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[Seba Amel et Merabet Nour el Imane]

Sur le thème

Notions sur la quasi-convexité et inégalités Intégrales classiques et fractionnaires

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Mr Souid Mohammed Said	Pr	Université de Tiaret	Président
Mr Benali Halim	Pr	Université de Tiaret	Encadreur
Mr Sofrani Mohammed	Pr	Université de Tiaret	Examinateur

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Introduction

The convexity of functions is an important tool in different disciplines of applied mathematics. In a number of cases, non-convex functions are used in the modeling of real-world problems. As a result, it's critical to determine whether these functions, despite not being convex, retain certain properties common to convex functions. This led to the invention of several generalizations of the classic concept of convex functions, which can be used to a variety of fields including economics, probability theory, and other scientific fields.

In recent years, important generalizations have been made in the context of convexity: pseudo-convex, invex and preinvex, strongly convex, approximately convex, MT-convex, (α;m)-convex, and strongly (s;m)-convex.

A vast class of quasi-convex functions is introduced.

This idea is stated to De Finetti [4](1949), while its use dates back to 1928 as a technical hypothesis in John Von Neumann's minimaximization theory.

Hadamard's inequality is connected tenaciously with convexity and versions, this inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalisations have been found, for example, we mention the works of Dragomir ([5]1992, [6]1995), 'On Hadamard's inequalities for convex functions', 'Some inequalities of Hadamard type', Alomari et al [1](2010) dealt with Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means.

In recent years, some other kinds of Hermite-Hadamard type inequalities were generated, we refer the researches of Bai et al ([2]2012, [3]2013), Wang et al [14](2013), Xi et al [13](2013).

Many researchers were interested on the study of Hermite-Hadamard's inequality for quasi-convex functions for example: Pearce et al [7](2000), Kirmaci et al [11](2004), we mention also the work of Ion [10](2007) Some estimates on the Hermite-Hadamard inequality through quasi-convex functions.

In our work we shed light on some properties of Hermite-Hadamard's inequality for quasi-convex functions and its application to the fractional case by considering the k-Riemann-Liouville integrals and the Katugampula integrals, some fractional inequalities of Hermite-Hadamard type are obtained.

Chapter 1

Preliminaries

In this chapter we define convex and quasi-convex function, then we recall some of their properties, furthermore we present Hermite-Hadamard's type inequalities via convexity and quasi-convexity.

1.1 Convex function

1.1.1 Definitions and properties

Let I be an interval in $\mathbb{R}$,

Definition 1.1.1. $f : I \to \mathbb{R}$ is said to be convex function on I if only if $\forall (a, b) \in I^2, \forall \lambda \in [0, 1]$, we have:

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b). \tag{1.1}$$

Definition 1.1.2. $f : I \to \mathbb{R}$ is said to be strictly convex function on I if only if $\forall (a, b) \in I^2, \forall \lambda \in]0, 1[$, we have:

$$f(\lambda a + (1 - \lambda)b) < \lambda f(a) + (1 - \lambda)f(b). \tag{1.2}$$

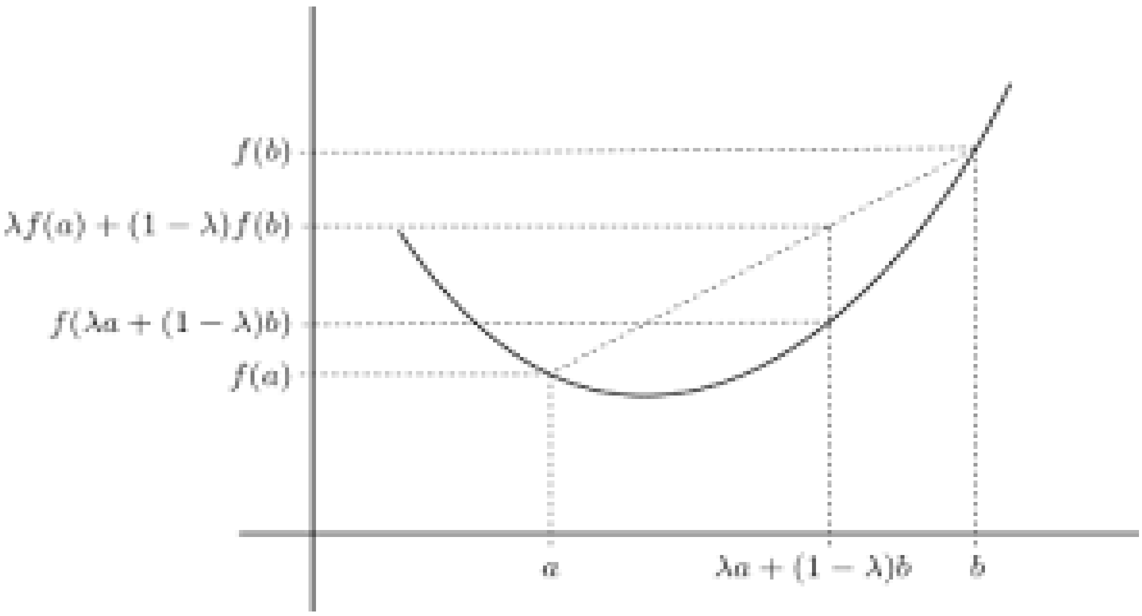

Figure 1.1: Convex Function

Examples

Let f, g be functions on $\mathbb{R}$ defined as follows:

$f(x) = e^x$ and $g(x) = x^a, a > 1$.

f, g are convex functions on $\mathbb{R}$.

Remark 1.1.1. We say that $f : \mathbb{I} \to \mathbb{R}$ is a concave function (strictly concave) if only if $(-f)$ is convex function (strictly convex), thus (1.1) and (1.2) are reversed.

Examples

Let h, k be functions defined as follows:

$h(x) = \ln(x)$ is a concave function on $]0, +\infty[$.

$k(x) = x^a$ with $0 < a < 1$ is a concave function on $\mathbb{R}$.

Properties

[8] Let f, g convex functions and $\alpha \in \mathbb{R}$,

1) αf is convex function for $\alpha \geq 0$ and concave function for $\alpha < 0$.

2) $f + \alpha$ and $f - \alpha$ are convex functions.

3) $f + g$ is convex function.

4) f convex function, g increasing convex function so $g \circ f$ is convex function.

Theorem 1.1.1. *let $f : \mathbb{I} \to \mathbb{R}$ a differentiable function, we say that f is convex if only if f' is increasing.*

Corollary 1 *let $f : \mathbb{I} \to \mathbb{R}$ a differentiable function, we say that f is convex if only if f'' is positive.*

1.2 Quasi-convex function

1.2.1 Definitions

Let I be an interval of $\mathbb{R}$,

Definition 1.2.1. $f : I \to \mathbb{R}$ is said quasi-convex function on I if

$$f(\lambda a + (1-\lambda)b) \leq \max\{f(a), f(b)\}. \tag{1.3}$$

for any $a, b \in I$ and $\lambda \in [0, 1]$.

Definition 1.2.2. $f : I \to \mathbb{R}$ is said strictly quasi-convex function on I if

$$f(\lambda a + (1-\lambda)b) < \max\{f(a), f(b)\}. \tag{1.4}$$

for any $a, b \in I$ and $\lambda \in]0, 1[$.

Remark 1.2.1. Quasi-convexity is a weaker convexity, that is it generalizes the notion of convexity, therefore every convex function is quasi-convex, there are quasi-convex functions which are not convex.

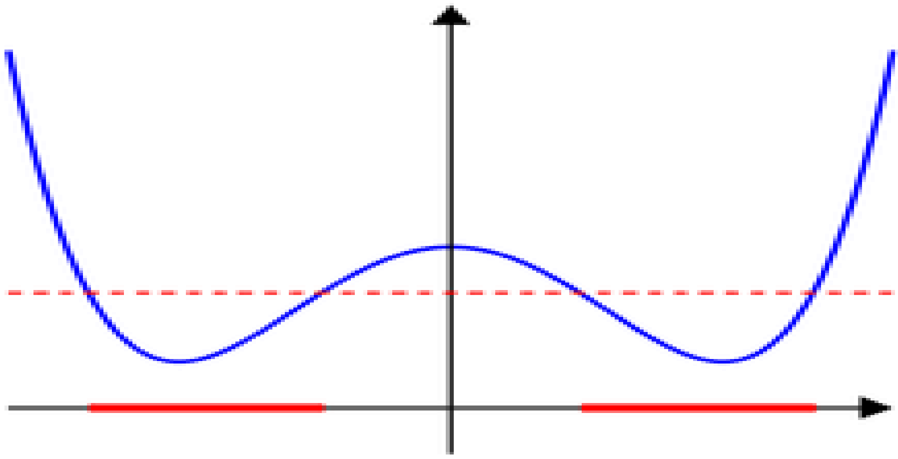

Figure 1.2: Quasi-Convex Function .

Example

Let f be function, $f:[-2,2]\to\mathbb{R}$, defined by:

$$f(x)=\begin{cases}1 & \text{if}, x\in[-2,-1]\\ x^2 & x\in]-1,2]\end{cases}$$

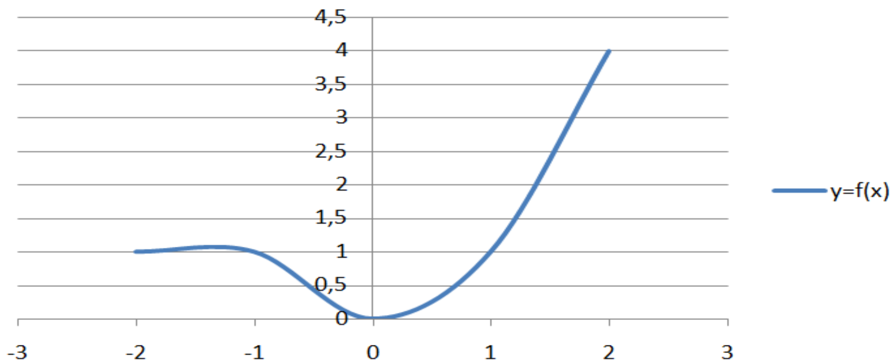

Figure 1.3: quasi-convex functions which are not convex.

Remark 1.2.2. We say that $f : I \to \mathbb{R}$ a quasi-concave function (strictly quasi-concave) if and only if $(-f)$ is quasi-convex function(strictly quasi-convex), thus (1.3) and (1.4) are reversed.

1.2.2 Definition of class P(I)

Definition 1.2.3. [7] Let I an interval in $\mathbb{R}$, we say that a function $f : I \to \mathbb{R}$ is of P type, or that f belongs to the class $P(I)$, if f is nonnegative and for all $a, b \in I$ and $\lambda \in [0, 1]$ we have:

$$f(\lambda a + (1 - \lambda)b) \leq f(a) + f(b). \tag{1.5}$$

Definition 1.2.4. [7] Let I an interval in $\mathbb{R}$, we say that a function $f : I \to \mathbb{R}$ is Jensen-convex function or shortly (J-convex), that is function satisfying the condition:

$$\forall a, b \in I, f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2}. \tag{1.6}$$

Definition 1.2.5. [7] Let I an interval of $\mathbb{R}$. We say that a function $f : I \to \mathbb{R}$ is Jensen-quasi-convex function, that is, function satisfying the condition:

$$\forall a, b \in I, f\left(\frac{a+b}{2}\right) \leq \max\{f(a), f(b)\}. \tag{1.7}$$

1.3 Some classical inequalities

We present the diagram of classical Hermite-Hadamard's inequalities

1.3.1 Hermite-Hadamard's inequality

Let $f : [a, b] \to \mathbb{R}$ be a convex function defined on the interval of real numbers, the following inequality holds

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}. \tag{1.8}$$

This inequality is known as the Hermite-Hadamard's inequality for convex function [1].

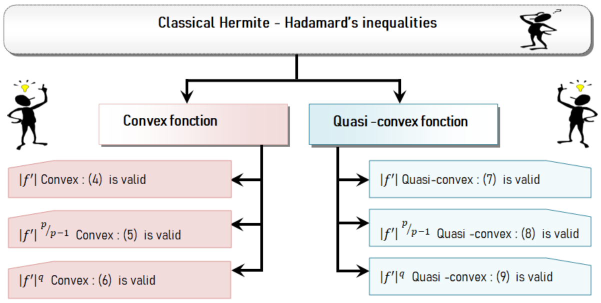

Figure 1.4: Hermite-Hadamard Diagram.

