n]$,

then $H(x_n, \lambda_n)(t) = H(x, \lambda)(t) = x_n(t) - x(t)$,

$$x_n(t) = T(t)(x_0 - g(x_n)) + \int_0^t T(t - s)\widetilde{f}(s, x_n(s))ds,$$

since f *is continuous function,*

$$x(t) = T(t)(x_0 - g(x)) + \int_0^t T(t - s)\widetilde{f}(s, x(s))ds$$

$$\|H(x_n, \lambda_n) - H(x, \lambda)\|_X \longrightarrow 0 \text{ as } n \longrightarrow +\infty,$$

- $t \in (\lambda_n, 1]$,

then $H(x_n, \lambda_n)(t) = H(x, \lambda)(t) = \overline{x}(t)$, *thus*

$$\|H(x_n, \lambda_n) - H(x, \lambda)\|_X \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

Therefore H is a continuous function, proving that $S(f, x_0, g)$ is contractible to the point $\overline{x}$.

4.3 Toplogical structure for nonlinear impulsive differential evolution equation

Our aim in this section is to study the compactness of solution sets for the nonlinear impulsive problem,

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \in [0, b] \tag{4.6}$$

$$\Delta x|_{t_k} = I_k(x(t_k^-)), \tag{4.7}$$

$$x(0) = x_0 - g(x), \tag{4.8}$$

$I_k : X \to X$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$ and $\Delta x|_{t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+) = \lim_{t \longrightarrow t_k^+} x(t)$ and $x(t_k^-) = \lim_{t \longrightarrow t_k^-} x(t)$.

Firstly , we give our main existence result for problem (4.6)–(4.8). Before stating and proving this result, we give the definition of its mild solution.

Definition 4.3.1. *A function $x \in PC([0, b], X)$ is said to be a mild solution of problem (4.6)–(4.8) if $x(t) = x_0 - g(x)$ and x is a solution of impulsive integral equation*

$$x(t) = T(t)(x_0 - g(x)) + \int_0^t T(t-s)f(t, x(s))ds + \sum_{0<t_k<t} T(t - t_k)I_k(x(t_k^-)), \quad t \in [0, b].$$

our result is based on the nonlinear alternative of Leray-Schauder type.
We assume the following hypotheses :

(H1) $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}$, $t \in [0, b]$ which is compact for $t > 0$ in the Banach space X. Let $M = \sup\{\|T(t)\|_{B(X)} : \ t \in [0, b]\}$;

(H2) $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

(H3) There exist constants a , c $\in \mathbb{R}_+$

$$|f(t, u)| \leq a|u| + c$$

for $t \in [0, b]$ and $u \in X$.

(H4) There exist constants $a_k, b_k \in R_+$ such that

$$|I_k(u)| \leq a_k|u| + b_k \quad \text{for } u \in X.$$

(H5) There exists a constant $l' > 0$ such that

$$|g(u_2) - g(u_1)| \leq l'|u_2 - u_1|,$$

for each $t \in [0, b]$, and each $u_2, u_1 \in X$.

(H6) There exists a constant $\xi > 0$ such that

$$|g(u)| \leq \xi,$$

for each $u \in X$.

Theorem 4.3.1. *Under assumptions* $(H1) - (H6)$*, the solution set* $S_4(f, x_0, g)$ *is an* R_δ*- set .*

Transform the problem (4.6)-(4.8) into a fixed point problem. Consider the operator $N : PC([0, b], X) \longrightarrow PC([0, b], X)$ defined by Our main result in this section is based upon the following fixed point theorem .

Theorem 4.3.2. *Let* X *be a Banach space, and* $\mathcal{A},\ \mathcal{B} : X \to X$ *two operators satisfying:*

(i) $\mathcal{A}$ *is a contraction, and*

(ii) $\mathcal{B}$ *is completely continuous.*

Then either

(a) the operator equation $x = \mathcal{A}(x) + \mathcal{B}(x)$ *has a solution, or*

(b) the set $\mathcal{E} = \left\{u \in X : \lambda\mathcal{A}\left(\frac{u}{\lambda}\right) + \lambda\mathcal{B}(u) = u\right\}$ *is unbounded for* $\lambda \in (0, 1)$.

Theorem 4.3.3. *Assume that (H1)-(H6) hold. Then the problem (4.6)–(4.8) has at least one mild solution on* $[0, b]$.

