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In Tiaret, in Front of the Jury Composed of:

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Dedication

I dedicate this project work to my parents.

Abstract

In this dissertation, we present the study of some fuzzy fractional differential equations, in the first one, we prove that fuzzy fractional differential equation is equivalent to the fuzzy integral equation and then using this equivalence existence and uniqueness result is establish. Fuzzy derivative is consider in the Goetschel-Voxman sense and fractional derivative is consider in the Riemann Liouville sense. Also we give some applications of the main result. Secondly the work is concerned with the existence and uniqueness of solutions of fuzzy fractional differential equations using fixed point theory. We provide some results answering when we can expect a solution of the problem. Thirdly we primarily focused on the existence and uniqueness of the initial value problem for fractional order fuzzy ordinary differential equations in a fuzzy metric space.

Key words: fuzzy fractional differential equation, fuzzy integral equation, Fuzzy derivative, fractional derivative, fuzzy metric space.

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Introduction

Fractional order fuzzy differential equations provide us a tool for modelling which appears to be better than that of the ordinary differential equations in the sense that the predictions made using models involving fractional derivatives are more close to nature than the ones done using ordinary differential equations. To work with models involving fuzzy fractional derivatives one must be able to tell whether the model is well posed or not that is whether it has a unique solution or not. Therefore it is necessary to study the restrictions and the properties of the models and to give some sufficient conditions on them so they possess a unique solution.

Fuzzy fractional differential equations (FFDEs) are a fascinating blend of fuzzy logic and fractional calculus, used to model systems with uncertainty and memory effects. These equations extend classical differential equations by incorporating fractional derivatives, which allow for more flexible and accurate representations of real-world phenomena.

In 2010 Agarwal et. al.[1] merged fractional and fuzzy differential equation. Despite being new, this topic is growing very fast and many works related to this are published. Useful surveys and collection of the literature for fuzzy fractional differential equations is given in [4]. While, the literature for fixed point theory and its application is vast, some of which are referred in [11].

The study of theory of the fuzzy differential equations has been growing rapidly. In many cases of the modeling of real world phenomena, fuzzy initial value problems appear naturally, because information about the behavior of a dynamical system is uncertain. In order to obtain a more adequate model, we have to take into account these uncertainties. Significant results from the theory of fuzzy differential equations and their applications can be found in [7].

Letting fuzzy sets involve in our model allows us to harness the ability to handle the vagueness present in the nature. Since its introduction in 1965 by Lotfy Zadeh the literature for fuzzy set theory has only grown and a useful amount of it can be found in [5].

Fractional calculus stems from the beginning of theory of differential and integral calculus [11]. Fractional differential equations find its applications in the problems arising in the fields including but not limited to electrical and mechanical properties of materials, dynamics of turbulence, electro-chemistry, visco-elasticity. Literature for fractional calculus can be found in [6].

Fractional differential equations have played an important role in many fields such as astrophysics, electronics, diffusion, material theory, chemistry, control theory, wave propagation, signal theory, electricity and thermodynamics (see [6],[8]).

The concept of solution of fuzzy fractional differential equations was first introduced in [1]. Generally, in fuzzy case, the fuzzy fractional differential equation

$$D^q y(t) = f(t, y(t)), \qquad \lim_{t\to 0^+} t^{1-q} y(t) = y_0 \tag{1}$$

is not equivalent to fuzzy integral equation

$$y(t) = y_0 t^{q-1} + I^q f(t, y(t)) \tag{2}$$

We established the existence and uniqueness of the solution of fuzzy fractional integral equation (2). Evidently, each solution of integral equation (2) is also a solution of (1). Furthermore, we give the existence and uniqueness of fractional differential equation with fuzzy initial condition. Fuzzy laplace transform method and modified Euler Method have been presented to solve fuzzy fractional differential equations.

In this work, we give the definition of the concept of Goetschel-Voxman derivative. In this definition of the derivative, non-standard fuzzy subtraction is used. The advantage of this derivative is that we can use integration by parts formula and this formula does not work in case of other kind of fuzzy derivatives.

In the context of Hukuhara derivative (H-derivative), the diameter of the solutions is increases as time progresses. This characteristic presents a significant challenge in applying H-differentiability to fuzzy mathematical modeling.