1.3.2 Hermite-Hadamard's inequality via convexity

[5] The following lemma allows us to prove Theorems (1.3.1), (1.3.2) and (1.3.3).

Lemma 1.3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I° where $a, b \in I$ with $a < b$. If $f' \in L_1[a, b]$, the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{1.9}$$

Proof

We set

$$J = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt.$$

By applying integration by parts, we get:

$$\begin{aligned} J &= \frac{b-a}{2}\left[(1-2t)\frac{f(ta+(1-t)b)}{a-b}|_0^1 + 2\int_0^1 \frac{f(ta+(1-t)b)}{a-b}dt\right] \\ &= \frac{b-a}{2}\left[\frac{f(a)+f(b)}{b-a} + 2\int_0^1 \frac{f(ta+(1-t)b)}{a-b}dt\right] \\ &= \frac{f(a)+f(b)}{2} + (b-a)\int_0^1 \frac{f(ta+(1-t)b)}{a-b}dt \end{aligned}$$

Making use of change of the variable $x = ta + (1-t)b$, we get:

$$\begin{aligned} J &= \frac{f(a)+f(b)}{2} + (b-a)\int_b^a \frac{f(x)}{(a-b)^2}dx \\ &= \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \end{aligned}$$

So we get

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt$$

The following inequalities of the Hermite-Hadamard type were established for the above convex function.

Theorem 1.3.1. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on $[a,b]$. If $|f'|$ is convex on $[a,b]$. Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{1.10}$$

Proof

By using (1.3.1), we have:

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &= \left|\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt\right|\\
&\leq \frac{b-a}{2}\int_0^1|1-2t||f'(ta+(1-t)b)|dt\\
&\leq \frac{b-a}{2}\int_0^1|1-2t|\left[t|f'(a)|+(1-t)|f'(b)|\right]dt\\
&\leq \frac{b-a}{2}\left[\int_0^1|1-2t|t|f'(a)|dt+\int_0^1|1-2t|(1-t)|f'(b)|dt\right]\\
&\leq \frac{b-a}{2}\left[\int_0^{\frac{1}{2}}(1-2t)t|f'(a)|dt+\int_{\frac{1}{2}}^1(2t-1)t|f'(a)|dt\right]\\
&+ \frac{b-a}{2}[\int_0^{\frac{1}{2}}(1-2t)(1-t)|f'(b)|dt\\
&+ \int_{\frac{1}{2}}^1(2t-1)(1-t)|f'(b)|dt]\\
&\leq \frac{b-a}{2}\left[\frac{1}{4}|f'(a)|+\frac{1}{4}|f'(b)|\right]\\
&\leq \frac{b-a}{8}\left[|f'(a)|+|f'(b)|\right].
\end{aligned}$$

Theorem 1.3.2. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on $[a,b]$. If $|f'|^q$ is convex on $[a,b]$ with $q\geq 1$. Then:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{1.11}$$

Proof

By using (1.3.1), we have:

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &= \left|\frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt\right|\\ &\leq \frac{b-a}{2}\int_0^1 |1-2t||f'(ta+(1-t)b)|dt.\end{aligned}$$

And by the power-mean inequality:

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|dt\right)^{\frac{q-1}{q}}\left(\int_0^1 |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\\ &\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|dt\right)^{\frac{q-1}{q}}\left[\int_0^1 t|f'(a)|^q dt+\int_0^1 (1-t)||f(b)|^q dt\right]^{\frac{1}{q}}\\ &\leq \frac{b-a}{4}\left[|f'(a)|^q\int_0^1 tdt+|f(b)|^q\int_0^1 (1-t)dt\right]^{\frac{1}{q}}\\ &\leq \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f(b)|^q}{2}\right]^{\frac{1}{q}}.\end{aligned}$$

which completes the proof.

Theorem 1.3.3. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on $[a,b]$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$ with $p>1$. Then:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}. \tag{1.12}$$

Proof

By using (1.3.1), we have:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| = \left|\frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt\right|.$$

And by the power-mean inequality:

$$\frac{b-a}{2}\int_0^1 |1-2t||f'(ta+(1-t)b)|dt \leq \frac{b-a}{2}\left(\int_0^1 |1-2t|dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}},$$

For $\frac{1}{q}=1-\frac{1}{p}$ and similarly to (1.3.2), we get:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}.$$

1.3.3 Hermite-Hadamard's inequality via quasi-convexity

[1] The proof of the theorem below is based on Theorem (1.3.1).

Theorem 1.3.4. *Let $f : [a, b] \to \mathbb{R}$ be differentiable function on $[a, b]$, if $|f'|$ is quasi-convex on $[a, b]$. The inequality is valid:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}\max\{|f'(a)|, |f'(b)|\}. \tag{1.13}$$

Proof 1 *Since*

$$\frac{f(a)+f(b)}{2} \leq max\{f(a), f(b)\}.$$

Because f is convex, we have:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2} \leq max\{f(a), f(b)\}.$$

We have:

$$\frac{|f'(a)|+|f'(b)|}{2} \leq max\{|f'(a)|, |f'(b)|\},$$

Then

$$\frac{b-a}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}\right) \leq \frac{b-a}{4} max\{|f'(a)|, |f'(b)|\}.$$

From (1.10) we get:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}\max\{|f'(a)|, |f'(b)|\}.$$

Theorem 1.3.5. *Let $f : I^0 \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on $I^0, a, b \in I^0$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a, b]$, with $q \geq 1$. Then:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}\left(\max\{|f'(a)|^q, |f'(b)|^q\}\right)^{\frac{1}{q}}. \tag{1.14}$$

Proof

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &= \left|\frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt\right| \\ &\leq \frac{b-a}{2}\int_0^1 |1-2t||f'(ta+(1-t)b)|dt.\end{aligned}$$

That is:

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \left(\frac{b-a}{2}\int_0^1|1-2t|dt\right)^{\frac{q}{q-1}}\left(\int_0^1|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\\
&\leq \frac{b-a}{4}\left[\int_0^1 t|f'(a)|^q dt+\int_0^1(1-t)|f'(b)|^q dt\right]^{\frac{1}{q}}\\
&\leq \frac{b-a}{4}\left[\int_0^1\frac{1}{2}|f'(a)|^q dt+\int_0^1\frac{1}{2}|f'(b)|^q dt\right]^{\frac{1}{q}}\\
&\leq \frac{b-a}{4}\left[\frac{1}{2}\max\{|f'(a)|^q,|f'(b)|^q\}+\frac{1}{2}\max\{|f'(a)|^q,|f'(b)|^q\}\right]^{\frac{1}{q}}\\
&\leq \frac{b-a}{4}\left[\max\{|f'(a)|^q,|f'(b)|^q\}\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 1.3.6. *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on $[a,b]$. If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on $[a,b]$ with $p>1$. Then:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\max\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\}\right)^{\frac{p-1}{p}}. \tag{1.15}$$

Proof

The proof is similarly to the proof of 1.3.5.

Lemma 1.3.2. *Let $f:I\subset R\to R$ be differentiable function on I° where $a,b\in I$ with $a<b$. If $f'\in L_1[a,b]$, the following equality holds:*

$$\begin{aligned}
\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt &= \frac{b-a}{4}[\int_0^1(-t)f'(\frac{1+t}{2}a+\frac{1-t}{2}b)dt\\
&\quad+\int_0^1 tf'(\frac{1+t}{2}b+\frac{1-t}{2}a)dt].
\end{aligned} \tag{1.16}$$

Proof

It suffices to note that

$$\begin{aligned}
I_1 &= \int_0^1(-t)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&= -\frac{2}{a-b}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)t|_0^1+\frac{2}{a-b}\int_0^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&= -\frac{2}{a-b}f(a)+\frac{2}{a-b}\int_0^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt
\end{aligned}$$

Setting $x = \frac{1+t}{2}a + \frac{1-t}{2}b$ and $dx = \frac{a-b}{2}dt$, which gives

$$I_1 = \frac{2}{b-a}f(a) - \frac{4}{(a-b)^2}\int_a^{\frac{a+b}{2}} f(x)dx.$$

Similarly, we can show that

$$\begin{aligned} I_2 &= \int_0^1 tf'(\frac{1+t}{2}b + \frac{1-t}{2}a)dt \\ &= \frac{2}{b-a}f(b) - \frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^{b} f(x)dx \end{aligned}$$

Thus

$$\frac{b-a}{4}[I_1 + I_2] = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx.$$

Theorem 1.3.7. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be differentiable function on I° such that $f' \in L_1[a,b]$, where $a, b \in I$ with $a < b$.if $|f'|$ is quasi-convex function on $[a,b]$. The subsequent inequality is valid*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{b-a}{8}$$
$$[\max\{|f'(\frac{a+b}{2})|, |f'(a)|\} + \max\{|f'(\frac{a+b}{2})|, |f'(b)|\}]. \quad (1.17)$$

Proof

From (1.3.2), we have

$$\begin{aligned} |\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| &= \frac{b-a}{4}|\int_0^1 (-t)f'(\frac{1+t}{2}a + \frac{1-t}{2}b)dt \\ &\quad + \int_0^1 tf'(\frac{1+t}{2}b + \frac{1-t}{2}a)dt| \end{aligned}$$

Since $|f'|$ is quasi-convex on $[a,b]$, for any $t \in [0,1]$ we have

$$\begin{aligned} |\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| &\leq \frac{b-a}{4}[\int_0^1 |(-t)||f'(\frac{1+t}{2}a + \frac{1-t}{2}b)|dt \\ &\quad + \int_0^1 |t||f'(\frac{1+t}{2}b + \frac{1-t}{2}a)|dt] \\ &\leq \frac{b-a}{4}[\int_0^1 t\max\{|f'(\frac{a+b}{2})|, |f'(a)|\}dt \\ &\quad + \int_0^1 t\max\{|f'(\frac{a+b}{2})|, |f'(b)|\}dt \\ &= \frac{b-a}{8}[\max\{|f'(\frac{a+b}{2})|, |f'(a)|\} + \max\{|f'(\frac{a+b}{2})|, |f'(b)|\}] \end{aligned}$$

Which completes the proof.

Corollary 2 *1) If $|f'|$ is increasing with $a<b$, we have: $|f'(a)|\leq|f'(b)|$. Then*

$$|f'(a)|\leq|f'(\frac{a+b}{2})|\leq|f'(b)|.$$

From (1.17), we get

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx|\leq\frac{b-a}{8}[|f'(b)|+|f'(\frac{a+b}{2})|].$$

2) If $|f'|$ is decreasing with $a<b$, we have: $|f'(a)|\geq|f'(b)|$ Then

$$|f'(a)|\geq|f'(\frac{a+b}{2})|\geq|f'(b)|.$$

From (1.17) we get

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx|\leq\frac{b-a}{8}[|f'(a)|+|f'(\frac{a+b}{2})|].$$

Theorem 1.3.8. *Let $f:I^0\subset\mathbb{R}\to\mathbb{R}$ be differentiable function on $I^0,a,b\in I^0$ with $a<b$. If $|f'|^q$ is quasi-convex on $[a,b]$, with $q\geq1$. The following inequality is valid*

$$\begin{aligned}|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt| &\leq \frac{b-a}{8}[(\max\{|f'(\frac{a+b}{2})|^q,|f'(b)|^q\})^{\frac{1}{q}}\\ &+(\max\{|f'(a)|^q,|f'(\frac{a+b}{2})|^q\})^{\frac{1}{q}}]. \quad (1.18)\end{aligned}$$