Proof. Transform the problem (4.6)-(4.8) into a fixed point problem. Consider the two operators:

$$\mathcal{A}, \mathcal{B} : PC\left([0, b]\,, X\right) \to PC\left([0, b]\,, X\right)$$

defined by

$$\mathcal{A}(x)\,(t) = \sum_{0<t_k<t} T\left(t - t_k\right) I_k\left(x\left(t_k^-\right)\right), \quad \text{if } t \in [0, b],$$

and

$$\mathcal{B}(x)\,(t) = T(t)(x_0 - g(x)) + \int_0^t T(t-s) f\left(s, x(s)\right) ds, \quad \text{if } t \in [0, b].$$

Then the problem of finding the solution of problem (4.6)–(4.8) is reduced to finding the solution of the operator equation

$\mathcal{A}(x)(t) + B(x)(t) = x(t), t \in [0,b]$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfies all the conditions of Theorem 4.3.2. For better readability, we devide the proof into a sequence of steps.

Step 1: $\mathcal{B}$ is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC([0,b],X)$. Then for $t \in [0,b]$

$$\begin{aligned} |\mathcal{B}(x_n)(t) - \mathcal{B}(x)(t)| &= M|g(x_n) - g(x)| + \left| \int_0^t T(t-s)[f(s, x_{n(s)}) - f(s,x(s))]ds \right| \\ &\leq Ml'||x_n - x||_{PC} + M\int_0^b |f(s,x_n(s)) - f(s,x(s))|\, ds. \end{aligned}$$

Since $f(s,\cdot)$ is continuous for a.e. $s \in [0,b]$, we have by the Lebesgue dominated convergence theorem

$$||\mathcal{B}(x_n)(t) - \mathcal{B}(x)(t)||_{PC} \to 0 \ as \ n \to \infty.$$

Thus $\mathcal{B}$ is continuous.

Step 2: $\mathcal{B}$ maps bounded sets into bounded sets in $PC([0,b],X)$.
There exists a positive constant l then we have for each $t \in [0,b]$,

$$\begin{aligned} |\mathcal{B}(x)(t)| &= \left| T(t)(x_0 - g(x)) + \int_0^t T(t-s)f(s,x(s))ds \right| \\ &\leq M|x_0| + M|g(x) - g(0)| + M|g(0)| + aM|x|b + Mcb; \end{aligned}$$

Then we have

$$||\mathcal{B}(x)||_{PC} \leq M|x_0| + ||x||_{PC}|(Ml' + aMb) + M\xi + Mcb = l.$$

Step 3: $\mathcal{B}$ maps bounded sets into equi-continuous sets of $PC([0,b],X)$.

Let $\tau_1, \tau_2 \in [0,b]\setminus\{t_1, ..., t_m\}$, $\tau_1 < \tau_2$.Thus if $\epsilon > 0$ and $\epsilon \leq \tau_1 < \tau_2$ we have

$$\begin{aligned} |\mathcal{B}(x)(\tau_2) - \mathcal{B}(x)(\tau_1)| &\leq |T(\tau_2)(x_0 - g(x)) - T(\tau_1)(x_0 - g(x))| \\ &+ (a||x||_{PC} + c)\int_0^{\tau_1 - \epsilon} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(X)}ds \\ &+ (a||x||_{PC} + c)\int_{\tau_1 - \epsilon}^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(X)}ds \\ &+ (a||x||_{PC} + c)\int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(X)}ds. \end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ become sufficiently small, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology ([1]). This proves the equi-continuity for the case where $t \neq t_i, k = 1, 2, \ldots, m+1$. It remains to examine the equi-continuity at $t = t_i$.

First we prove equi-continuity at $t = t_i^-$. Fix $\delta_1 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$. For $0 < h < \delta_1$ we have

$$|\mathcal{B}(x)(t_i-h)-\mathcal{B}(x)(t_i)| \le |\left(T(t_i-h)-T(t_i)\right)(x_0-g(x))| \\ + \int_0^{t_i-h} |\left(T(t_i-h-s)-T(t_i-s)\right)f(s,x(s))|ds \\ + M(a||x||_{PC}+c)h;$$

which tends to zero as $h \to 0$.

Define

$$\widehat{\mathcal{B}}_0(x)(t)=\mathcal{B}(x)(t), \quad t\in[0,t_1]$$

and

$$\widehat{\mathcal{B}}_i(x)(t)=\begin{cases} \mathcal{B}(x)(t), & \text{if } t\in(t_i,t_{i+1}] \\ \mathcal{B}(x)(t_i^+), & \text{if } t=t_i. \end{cases}$$

Next we prove equi-continuity at $t=t_i^+$. Fix $\delta_2>0$ such that $\{t_k : k\neq i\}\cap[t_i-\delta_2,t_i+\delta_2]=\emptyset$. For $0<h<\delta_2$ we have

$$|\widehat{\mathcal{B}}(x)(t_i+h)-\widehat{\mathcal{B}}(x)(t_i)| \le |\left(T(t_i+h)-T(t_i)\right)(x_0-g(x))| \\ + \int_0^{t_i} |\left(T(t_i+h-s)-T(t_i-s)\right)f(s,x(s))|ds \\ + M(a||x||_{PC}+c)h.$$

The right-hand side tends to zero as $h\to 0$. The equi-contnuity for the cases $\tau_1<\tau_2\le 0$.