Proof

From (1.3.2) and using known power mean inequality, we have

$$\begin{aligned}|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt| &\leq \frac{b-a}{4}[\int_0^1|(-t)||f'(\frac{1+t}{2}a+\frac{1-t}{2}b)|dt\\ &+\int_0^1|(t)||f'(\frac{1+t}{2}b+\frac{1-t}{2}a)|dt]\\ &\leq \frac{b-a}{4}[(\int_0^1 tdt)^{1-\frac{1}{q}}(\int_0^1 t|f'(\frac{1+t}{2}a+\frac{1-t}{2}b)|^q dt)^{\frac{1}{q}}\\ &+(\int_0^1 tdt)^{1-\frac{1}{q}}(\int_0^1 t|f'(\frac{1+t}{2}b+\frac{1-t}{2}a)|^q dt)^{\frac{1}{q}}]\\ &\leq \frac{b-a}{8}[(\max\{|f'(\frac{a+b}{2})|^q,|f'(a)|^q\})^{\frac{1}{q}}\\ &+(\max\{|f'(\frac{a+b}{2})|^q,|f'(b)|^q\})^{\frac{1}{q}}].\end{aligned}$$

Theorem 1.3.9. *Let $f : [a, b] \to \mathbb{R}$ be differentiable function on $[a, b]$. If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on $[a, b]$ with $p > 1$. The following inequality is valid*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}[(\max\{|f'(\frac{a+b}{2})|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}}\})^{\frac{p-1}{p}} + (\max\{|f'(a)|^{\frac{p}{p-1}}, |f'(\frac{a+b}{2})|^{\frac{p}{p-1}}\})^{\frac{p-1}{p}}] \quad (1.19)$$

Proof

From (1.3.2) and using Holder's integral inequality, we have

$$\begin{aligned} |\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt| &\leq \frac{b-a}{4}[\int_0^1 |(-t)||f'(\frac{1+t}{2}a + \frac{1-t}{2}b)|dt \\ &+ \int_0^1 |(t)||f'(\frac{1+t}{2}b + \frac{1-t}{2}a)|dt] \\ &\leq \frac{b-a}{4}[(\int_0^1 (t)^p dt)^{\frac{1}{p}}(\int_0^1 |f'(\frac{1+t}{2}a + \frac{1-t}{2}b)|^q dt)^{\frac{1}{q}} \\ &+ (\int_0^1 (t)^p dt)^{\frac{1}{p}}(\int_0^1 |f'(\frac{1+t}{2}b + \frac{1-t}{2}a)|^q dt)^{\frac{1}{q}}] \\ &\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}[(\max\{|f'(\frac{a+b}{2})|^q, |f'(a)|^q\})^{\frac{1}{q}} \\ &+ (\max\{|f'(\frac{a+b}{2})|^q, |f'(b)|^q\})^{\frac{1}{q}}] \end{aligned}$$

Corollary 3 *1) If $|f'|^{\frac{p}{p-1}}$ is increasing, then we have:*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}[|f'(b)| + |f'(\frac{a+b}{2})|].$$

2) If $|f'|^{\frac{p}{p-1}}$ is decreasing, then we have:

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}[|f'(a)| + |f'(\frac{a+b}{2})|].$$

Chapter 2

k-fractional inequalities

This chapter brings together some inequalities associated with Hermite-Hadamard's inequality by way of k-Riemann-Liouville and Riemann-Liouville fractional integrals.

Special-functions

k-Gamma function given as $\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt$ If $k = 1$ then $\Gamma_1(\alpha) = \Gamma(\alpha)$ Beta function .[14]

$$B(\alpha, \beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt \ (\mathcal{R}e(\alpha) > 0, \mathcal{R}e(\beta) > 0) \\ \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \ (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{cases} \tag{2.1}$$

2.1 k-Riemann-Liouville fractional integral

[5] We give the definition of k-Riemann-Liouville fractional integrals,

Definition 2.1.1. Let $f \in L_1[a, b]$, the k-Riemann-Liouville fractional integrals ${}_kJ^{\alpha}_{a+}f(u)$ and ${}_kJ^{\alpha}_{b-}f(u)$ of order $\alpha > 0$ with $a \geqslant 0, k > 0$, are defined by

$${}_kJ^{\alpha}_{a+}f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_a^u (u-t)^{\frac{\alpha}{k}-1} f(t) dt, 0 \leq a < u < b.$$

and

$${}_kJ^{\alpha}_{b-}f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_u^b (t-u)^{\frac{\alpha}{k}-1} f(t) dt, 0 \leq a < u < b.$$

respectively , where $\Gamma_k(\alpha)$ is the k-gamma function .

Remark 2.1.1. If $k = 1$, we get the Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f(u)$ and $J^{\alpha}_{b-}f(u)$ of order $\alpha > 0$ with $a \geqslant 0$, are defined by

$$J^{\alpha}_{a+}f(u) = \frac{1}{\Gamma(\alpha)} \int_{a}^{u} (u-t)^{\alpha-1} f(t)dt, 0 \leq a < u < b.$$

and

$$J^{\alpha}_{b-}f(u) = \frac{1}{\Gamma(\alpha)} \int_{u}^{b} (t-u)^{\alpha-1} f(t)dt, 0 \leq a < u < b.$$

If we put $\alpha = 1$ we obtain a inequality classic of Hermite-Hadamard

$$J_{a+}f(u) = \int_{a}^{u} f(t)dt, 0 \leq a < u < b.$$

and

$$J_{b-}f(u) = \int_{u}^{b} f(t)dt, 0 \leq a < u < b.$$

2.2 Fractional integral via convexity

Theorem 2.2.1. *Let $f : [a,b] \to \mathbb{R}$ be a positive convex function. Then for $\alpha, \beta \geqslant k$, the following inequality for the k-Riemann-Liouville fractional integrals holds :*

$$\begin{aligned} [{}_kJ^{\alpha}_{a+}f(u) +_k J^{\alpha}_{b-}f(u)] \leq \; & \frac{(u-a)^{\frac{\alpha}{k}} f(a) + (b-u)^{\frac{\beta}{k} f(b)}}{2k\Gamma_k(\alpha)} \\ & + f(u) \left(\frac{(u-a)^{\frac{\alpha}{k}} + (b-u)^{\frac{\beta}{k}}}{2k\Gamma_k(\beta)} \right). \end{aligned} \tag{2.2}$$

Proof

We observe that for $\alpha > k$ and $u \in [a,b]$ and $t \in [a,u]$:

$$u - t \leq u - a$$

and for $\alpha > k$ we obtain:

$$(u-t)^{\frac{\alpha}{k}-1} \leq (u-a)^{\frac{\alpha}{k}-1}, t \in [a,u]. \tag{2.3}$$

Writing:

$$t = \frac{u-t}{u-a} a + \frac{t-a}{u-a} u.$$

since f is convex, we have

$$f(t) \leq \frac{u-t}{u-a} f(a) + \frac{t-a}{u-a} f(u), t \in [a,u], u \in (a,b). \tag{2.4}$$

From (2.3) and (2.4) , we obtain

$$\begin{aligned}(u-t)^{\frac{\alpha}{k}-1} f(t) &\leq (u-t)^{\frac{\alpha}{k}-1}\left(\frac{u-t}{u-a} f(a) + \frac{t-a}{u-a} f(u)\right) \\ &\leq (u-a)^{\frac{\alpha}{k}-1}\left(\frac{u-t}{u-a} f(a) + \frac{t-a}{u-a} f(u)\right).\end{aligned}$$

By integrating with respect to t over [a,u] , we get :

$$\int_a^u (u-t)^{\frac{\alpha}{k}-1} f(t)dt \leq (u-a)^{\frac{\alpha}{k}-1}\left[\frac{f(a)}{u-a}\int_a^u (u-t)dt + \frac{f(u)}{u-a}\int_a^u (t-a)dt\right],$$

Then

$$\begin{aligned}\int_a^u (u-t)^{\frac{\alpha}{k}-1} f(t)dt &\leq \frac{(u-a)^{\frac{\alpha}{k}-1}}{u-a}\left(f(a)\int_a^u (u-t)dt + f(u)\int_a^u (t-a)dt\right). \\ &= \frac{(u-a)^{\frac{\alpha}{k}-1}}{u-a}\left[\left(-\frac{(u-t)^2}{2}|_a^u\right) + \frac{f(u)}{u-a}\left(\frac{(t-a)^2}{2}|_a^u\right)\right] \\ &= \frac{(u-a)^{\frac{\alpha}{k}}}{2} f(a) + \frac{(u-a)^{\frac{\alpha}{k}}}{2} f(u) \\ &= \frac{(u-a)^{\frac{\alpha}{k}}}{2}[f(a)+f(u)]\end{aligned}$$

Therefore, in view of the definition of the k-Riemann-Liouville fractional integrals, we get

$${}_kJ^{\alpha}_{a+} f(u) \leq \frac{(u-a)^{\frac{\alpha}{k}}}{2k\Gamma_k(\alpha)}(f(a)+f(u)). \tag{2.5}$$

Now, for $u \in [a,b]$ and $\beta > k$, and $t \in [u,b]$

$$t-u < b-u$$

the following inequality can be observed :

$$(t-u)^{\frac{\beta}{k}-1} \leq (b-u)^{\frac{\beta}{k}-1}. \tag{2.6}$$

We have

$$t = \frac{t-u}{b-u} b + \frac{b-t}{b-u} u,$$

By the convexity of f , we also have

$$f(t) \leq \frac{t-u}{b-u} f(b) + \frac{b-t}{b-u} f(u), t \in [u, b]. \tag{2.7}$$

From the inequalities (2.6) and (2.7) , we obtain

$$\int_u^b (t-u)^{\frac{\beta}{k}-1} f(t)dt \leq \frac{(b-u)^{\frac{\beta}{k}-1}}{b-u} \left(f(b) \int_u^b (t-u)dt + f(u) \int_u^b (b-t)dt \right).$$

Therefore, in view of the definition of the k-Riemann-Liouville fractional integrals, we conclude that

$$_kJ^{\beta}_{b-} f(u) \leq \frac{(b-u)^{\frac{\beta}{k}}}{2k\Gamma_k(\beta)} (f(b) + f(u)). \tag{2.8}$$

Adding (2.5) and (2.8), we get the required inequality (2.2).

2.2.1 Particular cases of k-Riemann-Liouville fractional integral via convexity

Corollary 4 *By setting $\alpha, \beta \geqslant 1$ and $k = 1$, the following inequality for the Riemann-Liouville fractional integrals holds :*

$$[J^{\alpha}_{a+} f(u) + J^{\alpha}_{b-} f(u)] \leq \frac{(u-a)^{\alpha} f(a) + (b-u)^{\beta} f(b)}{2\Gamma(\alpha)} + f(u) \left(\frac{(u-a)^{\alpha} + (b-u)^{\beta}}{2\Gamma(\beta)} \right), \tag{2.9}$$

Corollary 5 *By setting $\alpha = \beta$ in (2.2), this inequality reduces to the fractional integral inequality*

$$_kJ^{\alpha}_{a+} f(u) + {}_kJ^{\alpha}_{b-} f(u) \leq \frac{1}{2k\Gamma_k(\alpha)} \left((u-a)^{\frac{\alpha}{k}} f(a) + (b-u)^{\frac{\alpha}{k}} f(b) + f(u) \left((u-a)^{\frac{\alpha}{k}} + (b-u)^{\frac{\alpha}{k}} \right) \right).$$

Corollary 6 *By setting $\alpha = \beta = k = 1$ and taking $u = b$ or $u = a$ in (2.2), we get the inequality*

$$\frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}.$$

Corollary 7 *By setting $\alpha = \beta = 1$ and taking $u = (a+b)/2$ in (2.2), we have the inequalities*

$$0 \leq \frac{1}{b-a} \int_a^b f(t) - f\left(\frac{a+b}{2}\right) dt \leq \frac{f(a) + f(b)}{2}.$$

Remark 2.2.1. It is interesting to see that if in Theorem 2.2.1, the function f is concave and $0 < \alpha,\ \beta$ and $\alpha,\ \beta \geqslant k$, then the reverse of inequality (2.2) holds.

Theorem 2.2.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If $f^{'}$ is convex, then for $a < b$, and $\alpha, \beta > 0$, the following inequality for the k-Riemann-Liouville fractional integrals holds :*

$$\left|\Gamma_k(\alpha+k)_kJ^{\alpha}_{a+}f(u)+\Gamma_k(\beta+k)_kJ^{\beta}_{b-}f(u)-\left((u-a)^{\frac{\alpha}{k}}f(a)+(b-u)^{\frac{\alpha}{k}}f(b)\right)\right|$$
$$\leq \frac{1}{2}\left((u-a)^{\frac{\alpha}{k}+1}|f^{'}(a)|+(b-u)^{\frac{\alpha}{k}+1}|f^{'}(b)|+|f^{'}(u)|\left((u-a)^{\frac{\alpha}{k}+1}+(b-u)^{\frac{\alpha}{k}+1}\right)\right). \tag{2.10}$$

Proof

By the convexity of $|f^{'}|$, we have:

$$|f^{'}(t)| \leq \frac{u-t}{u-a}|f^{'}(a)|+\frac{t-a}{u-a}|f^{'}(u)|, t\in[a,u], u\in(a,b),$$

Then, it follows that

$$-\left(\frac{u-t}{u-a}|f^{'}(a)|+\frac{t-a}{u-a}|f^{'}(u)|\right)\leq f^{'}(t)\leq \frac{u-t}{u-a}|f^{'}(a)|+\frac{t-a}{u-a}|f^{'}(u)|. \tag{2.11}$$

We firstly consider the right hand side of (2.11) :

$$f^{'}(t)\leq \frac{u-t}{u-a}|f^{'}(a)|+\frac{t-a}{u-a}|f^{'}(u)|, \tag{2.12}$$

Now, using the inequality

$$(u-t)^{\frac{\alpha}{k}} \leq (u-a)^{\frac{\alpha}{k}}, t\in[a,u], \alpha, k>0, \tag{2.13}$$

By multiplying (2.12) and (2.13) side to side and integrating over [a,u] , we get

$$\begin{aligned}\int_a^u (u-t)^{\frac{\alpha}{k}}f^{'}(t)dt &\leq (u-a)^{\frac{\alpha}{k}+1}\left(|f^{'}(a)|\int_a^u(u-t)dt+|f^{'}(u)|\int_a^u(t-a)dt\right)\\ &= (u-a)^{\frac{\alpha}{k}+1}\left(\frac{|f^{'}(a)|+|f^{'}(u)|}{2}\right),\end{aligned}$$

By integrating by part, we have

$$\begin{aligned}\int_a^u (u-t)^{\frac{\alpha}{k}}f^{'}(t)dt &= f(t)(u-t)^{\frac{\alpha}{k}}|_a^u+\frac{\alpha}{k}\int_a^u(u-t)^{\frac{\alpha}{k}-1}f(t)dt,\\ &= -f(a)(u-a)^{\frac{\alpha}{k}}+\Gamma_k(\alpha+k)_kJ^{\alpha}_{a+}f(u)\end{aligned}$$

by the definition of the k-Riemann-Liouville fractional integral, and from (2.14), we have:

$$\Gamma_k(\alpha+k)_k J^{\alpha}_{a+} f(u) - f(a)(u-a)^{\frac{\alpha}{k}} \leq (u-a)^{\frac{\alpha}{k}+1}\left(\frac{|f^{'}(a)|+|f^{'}(u)|}{2}\right), \tag{2.14}$$

Now, considering the left hand side of (2.11) and proceeding as we did for (2.12), we get

$$f(a)(u-a)^{\frac{\alpha}{k}} - \Gamma_k(\alpha+k)_k J^{\alpha}_{a+} f(u) \leq (u-a)^{\frac{\alpha}{k}+1}\left(\frac{|f^{'}(a)|+|f^{'}(u)|}{2}\right). \tag{2.15}$$

From (2.14) and (2.15), we conclude that

$$\left|\Gamma_k(\alpha+k)_k J^{\alpha}_{a+} f(u) - f(a)(u-a)^{\frac{\alpha}{k}}\right| \leq (u-a)^{\frac{\alpha}{k}+1}\left(\frac{|f^{'}(a)|+|f^{'}(u)|}{2}\right). \tag{2.16}$$

On the other hand, using the convexity of $|f^{'}|$, for $t \in [u,b]$ we have

$$|f^{'}(t)| \leq \frac{t-u}{b-u}|f^{'}(b)| + \frac{b-t}{b-u}|f^{'}(u)|, \tag{2.17}$$

for $t \in [u,b]$ and $\beta, k > 0$, one has

$$(t-u)^{\frac{\beta}{k}} \leq (b-u)^{\frac{\beta}{k}}, \tag{2.18}$$

by adapting the same approach as we did for (2.12) and (2.13) from (2.17) and (2.18) we obtain the inequality

$$\left|\Gamma_k(\beta+k)_k J^{\beta}_{a+} f(a) - f(b)(b-u)^{\frac{\beta}{k}}\right| \leq (b-u)^{\frac{\beta}{k}+1}\left(\frac{|f^{'}(a)|+|f^{'}(u)|}{2}\right). \tag{2.19}$$

Combining (2.16) and (2.19) via the triangular inequality, we get the required result .

Particular cases

Corollary 8 *For $a < b$, $\alpha, \beta > 0$ and $k = 1$, the following inequality for the Riemann-Liouville fractional integrals holds:*

$$\begin{aligned} &\left|\Gamma(\alpha)J^{\alpha}_{a+} f(u) + \Gamma(\beta)J^{\beta}_{b-} f(u) - ((u-a)^{\alpha} f(a) + (b-u)^{\alpha} f(b))\right| \\ &\leq \frac{1}{2}\left((u-a)^{\alpha+1}|f^{'}(a)| + (b-u)^{\alpha+1}|f^{'}(b)| + |f^{'}(u)|\left((u-a)^{\alpha+1} + (b-u)^{\alpha+1}\right)\right). \end{aligned} \tag{2.20}$$

Corollary 9 *By setting $\alpha = \beta$ in 2.2.2, this inequality reduces to the fractional integral inequality:*

$$\left|\Gamma_k(\alpha+k)[{}_kJ^{\alpha}_{a+}f(u) +_k J^{\alpha}_{b-}f(u)] - \left((u-a)^{\frac{\alpha}{k}}f(a) + (b-u)^{\frac{\alpha}{k}}f(b)\right)\right|$$

$$\leq \frac{1}{2}\left((u-a)^{\frac{\alpha}{k}+1}|f'(a)| + (b-u)^{\frac{\alpha}{k}+1}|f'(b)| + |f'(u)|\left((u-a)^{\frac{\alpha}{k}+1} + (b-u)^{\frac{\alpha}{k}+1}\right)\right).$$

Corollary 10 *By setting $\alpha = \beta = k = 1$ and $u = (a+b)/2$ in 2.2.2 , we get the inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \leq \frac{b-a}{8}\left(|f'(a)| + |f'(b)| + 2f'(\frac{a+b}{2})\right).$$

We use the following lemma to prove our next theorem

Lemma 2.2.1 *Let $f : [a,b] \to \mathbb{R}$ be a convex function. If f is symmetric with respect to $(a+b)/2$, then*

$$f\left(\frac{a+b}{2}\right) \leq f(u), u \in [a,b]. \tag{2.21}$$

Proof

We have, for all $u \in [a,b]$;

$$\frac{a+b}{2} = 1/2\left\{\frac{u-a}{b-a}a + \frac{b-u}{b-a}b\right\} + 1/2\left\{\frac{u-a}{b-a}b + \frac{b-u}{b-a}a\right\},$$

We have,

$$\frac{au - a^2 + b^2 - bu}{b-a} = \frac{(b-a)(a+b) - u(b-a)}{b-a} = \frac{(b-a)(a+b-u)}{b-a} = a+b-u,$$

and

$$\frac{bu - ba + ab - au}{b-a} = \frac{u(b-a)}{b-a} = u,$$

Then :

$$1/2f(a+b-u) + 1/2f(u) = 1/2f(u) + 1/2f(u) = f(u),$$

Since f is convex and symmetric

$$f\left(\frac{a+b}{2}\right) \leq f(u).$$

Theorem 2.2.3. *Let $f:[a,b]\to\mathbb{R}$ be a positive convex function. If f is symmetric with respect to $(a+b)/2$, then the following inequalities for fractional integrals holds :*

$$\begin{aligned}\frac{1}{2k}\left(\frac{1}{\frac{\alpha}{k}+1}+\frac{1}{\frac{\beta}{k}+1}\right)f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma_k(\beta+k)_kJ_{b-}^{\beta+k}f(a)}{2(b-a)^{\frac{\beta}{k}+1}}+\frac{\Gamma_k(\alpha+k)_kJ_{a+}^{\alpha+k}f(b)}{2(b-a)^{\frac{\alpha}{k}+1}}\\ &\leq \frac{f(a)+f(b)}{2k}.\end{aligned}$$

Proof

For $u\in[a,b]$ and $\beta,k>0$, we have:

$$(u-a)^{\frac{\beta}{k}}\leq(b-a)^{\frac{\beta}{k}}. \tag{2.22}$$

By the convexity of f , we have

$$f(u)\leq\frac{u-a}{b-a}f(b)+\frac{b-u}{b-a}f(a),u\in[a,b], \tag{2.23}$$

From the inequalities (2.22) and (2.23) it follows that:

$$\int_a^b(u-a)^{\frac{\beta}{k}}f(u)du\leq\frac{(b-a)^{\frac{\beta}{k}}}{b-a}\left(f(b)\int_a^b(u-a)du+f(a)\int_a^b(b-u)du\right).$$

Thus, by the definition of the k-fractional integral, we have

$$\frac{\Gamma_k(\beta+k)_kJ_{b-}^{\beta+k}f(a)}{(b-a)^{\frac{\beta}{k}+1}}\leq\frac{f(a)+f(b)}{2k}, \tag{2.24}$$

On the other hand, since

$$(b-u)^{\frac{\beta}{k}}\leq(b-a)^{\frac{\beta}{k}},u\in[a,b],\alpha,k>0,$$

From (2.23), we get

$$\int_a^b(b-u)^{\frac{\beta}{k}}f(u)du\leq(b-a)^{\frac{\beta}{k}+1}\frac{f(a)+f(b)}{2},$$

Thus, by the definition of the k-fractional integral, we have

$$\frac{\Gamma_k(\alpha+k)_kJ_{a+}^{\alpha+k}f(a)}{(b-a)^{\frac{\alpha}{k}+1}}\leq\frac{f(a)+f(b)}{2k}, \tag{2.25}$$

Adding (2.24) and (2.25), we get

$$\frac{\Gamma_k(\beta+k)_k J_{b-}^{\beta+k} f(a)}{2(b-a)^{\frac{\beta}{k}+1}} + \frac{\Gamma_k(\alpha+k)_k J_{b-}^{\alpha+k} f(a)}{2(b-a)^{\frac{\alpha}{k}+1}} \leq \frac{f(a)+f(b)}{2k}.$$

Using Lemma 2.2.1 ,and multiplying (2.21) by $(u-a)^{\frac{\beta}{k}}$, integrating over [a,b], gives

$$f\left(\frac{a+b}{2}\right)\int_a^b (u-a)^{\frac{\beta}{k}} du \leq \int_a^b (u-a)^{\frac{\beta}{k}} f(u) du, \tag{2.26}$$

$$f\left(\frac{a+b}{2}\right)\frac{1}{2k(\frac{\beta}{k}+1)} \leq \frac{\Gamma_k(\beta+k)_k J_{b-}^{\beta+k} f(a)}{2(b-a)^{\frac{\beta}{k}+1}}, \tag{2.27}$$

Using Lemma 2.2.1 and multiplying (2.21) , by $(b-u)^{\alpha}$, integrating over [a,b] gives

$$f\left(\frac{a+b}{2}\right)\frac{1}{2k(\frac{\alpha}{k}+1)} \leq \frac{\Gamma_k(\alpha+k)_k J_{a+}^{\alpha+k} f(b)}{2(b-a)^{\frac{\alpha}{k}+1}}. \tag{2.28}$$

adding (2.27) and (2.28) , and then combining with (2.26) , we obtain the required result.