As consequence of Steps 1 to 3 together with Arzelá-Ascoli theorem it suffices to show that $\mathcal{B}$ maps B into a precompact set in X.

Let $0<t<b$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t$.

$$\mathcal{B}_\epsilon(x)(t)=T(t)(x_0-g(x))+T(\epsilon)\int_0^{t-\epsilon}T(t-s-\epsilon)f(s,x(s))ds.$$

Since $T(t)$ is a compact operator, the set

$$X_\epsilon(t)=\{\mathcal{B}_\epsilon(x)(t): \quad x\in B_q\}$$

is precompact in X for every $\epsilon, \quad 0<\epsilon<t$. Moreover, for every $y\in B_q$ we have

$$|\mathcal{B}(x)(t)-\mathcal{B}_\epsilon(x)(t)| \le \int_{t-\epsilon}^{t}\|T(t-s)\|_{B(X)}(a|x(s)|+c)ds \\ \le M(a||x||_{PC}+c)\epsilon.$$

Therefore, there are precompact sets arbitrarily close to the set $X_\epsilon(t)=\{\mathcal{B}_\epsilon(x)(t): \quad x\in B_q\}$. Hence the set $X(t)=\{\mathcal{B}(x)(t): \quad x\in B_q\}$ is precompact in X. Hence the operator $\mathcal{B}: PC([0,b],X)\to PC([0,b],X)$ is completely continuous.

Step 4: $\mathcal{A}$ is a contraction

Let $x, y \in PC([0,b], X)$. Then for $t \in [0,b]$

$$\begin{aligned} |\mathcal{A}(y)(t) - \mathcal{A}(x)(t)| &= \left| \sum_{0<t_k<t} T(t-t_k)\left(I_k\left(y\left(t_k^-\right)\right) - I_k\left(x\left(t_k^-\right)\right)\right)\right| \\ &\leq M \sum_{0<t_k<t} \left|I_k\left(y\left(t_k^-\right)\right) - I_k\left(x\left(t_k^-\right)\right)\right| \\ &\leq M \sum_{k=1}^{m} a_k \left|y\left(t_k^-\right) - x\left(t_k^-\right)\right| \\ &\leq M \sum_{k=1}^{m} a_k \left\|y - x\right\|. \end{aligned}$$

Then

$$\|\mathcal{A}(y) - \mathcal{A}(x)\| \leq M \sum_{k=1}^{m} a_k \left\|y - x\right\|,$$

which is a contraction, since $M \sum\limits_{k=1}^{m} a_k < 1$.

Step 5: A priori bounds.

Now it remains to show that the set

$$\mathcal{E} = \left\{x \in PC([0,b], X) : x = \lambda\mathcal{B}(x) + \lambda\mathcal{A}\left(\frac{x}{\lambda}\right) \quad \text{for some} \quad 0 < \lambda < 1\right\}$$

is bounded.

Let $x \in \mathcal{E}$, then $x = \lambda\mathcal{B}(x) + \lambda\mathcal{A}\left(\frac{x}{\lambda}\right)$ for some $0 < \lambda < 1$. Thus, for each $t \in [0,b]$,

$$x(t) = \lambda T(t)(x_0 - g(x)) + \lambda \int_0^t T(t-s)f(s,x_s)ds + \lambda \sum_{0<t_k<t} T(t-t_k)\, I_k\left(\frac{x}{\lambda}\left(t_k^-\right)\right).$$

This implies by (H3), (H4), (H5) and (H6) that, for each $t \in [0, b]$, we have

$$\begin{aligned}
|x(t)| &\leq \lambda M|(x_0 - g(x))| + \lambda M \int_0^t (a|x(s)| + c)ds + \lambda M \sum_{k=1}^{m} \left| I_k\left(\frac{x}{\lambda}\left(t_k^-\right)\right)\right| \\
&\leq \lambda M|x_0| + \lambda M|g(x) - g(0)| + \lambda M|g(0)| + \lambda M(a||x||_{PC} + c)b \\
&+ \lambda M \sum_{k=1}^{m} \left| I_k\left(\frac{x}{\lambda}\left(t_k^-\right)\right)\right| \\
&\leq \lambda M\left(|x_0| + |g(0)| + b_k\right) + \lambda M(a||x||_{PC} + c)b + \lambda M \sum_{k=1}^{m} a_k \left|\frac{x}{\lambda}\left(t_k^-\right)\right| + \lambda M l'||x||_{PC} \\
&\leq M\left(|x_0| + |g(0)| + b_k\right) + M(a||x||_{PC} + c)b + M \sum_{k=1}^{m} a_k \left|x\left(t_k^-\right)\right| + M l'||x||_{PC} \\
&\leq M\left(|x_0| + |g(0)| + b_k\right) + Mab||x||_{PC} + Mbc + M||x||_{PC} \sum_{k=1}^{m} a_k + M l'||x||_{PC} \\
&\leq M\left(|x_0| + \xi + b_k\right) + ||x||_{PC}(Mab + Ml' + M \sum_{k=1}^{m} a_k) + Mbc
\end{aligned}$$