Particular cases

Corollary 11 *If we put $k=1$, then the following inequalities for the Riemann-Liouville fractional integrals holds:*

$$\begin{aligned}\frac{1}{2}\left(\frac{1}{\alpha+1}+\frac{1}{\beta+1}\right) f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\beta) J_{b-}^{\beta} f(a)}{2(b-a)^{\beta+1}} + \frac{\Gamma(\alpha) J_{a+}^{\alpha} f(b)}{2(b-a)^{\alpha+1}} \\ &\leq \frac{f(a)+f(b)}{2}\end{aligned}$$

Corollary 12 *If we put $\alpha=\beta$ in (2.22) , then this inequality reduces to the fractional integral inequality .*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)\frac{1}{k(\frac{\alpha}{k}+1)} &\leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}+1}}\left({}_k J_{b-}^{\alpha+k} f(a) +_k J_{a+}^{\alpha+k} f(b)\right) \\ &\leq \frac{f(a)+f(b)}{2k}.\end{aligned}$$

2.3 Fractional integral via quasi-convexity

Lemma 2.3.1 *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a,b), if $f^{'} \in L_1[a, b]$, the following equality for k-fractional integrals is valid*

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}}}[_kJ^{\alpha}_{a+}f(b) +_k J^{\alpha}_{b-}f(a)]$$

$$= \frac{b-a}{2}\int_0^1 [(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}]f^{'}(ta + (1-t)b)dt.$$

Proof

We give the proof for $k = 1$:

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)]$$

$$= \frac{b-a}{2}\int_0^1 [(1-t)^{\alpha} - t^{\alpha}]f^{'}(ta + (1-t)b)dt.$$

set

$$I = \int_0^1 [(1-t)^{\alpha} - t^{\alpha}]f^{'}(ta + (1-t)b)dt,$$

We divide the field

$$I = I_1 - I_2$$

where

$$I_1 = \int_0^1 (1-t)^{\alpha} f^{'}(ta + (1-t)b)dt,$$

and

$$I_2 = \int_0^1 t^{\alpha} f^{'}(ta + (1-t)b)dt.$$

We have

$$I_1 = \int_0^1 (1-t)^{\alpha} f^{'}(ta + (1-t)b)dt.$$

We integrate by parts :

$$\begin{aligned} I_1 &= (1-t)^{\alpha}\frac{f(ta + (1-t)b)}{a-b}|_0^1 - \int_0^1 -\alpha(1-t)^{\alpha-1}\frac{f(ta + (1-t)b)}{a-b}dt \\ &= -\frac{f(b)}{a-b} + \alpha\int_0^1 (1-t)^{\alpha-1}\frac{f(ta + (1-t)b)}{a-b}dt, \end{aligned}$$

With a change of variable

$$\begin{cases} v = ta + (1-t)b \Rightarrow v = (a-b)t + b \\ t = 0 \Rightarrow v = b \\ dv = (a-b)dt \\ t = \dfrac{v-b}{a-b} \\ 1-t = \dfrac{v-a}{b-a} \\ (1-t)^{\alpha-1} = \dfrac{(v-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \end{cases}$$

We obtain :

$$\begin{aligned} I_1 &= \frac{f(b)}{b-a} - \frac{\alpha}{b-a}\int_b^a \frac{(v-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{f(v)}{a-b}dv \\ &= \frac{f(b)}{b-a} + \frac{\alpha}{b-a}\int_b^a \frac{(v-a)^{\alpha-1}}{(b-a)^{\alpha}} f(v)dv \\ &= \frac{f(b)}{b-a} - \frac{\alpha}{(b-a)^{\alpha+1}}\int_a^b (v-a)^{\alpha-1} f(v)dv \end{aligned}$$

Then, we have

$$\begin{aligned} I_1 &= \frac{f(b)}{b-a} - \frac{\alpha\Gamma(\alpha)}{(b-a)^{\alpha+1}}\int_a^b \frac{1}{\Gamma(\alpha)}(v-a)^{\alpha-1} f(v)dv \\ &= \frac{f(b)}{b-a} - \frac{\alpha\Gamma(\alpha)}{(b-a)^{\alpha+1}} J_{b-}^{\alpha} f(a) \end{aligned}$$

Now, let

$$I_2 = \int_0^1 t^{\alpha} f'(ta + (1-t)b)dt,$$

We intégrateur I, we obtain :

$$\begin{aligned} I_2 &= t^{\alpha}\frac{f(ta+(1-t)b)}{a-b}\Big|_0^1 - \int_0^1 \alpha t^{\alpha-1}\frac{f(ta+(1-t)b)}{a-b}dt \\ &= \frac{f(a)}{a-b} - \alpha\int_0^1 t^{\alpha-1}\frac{f(ta+(1-t)b)}{a-b}dt. \end{aligned}$$

With a change of variable

$$\begin{cases} v = ta + (1-t)b \Rightarrow v = (a-b)t + b \\ t = 0 \Rightarrow v = b \\ t = 1 \Rightarrow v = a \\ dv = (a-b)dt \\ t = \dfrac{v-b}{a-b} \\ t^{\alpha-1} = \dfrac{(b-v)^{\alpha-1}}{(b-a)^{\alpha-1}} \\ (1-t)^{\alpha-1} = \dfrac{(v-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \end{cases}$$

Thus:

$$\begin{aligned} I_2 &= -\frac{f(a)}{b-a} + \frac{\alpha}{b-a}\int_b^a \frac{(b-v)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{f(v)}{a-b}dv \\ &= -\frac{f(a)}{b-a} - \frac{\alpha}{b-a}\int_b^a \frac{(b-v)^{\alpha-1}}{(b-a)^{\alpha}} f(v)dv \\ &= -\frac{f(a)}{b-a} + \frac{\alpha}{(b-a)^{\alpha+1}}\int_a^b (b-v)^{\alpha-1} f(v)dv. \end{aligned}$$

That is

$$\begin{aligned} I_2 &= -\frac{f(a)}{b-a} + \frac{\alpha\Gamma(\alpha)}{(b-a)^{\alpha+1}}\int_a^b \frac{1}{\Gamma(\alpha)}(b-v)^{\alpha-1} f(v)dv \\ &= -\frac{f(a)}{b-a} + \frac{\alpha\Gamma(\alpha)}{(b-a)^{\alpha+1}} J_{a+}^{\alpha} f(b) \end{aligned}$$

So, we have

$$I_1 - I_2 = \frac{f(b)}{b-a} - \frac{\alpha\Gamma(\alpha)}{(b-a)^{\alpha+1}} J_{b-}^{\alpha} f(a) + \frac{f(a)}{b-a} - \frac{\alpha\Gamma(\alpha)}{(b-a)^{\alpha+1}} J_{a+}^{\alpha} f(b)$$

Then

$$\frac{b-a}{2}(I_1 - I_2)$$

$$= \frac{b-a}{2} \times \left(\frac{f(b)}{b-a} - \frac{\alpha\Gamma(\alpha)}{(b-a)^{\alpha+1}} J_{b-}^{\alpha} f(a) + \frac{f(a)}{b-a} - \frac{\alpha\Gamma(\alpha)}{(b-a)^{\alpha+1}} J_{a+}^{\alpha} f(b) \right)$$

$$= \frac{f(a)+f(b)}{2} - \frac{\alpha\Gamma(\alpha)}{(b-a)^{\alpha}}[J_{b-}^{\alpha} f(a) + J_{a+}^{\alpha} f(b)].$$

2.3.1 Particular cases of k-Riemann-Liouville fractional integral via quasi-convexity

Corollary 13 *If we take $k = 1$, the k-Riemann-Liouville fractinal integrals of order α turn out to be Riemann-Liouville fractional integrals of order α .*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)]$$

$$= \frac{b-a}{2}\int_0^1 [(1-t)^{\alpha} - t^{\alpha}]f'(ta + (1-t)b)dt.$$

Corollary 14 *If we put $\alpha = 1$, we obtain :*

$$\frac{f(a)+f(b)}{2} - \frac{1}{(b-a)}\int_a^b f(t)dt$$

$$= \frac{b-a}{2}\int_0^1 (1-2t)f'(ta + (1-t)b)dt.$$

Lemma 2.3.2 *Let $f : [a, b] \to \mathbb{R}$ be positive function and $f \in L_1[a, b]$. If f is quasi-convex on $[a, b]$, the subsequent inequality for k-fractional integrals is valid*

$$\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a+}f(b) +_k J^{\alpha}_{b-}f(a)] \leq \max\{f(a), f(b)\}.$$

with $\dfrac{\alpha}{k} > 0.$

Proof

Since f is quasi-convex on [a,b], we have

$$f(ta + (1-t)b) \leq max\{f(a), f(b)\},$$

and

$$f((1-t)a + tb) \leq max\{f(a), f(b)\},$$

by adding these inequalities, we get

$$\frac{1}{2}[f(ta + (1-t)b) + f((1-t)a + tb)] \leq max\{f(a), f(b)\},$$

now multiplying both sides by $t^{\frac{\alpha}{k}-1}$ and integrating the resulting inequality with respect to t over [0,1] , we obtain:

$$\int_0^1 t^{\frac{\alpha}{k}-1} f(ta+(1-t)b)dt + \int_0^1 t^{\frac{\alpha}{k}-1} f((1-t)a+tb)dt$$
$$= \int_a^b (\frac{b-u}{b-a})^{\frac{\alpha}{k}-1} f(u)\frac{du}{a-b} + \int_a^b (\frac{v-a}{b-a})^{\frac{\alpha}{k}-1} f(v)\frac{dv}{b-a}$$
$$\leq \frac{2k}{\alpha} max\{f(a), f(b)\}.$$

by using the definition of k-Riemann-Liouville fractional integrals, we get

$$\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a+}f(b) +_k J^{\alpha}_{b-}f(a)] \leq max\{f(a), f(b)\}.$$

hence the proof is complete.

Particular cases

Corollary 15 *If $k = 1$, we have an inequality for Riemann fractional integrals is valid:*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)] \leq \max\{f(a), f(b)\},$$

with $\alpha > 0$.

Corollary 16 *If we put $\alpha = 1$, then*

$$\frac{1}{(b-a)} \int_a^b f(t)dt \leq \max\{f(a), f(b)\}.$$

Theorem 2.3.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a,b), if $|f'|$is quasi-convex on $[a, b], \alpha > 0$, the inequality for k-Riemann fractional integrals is valid:*

$$| \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a+}f(b) +_k J^{\alpha}_{b-}f(a)] |$$
$$\leq \frac{b-a}{(\frac{\alpha}{k}+1)}(1-\frac{1}{2^{\frac{\alpha}{k}}}) \max\{| f'(a) |, | f'(b) |\}.$$

Proof

Using Lemma 2.3.1 , the fact that $|f^{'}|$ is quasi-convex and properties of modulus, we have

$$|\frac{f(a)+f(b)}{2} - \alpha\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a+}f(b) +_k J^{\alpha}_{b-}f(a)] \,|$$
$$\leq \frac{b-a}{2}\int_0^1 |\,(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\,||\,f^{'}(ta+(1-t)b)\,|\,dt.$$
$$\leq \frac{b-a}{2}\int_0^1 |\,(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\,|\,max\{|\,f^{'}(a)\,|,|\,f^{'}(b)\,|\}dt$$
$$= \frac{b-a}{2}\{\int_0^{\frac{1}{2}}[(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}]\int_{\frac{1}{2}}^1[t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}]dt\}max\{|\,f^{'}(a)\,|,|\,f^{'}(b)\,|\}$$
$$= \frac{b-a}{(\frac{\alpha}{k}+1)}(1-\frac{1}{2^{\frac{\alpha}{k}}})max\{|\,f^{'}(a)\,|,|\,f^{'}(b)\,|\}$$

Here we have used

$$\begin{aligned}\int_0^1 |\,(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}\,|\,dt &= \int_0^{\frac{1}{2}}[(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}]dt + \int_{\frac{1}{2}}^1[t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}]dt \\ &= \frac{2}{(\frac{\alpha}{k}+1)}(1-\frac{1}{2^{\frac{\alpha}{k}}}).\end{aligned}$$

which completes the proof.

Particular cases

Corollary 17 *If $k = 1$, the inequality for Riemann-Liouville fractional integrals is valid*

$$|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)]|$$
$$\leq \frac{b-a}{\alpha+1}(1-\frac{1}{2^{\alpha}})\max\{|f'(a)|,|f'(b)|\}.$$

Corollary 18 *If we put $\alpha = 1$, we obtain*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{(b-a)}\int_a^b f(t)dt|$$
$$\leq \frac{b-a}{4}\max\{|f'(a)|,|f'(b)|\}.$$

Theorem 2.3.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L_1[a, b]$. If $| f' |^q$ is quasi-convex on $[a, b]$ and $q > 1$, the sebsequent inequality for k-fractional integrals is valid:*

$$| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b - a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a+}f(b) +_k J^{\alpha}_{b-}f(a)] |$$

$$\leq \frac{b - a}{2(\frac{\alpha}{k}p + 1)^{\frac{1}{p}}}(\max\{| f'(a) |^q, | f'(b) |^q\})^{\frac{1}{q}}.$$

Where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\frac{\alpha}{k} \in [0, 1]$.

Proof

From Lemma 2.3.1 and using Hölder's inequality with properties of modulus, we have

$$| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b - a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a+}f(b) +_k J^{\alpha}_{b-}f(a)] |$$

$$\leq \frac{b - a}{2}\int_0^1 | (1 - t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} || f'(ta + (1 - t)b) | dt$$

$$\leq \frac{b - a}{2}(\int_0^1 | (1 - t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} |^p dt)^{\frac{1}{p}}(\int_0^1 | f'(ta + (1 - t)b) |^q dt)^{\frac{1}{q}}$$

We know that for $\frac{\alpha}{k} \in [0, 1]$ and for all $t_1, t_2 \in [0, 1]$, $| t_1^{\frac{\alpha}{k}} - t_2^{\frac{\alpha}{k}} | \leq | t_1 - t_2 |^{\frac{\alpha}{k}}$ therefore

$$\begin{aligned} \int_0^1 | (1 - t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} |^p dt &\leq \int_0^1 | 1 - 2t |^{\frac{\alpha}{k}p} dt \\ &= \int_0^{\frac{1}{2}} | 1 - 2t |^{\frac{\alpha}{k}p} dt + \int_{\frac{1}{2}}^1 | 1 - 2t |^{\frac{\alpha}{k}p} dt \\ &= \frac{1}{\frac{\alpha}{k}p + 1}. \end{aligned}$$

Since $| f' |^q$ is quasi-convex on$[a, b]$, we have

$$| \frac{f(a) + f(b)}{2} - \alpha\frac{\Gamma_k(\alpha + k)}{2(b - a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a+}f(b) +_k J^{\alpha}_{b-}f(a)] |$$

$$\leq \frac{b - a}{2(\frac{\alpha}{k}p + 1)^{\frac{1}{p}}}(\max\{| f'(a) |^q, | f'(b) |^q\})^{\frac{1}{q}}.$$

Particular cases

Corollary 19 *If $k = 1$, the inequality for Riemann-Liouville fractional integrals is valid*

$$| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)] |$$

$$\leq \frac{b-a}{2(\alpha p+1)^{\frac{1}{p}}}(\max\{| f^{'}(a) |^{q}, | f^{'}(b) |^{q}\})^{\frac{1}{q}}.$$

Where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\alpha \in [0,1]$.

Corollary 20 *If we put $\alpha = 1$, we obtain :*

$$| \frac{f(a)+f(b)}{2} - \frac{1}{(b-a)} \int_{a}^{b} f(t)dt |$$

$$\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}(\max\{| f^{'}(a) |^{q}, | f^{'}(b) |^{q}\})^{\frac{1}{q}}.$$

Where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\alpha \in [0,1]$.

Theorem 2.3.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f^{'} \in L_1[a, b]$. If $| f^{'} |^{q}$ is quasi-convex on $[a, b]$ and $q \geqslant 1$, the following inequality for k-fractional integrals is valid:*

$$| \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a+}f(b) +_k J^{\alpha}_{b-}f(a)] |$$

$$\leq \frac{b-a}{(\frac{\alpha}{k}+1)}(1 - \frac{1}{2^{\frac{\alpha}{k}}})(\max\{| f^{'}(a) |^{q}, | f^{'}(b) |^{q}\})^{\frac{1}{q}}$$

with $\frac{\alpha}{k} > 0$

Proof

From Lemma 2.3.1, using power mean inequality with properties of modulus and using the fact that $| f^{'} |$ is quasi-convex, we have

$$| \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[{}_kJ^{\alpha}_{a+}f(b) +_k J^{\alpha}_{b-}f(a)] |$$

$$\leq \frac{b-a}{2} \int_{0}^{1} | (1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} || f^{'}(ta+(1-t)b) | dt$$

$$\leq \frac{b-a}{2}(\int_0^1 \mid (1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \mid dt)^{1-\frac{1}{q}}(\int_0^1 \mid (1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \mid\mid f^{'}(ta+(1-t)b) \mid^q dt)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2}(\int_0^1 \mid (1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \mid dt)(\max\{\mid f^{'}(a) \mid^q, \mid f^{'}(b) \mid^q\})^{\frac{1}{q}}$$

$$= \frac{b-a}{2}(\int_0^{\frac{1}{2}}[(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}]dt + \int_{\frac{1}{2}}^1 [t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}]dt)(\max\{\mid f^{'}(a) \mid^q, \mid f^{'}(b) \mid^q\})^{\frac{1}{q}}$$

$$= \frac{b-a}{(\frac{\alpha}{k}+1)}(1 - \frac{1}{2^{\frac{\alpha}{k}}})(\max\{\mid f^{'}(a) \mid^q, \mid f^{'}(b) \mid^q\})^{\frac{1}{q}}.$$

Which complete the proof.

Particular cases

Corollary 21 *If $k = 1$, the following inequality for Riemann-Liouville fractional integrals is valid:*

$$\mid \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a+}^\alpha f(b) + [J_{b-}^\alpha f(a)] \mid$$

$$\leq \frac{b-a}{(\alpha+1)}(1 - \frac{1}{2^\alpha})(\max\{\mid f^{'}(a) \mid^q, \mid f^{'}(b) \mid^q\})^{\frac{1}{q}}, \text{ with } \alpha \in [0,1].$$

Corollary 22 *If we put $\alpha = 1$, we put*

$$\mid \frac{f(a)+f(b)}{2} - \frac{1}{(b-a)}\int_a^b f(t)dt \mid \leq \frac{b-a}{4}(\max\{\mid f^{'}(a) \mid^q, \mid f^{'}(b) \mid^q\})^{\frac{1}{q}}.$$

We conclude the following diagram:

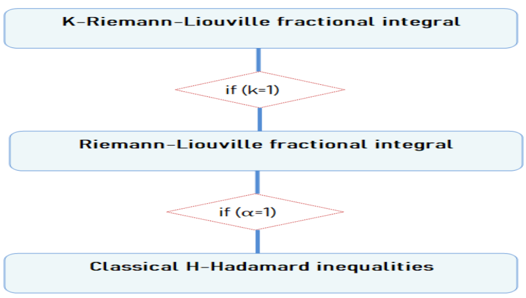

Figure 2.1: K-Riemann-Liouville Diagram.

Chapter 3

Hermite-Hadamard type inequalities for quasi-convex functions via katugampola fractional integrals

The main purpose of this chapter is to establish Hermite-Hadmard's inequalities for quasi-convex functions via Katugampola fractional integral. We also obtain Hermite-Hadmard type inequalities of these classes functions.[11]

3.1 Katugampola fractional integral

Definition 3.1.1. [12] Let $1 \leq p < \infty$. The set $X_c^p(a,b)$ is the set of the functions $f : [a,b] \to \mathbb{R}$, such that

1. f mesurable.
2. $\int_a^b |x^c f(x)|^p \frac{dx}{x} < \infty$.

Theorem 3.1.1. *The space $X_c^p(a,b)$ is a Banach space with the norm*

$$||f||_{X_c^p} = \left(\int_a^b |x^c f(x)|^p \frac{dx}{x} < \infty \right)^{1/p}.$$

Remark 3.1.1. If $c = 1/p$, we obtain the classical L_p space.

Katugampola gave a fractional integral that generalizes the Riemann-Liouville and the Hadamard fractional integral into a single form.

Definition 3.1.2. Let $[a,b] \subset \mathbb{R}$ be a finite interval $f \in X_c^p(a,b)$. Then, the left-and right-side Katugampola fractional integrals of order ($\alpha > 0$) are defined:

$$^\rho I_{a+}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x t^{\rho-1}(x^\rho - t^\rho)^{\alpha-1} f(t)dt. \tag{3.1}$$

and

$$^\rho I_{b-}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b t^{\rho-1}(t^\rho - x^\rho)^{\alpha-1} f(t)dt. \tag{3.2}$$

with $a \leq x \leq b$ and $\rho > 0$, if the integral exist.

Proposition 3.1.2. *Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$,*

$$\lim_{\rho \to 1} \left(^\rho I_{a+}^{\alpha} f(x) \right) = J_{a+}^{\alpha} f(x).$$

We have :

$$^\rho I_{a+}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt$$

When $\rho \to 1$, we obtain

$$\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \lim_{\rho \to 1} \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt \right)$$

$$\begin{aligned} &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \\ &= J_{a+}^{\alpha} f(x) \qquad 0 \leq a < x < b \end{aligned}$$

Similar results also hold for right-sided operators.

Lemma 3.1.1. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ) with $0 \leq a < b$. Thene the following inequality holds if the fractional integral exist:*

$$\begin{aligned} &\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha \rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left[^\rho I_{a+}^{\alpha}(fog)(b) +^\rho I_{b-}^{\alpha}(fog)(a) \right] \\ &= \frac{b^\rho - a^\rho}{2} \int_0^1 \left[(1-t^\rho)^\alpha - t^{\rho\alpha} \right] t^{\rho-1} f'(t^\rho a^\rho + (1-t^\rho) b^\rho) dt. \end{aligned} \tag{3.3}$$

3.2 Katugampula fractional integral via convexity

Theorem 3.2.1. *Let $\alpha_1, \alpha_2 \geq 1$, $\rho_1, \rho_2 \geq 0$, Let $f : [a, b] \to \mathbb{R}$ be a non-negative differentiable function. If $|f'|$ is convex function, then*

$$(\rho_1\Gamma(\alpha_1+1)^{\rho_1} I^{a^+}_{\alpha_1} - \rho_2\Gamma(\alpha_2+1)^{\rho_2} I^{b^-}_{\alpha_2}) + \rho_2^{\alpha_2}(b^{\rho_2} - x^{\rho_2})^{\alpha_2} f(b) - \rho_1^{\alpha}(b^{\rho_1} - x^{\rho_1})^{\alpha_1} f(a) \leq$$
$$|f'(x)|\frac{\rho_2^{\alpha}(b-x)(b^{\rho_2} - x^{\rho_2})^{\alpha_2} + \rho_1^{\alpha_1-1}(x-a)(x^{\rho_1} - a^{\rho_1})^{\alpha_1}}{2} +$$
$$|f'(b)|\frac{\rho_2^{\alpha}(b-x)(b^{\rho_2} - x^{\rho_2})^{\alpha_2}}{2} + |f'(a)|\frac{\rho_1^{\alpha_1}(x-a)(x^{\rho_1} - a^{\rho_1})^{\alpha_1}}{2}.$$

Proof

We consider the function f on the interval $[a, x]$, $x \in (a, b)$, then for $t \in [a, x]$ and $\alpha_1 \geq 0$, $\rho_1 \geq 0$ the following inequality holds

$$(x^{\rho_1} - t^{\rho_1})^{\alpha_1} \leq (x^{\rho_1} - a^{\rho_1})^{\alpha_1}. \tag{3.4}$$

Since f' is convex therefore for $t \in [a, x]$, we have

$$-\left(\frac{x-t}{x-a}|f'(a)| + \frac{t-a}{x-a}|f'(x)|\right) \leq f'(t) \leq \frac{x-t}{x-a}|f'(a)| + \frac{t-a}{x-a}|f'(x)|. \tag{3.5}$$

Multiplying inequalities (3.4) and the right hand side (3.5) and integrating the resulting inequality over $[a, x]$ with respect to t, we obtain:

$$\begin{aligned} \Gamma(\alpha_1-1)^{\rho_1} I^{\alpha_1}_{a+} f(x) - \rho_1^{\alpha_1-1}(x^{\rho_1} - a^{\rho_1})^{\alpha_1-1} f(a) &\leq \\ \rho_1^{\alpha_1-1}(x-a)((x^{\rho_1} - a^{\rho_1})^{\alpha_1})\frac{f'(x)+f'(a)}{2}&, \end{aligned}$$

on the other hand $\alpha_2 \leq 0$, $\rho_2 \leq 0$, $t \in [x, b]$, we have

$$(t^{\rho_2} - x^{\rho_2})^{\alpha_2} \leq (b^{\rho_2} - x^{\rho_2})^{\alpha_2} \tag{3.6}$$

Since f' is convex therefore for $t \in [x, b]$, we have

$$f'(t) \leq \frac{b-t}{b-x} f'(x) + \frac{t-x}{b-x} f'(b). \tag{3.7}$$

From (3.6) and (3.7) and integrating over $[x, b]$, we obtain:
$$\rho_2^{1-\alpha}\int_x^b (t^{\rho_2} - x^{\rho_2}) f'(t)dt \leq \rho_2^{1-\alpha_2}(b-x)(b^{\rho_2} - x^{\rho_2})^{\alpha_2+1}\frac{f'(x)+f'(b)}{2} \ ,$$
By integrating by parts, we obtain:

$$\begin{aligned} \rho_2\Gamma(\alpha_2+1)^{\rho_2} I^{\alpha_2}_{b-} f(x) - \rho_2^{1-\alpha_2}(b^{\rho_2} - x^{\rho_2})^{\alpha_2} f(b) &\leq \\ \rho_2^{\alpha}(x-a)((x^{\rho_2} - a^{\rho_2})^{\alpha_2})\frac{f'(x)+f'(a)}{2}& \end{aligned}$$

By adding inequalities (3.6) and (3.10), we get (3.2.1).