Therefore

$$\left[1 - Mab - Ml' - M \sum_{k=1}^{m} a_k\right] ||x||_{PC} \leq M\left(|x_0| + \xi + b_k\right) + Mcb$$

If we assume that

$$\left(Mab + Ml' + M \sum_{k=1}^{m} a_k\right) < 1$$

we'll have

$$||x||_{PC} \leq \frac{M\left(|x_0| + \xi + b_k\right) + Mcb}{\left[1 - Mab - Ml' - M \sum_{k=1}^{m} a_k\right]}$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 4.3.2 we deduce that $\mathcal{A} + \mathcal{B}$ has a fixed point which is a mild solution of problem (4.6)–(4.8).

4.3.1 Compactness of Solution Set

Now we show that the set

$$S = \{x \in C([0, b], X) : x \text{ is a solution of } (4.1) - (4.2)\} \text{ is compact.}$$

Let $(x_n)_{n\in N}$ be a sequence in S.

We put $B = \{x_n : n \in N\} \subseteq C([0, b], X)$. Then from earlier parts of the proof of this theorem, we conclude that B is bounded and equi-continuous. Then from the Ascoli-Arzela theorem, we can conclude that B is compact.

Consider the equation the problem (4.1)-(4.2) following :

$$x'(t) = Ax(t) + f(t, x(t)), \quad \text{a.e} \ \ t \in [0, b],$$
$$x(0) = x_0 - g(x).$$

Hence:

$x_n|_{[0,b]}$ has a subsequence $(x_{n_m})_{n_m \in N}$ converges to x with
x is a solution of (4.1)-(4.2)}.

Let

$$x(t) = T(t)(x_0 - g(x)) + \int_0^t T(t-s)f(s, x(s))ds,$$

and

$$|x_{n_m}(t) - x(t)| \leq |g(x_{n_m})| - g(x)| + \int_0^t T(t-s)|f(s, x_{n_m}(s)) - f(s, x(s))|ds.$$

As $n_m \to +\infty$, $x_{n_m}(t) \to x(t)$, and then

$$x(t) = T(t)(x_0 - g(x)) + \int_0^t T(t-s)f(s, x(s))ds.$$

Hence $S(f, x_0, g)$ is compact.

4.3.2 The solution set is an R_δ-set

Define

$$\tilde{f}(t, x(t)) = \begin{cases} f(t, x(t)), |x(t)| & \leq M', \\ f(t, x(\frac{M'x(t)}{|x(t)|})), & |x(t)| \geq M', \end{cases}$$

Since f is continuous, the function $\widetilde{f}$ is continuous and it is bounded by $(H1)$. So there exists $M' > 0$ such that

$$|\widetilde{f}(t, x)| \leq M', \ \ for \ a.e. \ t \text{ and all } \ x \in R. \tag{4.9}$$

We consider the following modified problem,

$$\begin{cases} x'(t) = Ax(t) + \tilde{f}(t, x(t)), & t \in (0, b], \\ x(0) = x_0 - g(x), \end{cases}$$

We can easily prove that $S(f, x_0, g) = S(\tilde{f}, x_0, g) = Fix\tilde{N}$, where

$$\tilde{N} : C((0, b], X) \longrightarrow C((0, b], X)$$

is defined by

$$\tilde{N}(x)(t) = T(t)(x_0 - g(x)) + \int_0^t T(t-s)\tilde{f}(s, x(s))ds,$$

$$\|\tilde{N}x\| \leq M|x_0 - g(x)| + MM'b =: M_*.$$

Finally we have

$$\|\tilde{N}(x)\|_X \leq M_*,$$

then $\tilde{N}$ is uniformly bounded, as in steps 1 to 2 we can prove that

$$\tilde{N} : C((0,b],X) \longrightarrow C((0,b],X),$$

is compact which allows us to define the compact perturbation of the identity $\widetilde{G}(x) = x - \tilde{N}(x)$ which is a proper map.

From the compactness of $\widetilde{N}$, we can easily prove that all conditions of Theorem (4.2.1) are satisfied. Therefore the solution set $S(\tilde{f}, x_0, g) = \tilde{G}^{-1}(0)$ is an R_δ-set so $S(f, x_0, g)$ is an R_δ-set.

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