By setting $\alpha_1 = \alpha_2 = \alpha$ and $\rho_1 = \rho_2 = \rho$ in (3.6), we get the following corollary

Corollary 23 *Under the assumptions of theorem (3.2.1), the inequality*

$$(\rho\Gamma(\alpha+1)(I_{\alpha}^{a^+} f(x) - I\!I_{\alpha}^{b^-} f(x)) + \rho_2^{\alpha}(b^{\rho} - x^{\rho})^{\alpha} f(b) - \rho^{\alpha}(x^{\rho} - a^{\rho})^{\alpha} f(a) \leq$$
$$|f'(x)|\frac{\rho^{\alpha}(b-x)(b^{\rho} - x^{\rho})^{\alpha} + \rho^{\alpha-1}(x-a)(x^{\rho} - a^{\rho})^{\alpha}}{2} +$$
$$|f'(b)|\frac{\rho_2^{\alpha}(b-x)(b^{\rho} - x^{\rho^2})^{\alpha}}{2} + |f'(a)|\frac{\rho^{\alpha}(x-a)(x^{\rho} - a^{\rho})^{\alpha}}{2}.$$

Corollary 24 *Letting $\rho \to 1$ in (23), we get*

$$\Gamma(\alpha+1)(J_{a+}^{\alpha} f(x) - J_{b-}^{\alpha} f(x)) - (f(b)(b-x)^{\alpha} + f(a)(x-a)^{\alpha}) \leq$$
$$f'(x)\frac{(b-x)^{\alpha+1} + (x+a)^{\alpha+1}}{2} + f'(b)\frac{(b-x)^{\alpha+1}}{2} + f'(a)\frac{(x+a)^{\alpha+1}}{2}.$$

holds

Corollary 25 *By setting $\alpha = 1$ and $x = \frac{b+a}{2}$ in (24), we get the following inequality*

$$\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)+f(a)}{2} \leq \frac{(b-a)}{8}\left(2f'\left(\frac{a+b}{2} + f'(b) + f'(a)\right)\right). \tag{3.8}$$

We need the following result

Lemma 3.2.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function If f symmetric about $\frac{a+b}{2}$, then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq f(x), \quad x \in [a,b]. \tag{3.9}$$

Theorem 3.2.2. *Let $\alpha_1 > 0, \alpha_2 > 0$, $\rho_1, \rho_2 \geq 0$, let $f : [a,b] \to \mathbb{R}$, be a convex function, If f is symmetric about $\frac{a+b}{2}$, then*

$$f\left(\frac{a+b}{2}\right)\left[a^{\rho_1}\int_a^b \left(\frac{t^{\rho_1} - a^{\rho_1}}{\rho_1}\right)^{\alpha_1} dt + a^{\rho_1}\int_a^b \left(\frac{t^{\rho_1} - a^{\rho_1}}{\rho_1}\right)\alpha_1 dt\right] \leq$$
$$\Gamma(\alpha_1)^{\rho_1} I_{a+}^{\alpha_1+1} f(b) + \Gamma(\alpha_1)^{\rho_2} I_{b-}^{\alpha_1+1} f(a) \leq$$
$$\left[b^{\rho_1}\left(\frac{b^{\rho_1} - a^{\rho_1}}{\rho_1}\right)^{\alpha_1} + b^{\rho_2}\left(\frac{b^{\rho_1} - a^{\rho_2}}{\rho_2}\right)\alpha_2 dt\right](b-a)\frac{f(b)+f(a)}{2}$$

holds

Proof

We have for $t \in [a,b]$

$$\rho_1^{-\alpha_1} t^{\rho_1}(t^{\rho_1}-a^{\rho_1})^{\alpha_1} \le \rho_1^{-\alpha_1} b^{\rho_1}(b^{\rho_1}-a^{\rho_1})^{\alpha_1}. \tag{3.10}$$

Since f is convex therefore for $t \in [a,b]$, we have

$$f(t) \le \frac{t-a}{b-a} f(b) + \frac{b-t}{b-a} f(a). \tag{3.11}$$

Multiplying inequalities (3.10), (3.11), and integrating over $[a,b]$, with respect to t, we get

$$\rho_1^{\alpha_1} \int_a^b (t^{\rho_1}-a^{\rho_1})^{\alpha_1} f(t)dt \le \rho_1^{\alpha_1}(b-a)^{\alpha_1+1}\frac{f(a)+f(b)}{2}. \tag{3.12}$$

We have

$$\Gamma(\alpha_1)^{\rho_1} I_{a+}^{\alpha_1} f(b) \le b^{\rho_1}(b-a)\left(\frac{b^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha_1}\frac{f(a)+fb}{2}. \tag{3.13}$$

On the other hand

$$\rho_1^{-\alpha_2} t^{\rho_2}(t^{\rho_2}-a^{\rho_2})^{\alpha_2} \le \rho_2^{-\alpha_2} b^{\rho_2}(b^{\rho_2}-a^{\rho_2})^{\alpha_2-1}. \tag{3.14}$$

Similarly, we get

$$\Gamma(\alpha_2)^{\rho_2} I_{a+}^{\alpha_2} f(b) \le b^{\rho_2}(b-a)\left(\frac{b^{\rho_2}-a^{\rho_2}}{\rho_2}\right)^{\alpha_2}\frac{f(a)+fb}{2}. \tag{3.15}$$

By adding (3.13) and (3.13), we obtain

$$\Gamma(\alpha_1)^{\rho_1} I_{a+}^{\alpha_1+1} f(b) + \Gamma(\alpha_2)^{\rho_2} I_{b-}^{\alpha_2} f(a) \le$$
$$\left[b^{\rho_1}\left(\frac{b^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha_1} + b^{\rho_2}\left(\frac{b^{\rho_1}-a^{\rho_2}}{\rho_2}\right)\alpha_2 dt\right](b-a)\frac{f(b)+f(a)}{2}$$

Using lemma (3.2.1), we have

$$f\left(\frac{a+b}{2}\right)\left[a^{\rho_1}\int_a^b \left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha_1} dt\right] \le \Gamma(\alpha_1+1)^{\rho_1} I_{a+}^{\alpha+1} f(b) \tag{3.16}$$

Analogously

$$f\left(\frac{a+b}{2}\right)\left[a^{\rho_1}\int_a^b \left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha_1} dt\right] \le \Gamma(\alpha_1+1)^{\rho_2} I_{b-}^{\alpha+1} f(a) \tag{3.17}$$

Combining (3.15), (3.16) and (3.17), we get the required inequality. By setting $\alpha_1 = \alpha_2 = \alpha$ and $\rho_1 = \rho_2 = \rho$

Corollary 26 *Under the assumptions of (3.2.2) the following inequality*

$$f\left(\frac{a+b}{2}\right)a^{\rho}\left[\int_a^b\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}+\left(\frac{b^{\rho}-t^{\rho}}{\rho}\right)^{\alpha}dt\right]\leq\Gamma(\alpha+1)I_{a+}^{\alpha+1}f(b)+I_{b-}^{\alpha+1}f(a)\ \leq b^{\rho}(b-a)\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)(f(b)+f(a)).$$

If we let $\rho\to 1$ in (3.18), we get the corollary.

Corollary 27 *Under the assumptions of (3.2.2) the following inequality*

$$\frac{1}{\alpha+1}f\left(\frac{a+b}{2}\right)\ \leq\ \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}(J_{\alpha+1}^{b-}f(a)+J_{\alpha+1}^{a+}f(b))\left(\frac{f(a)+f(b)}{2}\right).$$

3.3 Katugampula fractional integral via quasi-convexity

Theorem 3.3.1. *Let $\alpha>0$ and $\rho>0$. Let $f:[a^{\rho},b^{\rho}]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in X_c^p(a^{\rho},b^{\rho})$. If f is a quasi-convex function on $[a^{\rho},b^{\rho}]$, then the following inequality holds:*

$$\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^{\rho}-a^{\rho})^{\alpha}}\left[{}^{\rho}I_{a+}^{\alpha}(fog)(b)+{}^{\rho}I_{b-}^{\alpha}(fog)(a)\right]\leq max\{f(a^{\rho}),f(b^{\rho})\}\tag{3.18}$$

where $g(x)=x^{\rho}$.

Proof

Since f is quasi-convex function on $[a^{\rho},b^{\rho}]$, we get

$$f(t^{\rho}a^{\rho}+(1-t^{\rho})b^{\rho})\leq max\{f(a^{\rho}),f(b^{\rho})\}.$$

and

$$f(t^{\rho}b^{\rho}+(1-t^{\rho})a^{\rho})\leq max\{f(a^{\rho}),f(b^{\rho})\}.$$

by adding inequalities we have

$$\frac{1}{2}\left[f(t^{\rho}a^{\rho}+(1-t^{\rho})b^{\rho})+f(t^{\rho}b^{\rho}+(1-t^{\rho})a^{\rho})\right]\leq max\{f(a^{\rho}),f(b^{\rho})\},\tag{3.19}$$

Multiplying both sides of (3.19) by $t^{\alpha\rho-1}$ and integrating the resulting inequality with respect to t over $[a^\rho, b^\rho]$,we obtain

$$\int_0^1 t^{\alpha\rho-1} f(t^\rho a^\rho + (1-t^\rho)b^\rho) + \int_0^1 t^{\alpha\rho-1} f(t^\rho b^\rho + (1-t^\rho)a^\rho)$$

$$\begin{aligned}
&= \int_a^b \left(\frac{b^\rho - x^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} f(x^\rho)\frac{x^{\rho-1}}{b^\rho - a^\rho}dx + \int_a^b \left(\frac{x^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} f(x^\rho)\frac{x^{\rho-1}}{b^\rho - a^\rho}dx \\
&= \frac{1}{(b^\rho - a^\rho)^\alpha}\int_a^b \frac{x^{\rho-1}}{(b^\rho - x^\rho)^{1-\alpha}} f(x^\rho)dx + \frac{1}{(b^\rho - a^\rho)^\alpha}\int_a^b \frac{x^{\rho-1}}{(x^\rho - a^\rho)^{1-\alpha}} f(x^\rho)dx \\
&= \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}(fog)(b) +^\rho I^\alpha_{b-}(fog)(a)\right] \\
&\leq \frac{2}{\rho\alpha} max\{f(a^\rho), f(b^\rho)\}
\end{aligned}$$

So we get the desired result .

Particular cases

Corollary 28 *In theorem 3.3.1, taking limit $\rho \to 1$, we obtain inequality of Riemann-Liouville*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)] \leq \max\{f(a), f(b)\},$$

with $\alpha > 0$.

Corollary 29 *If we put $\alpha = 1$, we obtain :*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \max\{f(a), f(b)\}.$$

Theorem 3.3.2. *Let $\alpha > 0$ and $\rho > 0$. Let$f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on $[a^\rho, b^\rho]$ with $0 \leq a < b$. If $|f'|$ is a quasi-convex function on $[a^\rho, b^\rho]$, then the following inequalities holds:*

$$\begin{aligned}
&\left|\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}(fog)(b) +^\rho I^\alpha_{b-}(fog)(a)\right]\right| \\
&\qquad = \frac{b^\rho - a^\rho}{\rho(\alpha+1)}\left(1 - \frac{1}{2^{\rho(\alpha+1)}}\right) max\{|f'(a^\rho)|, |f'(b^\rho)|\}, \qquad (3.20)
\end{aligned}$$

where $g(x) = x^\rho$.

Proof

Using the Beta-function and quasi-convex of |f'| with modulus, we get:

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}(fog)(b)+^\rho I^\alpha_{b-}(fog)(a)\right]\right|$$
$$\leq \frac{b^\rho-a^\rho}{2}\int_0^1\left[(1-t^\rho)^\alpha-t^{\rho\alpha}\right]t^{\rho-1}f'(t^\rho a^\rho+(1-t^\rho)b^\rho)dt$$
$$\leq \frac{b^\rho-a^\rho}{2}\int_0^1\left[(1-t^\rho)^\alpha-t^{\rho\alpha}\right]t^{\rho-1}max\{|f'(a^\rho)|,|f'(b^\rho)|\}$$
$$=\frac{b^\rho-a^\rho}{2}max\{|f'(a^\rho)|,|f'(b^\rho)|\}$$
$$\times\left\{\int_0^{\frac{1}{2^{1/\rho}}}\left[(1-t^\rho)^\alpha-t^{\rho\alpha}\right]t^{\rho-1}dt+\int_{\frac{1}{2^{\frac{1}{\rho}}}}^1\left[t^{\rho\alpha}+(1-t^\rho)^\alpha\right]t^{\rho-1}dt\right\}$$

where

$$\int_0^{\frac{1}{2^{\frac{1}{\rho}}}}\left[(1-t^\rho)^\alpha-t^{\rho\alpha}\right]t^{\rho-1}dt+\int_{\frac{1}{2^{\frac{1}{\rho}}}}^1\left[t^{\rho\alpha}+(1-t^\rho)^\alpha\right]t^{\rho-1}dt$$
$$=\frac{1}{\rho}\left\{\int_0^{\frac{1}{2}}\left[(1-u)^\alpha-u^\alpha\right]du+\int_{\frac{1}{2}}^1\left[u^\alpha-(1-u)^\alpha\right]du\right\} \tag{3.21}$$
$$=\frac{2}{\rho(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)$$

The proof is completed.

3.3.1 Particular cases of Katugampula fractional integral via quasi-covexity

Corollary 30 *In theorem 3.3.2 , taking limit $\rho\to 1$ we obtain inequality of Riemann*

$$|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)]|$$
$$\leq\frac{b-a}{\alpha+1}(1-\frac{1}{2^\alpha})\max\{f'(a),f'(b)\}.$$

Corollary 31 *If we put $\alpha=1$ in previous corollary , we obtain*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(t)dt|\leq\frac{b-a}{4}\max\{|f'(a)|,|f'(b)|\}.$$

Theorem 3.3.3. *Let $\alpha>0$ and $\rho>0$. Let$f:[a^\rho,b^\rho]\to\mathbb{R}$ be a differentiable function on $[a^\rho,b^\rho]$ with $0\leq a<b$. If $|f'|^q$ is a quasi-convex function on $[a^\rho,b^\rho]$ and*

$s > 1$, then the following inequality holds:

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}(fog)(b) +^\rho I^\alpha_{b-}(fog)(a)\right]\right|, = \frac{b^\rho - a^\rho}{2}\left(max\{|f'(a)|^q, |f'(b)|^q\}\right)^{\frac{1}{q}}(K_1+K_2)^{\frac{1}{s}}$$

where

$$K_1 = \frac{1}{\rho 2^{s+\frac{1-a}{\rho}}} B\left(s+\frac{1-s}{\rho}, as+1\right), K_2 = \frac{as+1}{2\rho}\,{}_2F_1\left(1-s+\frac{s-1}{\rho}, 1; as+2; \frac{1}{2}\right),$$

$$\frac{1}{s}+\frac{1}{q} = 1$$

and $g(x) = x^\rho$.

Proof

Using Beta function, Hölder's inequality and quasi-convex of $|f^{'}|$ with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}(fog)(b) +^\rho I^\alpha_{b-}(fog)(a)\right]\right| \\
&\le \frac{b^\rho-a^\rho}{2}\int_0^1 |(1-t^\rho)^\alpha - t^{\rho\alpha}|^{\rho-1}|f'(t^\rho a^\rho + (1-t^\rho)b^\rho)|dt \\
&\le \frac{b^\rho-a^\rho}{2}\left(\int_0^1 |(1-t^\rho)^\alpha - t^{\rho\alpha}|^s t^{s(\rho-1)}dt\right)^{1/s}\left(\int_0^1 |f'(t^\rho a^\rho + (1-t^\rho)b^\rho)|^q dt\right)^{1/q} \\
&\le \frac{b^\rho-a^\rho}{2}\left(\int_0^1 |(1-2t^\rho)^{\alpha s}|^{\alpha s} t^{s(\rho-1)}dt\right)^{1/s}\left(max\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\}^{1/q}\right) \\
&= \frac{b^\rho-a^\rho}{2} max\{|f'(a)|^q, |f'(b)|^q\}^{1/q} \\
&\times \left\{\int_0^{\frac{1}{2^{1/\rho}}} (1-2t^\rho)^{\alpha s} t^{s(\rho-1)}dt + \int_{\frac{1}{2^{1/\rho}}}^1 (2t^\rho-1)^{\alpha s} t^{s(\rho-1)}dt\right\}^{1/s}
\end{aligned}$$

$$= \frac{b-a}{2}(max\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\})^{1/q}(K_1+K_2)^{1/s} \tag{3.22}$$

Where

$$\begin{aligned}
K_1 = \int_0^{1/2^{1/\rho}} (1-2t^\rho)^{\alpha s} t^{s(\rho-1)}dt &= \frac{1}{\rho 2^{s+\frac{1-s}{\rho}}}\int_0^1 u^{s-1+\frac{1-s}{\rho}}(1-u)^{\alpha s}du \\
&= \frac{1}{\rho 2^{s+\frac{1-s}{\rho}}} B\left(s+\frac{1-s}{\rho}, \alpha s+1\right)
\end{aligned}$$

$$K_2 = \int_{1/2^{1/\rho}}^{1} (2t^\rho - 1)^{\alpha s} t^{s(\rho-1)} dt \;=\; \frac{1}{2^{s+\frac{1-s}{\rho}}\rho} \int_0^1 u^{\alpha s}(1+u)^{s-1+\frac{1-s}{\rho}} du$$
$$= \frac{\alpha s + 1}{2\rho} 2F_1\left(1 - s + \frac{s-1}{\rho}, 1; \alpha s + 2; \frac{1}{2}\right)$$

So, if we use (3.23), (3.23) in(3.22) , we obtain desired result.

Particular cases

Corollary 32 *In theorem 3.3.3, taking limit $\rho \to 1$ we obtain inequality of Riemann-Liouville*

$$|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)]|$$
$$\leq \frac{b-a}{2(\alpha p+1)^{1/p}}\left(1 - \frac{1}{2^\alpha}\right)(\max\{|f'(a)|^q, |f'(b)|^q\})^{1/q}.$$

Corollary 33 *If we put $\alpha = 1$ in previous corollary , we obtain*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt$$
$$\leq \frac{b-a}{4(p+1)^{1/p}}(\max\{|f'(a)|^q, |f'(b)|^q\})^{1/q}.$$

Theorem 3.3.4. *Let $\alpha > 0$ and $\rho > 0$. Let$f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on $[a^\rho, b^\rho]$ with $0 \leq a < b$. If $|f'|^q$ is a quasi-convex function on $[a^\rho, b^\rho]$ and $q \geqslant 1$, then the following inequality holds:*

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}(fog)(b) +^\rho I^\alpha_{b-}(fog)(a)\right]\right|$$
$$\leq \frac{b-a}{\rho(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)(max\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\})^{1/q},$$

Where $g(x) = x^\rho$.

Proof

From Lemma 3.1.1, quasi-convex of $|f'|$ and using power-mean inequality with properties of modulus, we have

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}(fog)(b) +^\rho I^\alpha_{b-}(fog)(a)\right]\right|$$

$$\le \frac{b^\rho - a^\rho}{2}\int_0^1 |(1-t^\rho)^\alpha - t^{\rho\alpha}|t^{\rho-1}|f'(t^\rho a^\rho + (1-t^\rho)b^\rho)|dt$$
$$\le \frac{b^\rho - a^\rho}{2}\left(\int_0^1 |(1-t^\rho)^\alpha - t^{\rho\alpha}|t^{\rho-1}dt\right)^{1-1/q}$$
$$\times\left(\int_0^1 |(1-t^\rho)^\alpha - t^{\rho\alpha}|t^{\rho-1}|f'(t^\rho a^\rho + (1-t^\rho)b^\rho)|^q dt\right)^{1/q}$$
$$\le \frac{b^\rho - a^\rho}{2}\left(\int_0^1 |(1-t^\rho)^\alpha - t^{\rho\alpha}|t^{\rho-1}dt\right)^{1-1/q}$$
$$\times(max\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\})^{1/q}\left(\int_0^1 |(1-t^\rho)^\alpha - t^{\rho\alpha}|t^{\rho-1}dt\right)^{1/q}$$
$$= \frac{b^\rho - a^\rho}{2}\left(\int_0^1 |(1-t^\rho)^\alpha - t^{\rho\alpha}|t^{\rho-1}dt\right)(max\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\})^{1/q}$$

Using (3.21) we get desired result.

particular cases

Corollary 34 *In theorem 3.3.3 , taking limit* $p \to 1$*, we obtain inequality of Riemann-Liouville*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right|$$
$$\le \frac{b-a}{\alpha+1}(\max\{| f^{'}(a) |^q, | f^{'}(b) |^q\})^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\alpha \in [0,1]$.

Corollary 35 *If we put* $\alpha = 1$ *, we put*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{(b-a)}\int_a^b f(t)dt \right|$$

$$\le \frac{b-a}{4}(\max\{| f^{'}(a) |^q, | f^{'}(b) |^q\})^{\frac{1}{q}}.$$

We conclude the following diagram

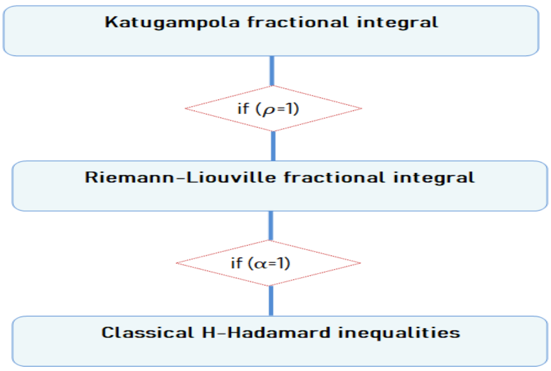

Figure 3.1: Katugampula Diagram.

General Conclusion

To sum up, In this work we have studied some notions on quasi-convexity and integral inequalities classic and fractional, we have dealt with Hermite-Hedamard's classic and fractional.
We have achieved the main objective which is the study of some classical integral inequalities and we have established some results on fractional calculus.
In the first chapter we have presented some basic notions and important tools of convexity and quasi-convexity, which led us to deal with Classical case of Hermite-Hadamard's inequalities via convexity and via quas-convexity . Then we have discussed in the second chapter the k-Riemann-Liouville Fractional Integral and some special cases when $k = 1$, this case is Hermite-Hadamard inequality. The last chapter shed light on Hermite-Hadamard type inequalities for quasi-convex functions via katugampola fractional integrals which generalizes the previous results of k-Riemann-Liouville Fractional Integral and Hermite-Hadamard fractional integral.

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