By the work initiated by Agarwal et al.[1] in which they viewed the set of all fuzzy numbers as a semi-linear space and constructed a fixed point theorem for the space in semi-linear sense. It is known that the set of all fuzzy numbers is also a metric space and hence Banach contraction principle can be straight forwardly used, so observing that the Banach contraction principle is not straight-forwardly used, we study the fuzzy fractional initial value problem given below:

$${}_{t_0}^{C}D_t^{\alpha} x(t) = f(t, x(t)), \quad x(t_0) = x_0 \in \mathbb{F}_{\mathbb{R}} \tag{3}$$

Where, $\mathbb{F}_{\mathbb{R}}$ denotes the collection of all fuzzy numbers with universe $\mathbb{R}$, $t \in [t_0, T]$, x is the unknown with codomain $\mathbb{F}_{\mathbb{R}}$, ${}_{t_0}^{C}D_t^{\alpha}$ denotes α order fractional derivative in Caputo sense with $0 < \alpha < 1$, f is also a fuzzy number valued function with the property that:

$$f \in C([t_0, T] \times B(x_0, \eta), \mathbb{F}_{\mathbb{R}}) \tag{4}$$

Here, $B(x_0, \eta)$ is a fuzzy ball with center x_0 and radius η.

Furthermore, we will use the fuzzy numerical function integral to study the existence and uniqueness of the initial value problems of fuzzy differential equations in fuzzy metric space based on the principle of power compression in fuzzy metric space. The initial value problem of the following equation will be studied

$$\begin{cases} {}^{c}D_t^d \overline{u}(t) = \widetilde{f}(t, \widetilde{u}(t)), \quad 0 < \lambda < 1, t > 0, \\ \widetilde{u}(0) = \widetilde{u}_0. \end{cases} \tag{5}$$

This Dissertation consists of an introduction and five chapters. The first chapter contains preliminaries of some basic definitions related to fuzzy numbers and fractional differential equations to be used in the work. In the second chapter, we proved the existence and uniqueness of the solution of fuzzy fractional differential equations, provided the applications of the existence and uniqueness theorem. Fuzzy derivative is consider in the Goetschel- Voxman sense and fractional derivative is consider in the Riemann Liouville sense. The third chapter contained the existence and uniqueness main results of the solution using Banach fixed point theorem and give an example supporting the results we established, in which we prove the existence and uniqueness. Finally the fourth chapter present the application of power compression mapping principle and Choquet integral of fuzzy numerical functions, we established the existence and uniqueness of solutions to initial value problems for fractional ordinary differential equations in fuzzy space. To show the validity of the derived results, an appropriate example and applications are also discussed.

Chapter 1

Preliminaries

1.1 Preliminaries

Let E denote the set of all fuzzy numbers. We recall that $y : \mathbb{R} \to [0,1]$ is a fuzzy number if it satisfies the following properties:
(i) There is a unique $\xi_0 \in \mathbb{R}$ such that $y(\xi_0) = 1$,
(ii) $[y]^0 = \{\xi \in \mathbb{R} | y(\xi) \geq 0\}$ is bounded in $\mathbb{R}$,
(iii) y is strictly fuzzy convex on $[y]^0$, i.e.,

$$y(\lambda\xi_1 + (1-\lambda)\xi_2) > \min\{y(\xi_1), y(\xi_2)\} \text{ for all } \xi_1, \xi_2 \in [y]^0, \xi_1 \neq \xi_2 \text{ for all } \lambda \in (0,1)$$

(iv) y is upper semi-continuous on $\mathbb{R}$.
Let $y \in E$. Then for each $\alpha \in (0,1]$, the set

$$[y]^\alpha = \{\xi \in \mathbb{R};\ y(\xi) \geq \alpha\},$$

is called the α-level set of y.

Theorem 1.1. [2] *Let $y \in E$ and for each $\alpha \in [0,1]$,*

$$y_1(\alpha) = min[y]^\alpha and y_2(\alpha) = max[y]^\alpha,$$

then we have
(i) $y_1, y_2 \in C[0,1] = \{u : [0,1] \to \mathbb{R};$ *u is continuous on* $[0,1]\}$,
(ii) y_1 *is monotone increasing and* y_2 *is monotone decreasing,*
(iii) $y_1(1) = y_2(1)$.
Conversely, if $x(\alpha), z(\alpha) : [0,1] \to \mathbb{R}$ *satisfy the above conditions* $(i)-(iii)$, *denote*

$$y(\xi) = \begin{cases} sup\{\alpha \in [0,1] : x(\alpha) \leq \xi \leq z(\alpha)\}, & \xi \in [x(0), z(0)], \\ 0, & \xi \in [x(0), z(0)]. \end{cases}$$

Then there exists $y \in E$ *such that* $[y]^\alpha = [x(\alpha), z(\alpha)], y_1(\alpha) = x(\alpha), y_2(\alpha) = z(\alpha), \alpha \in [0,1]$.

In [1] parametric representation of fuzzy number was introduced, i.e. $y \in E$ can be written as $y = (y_1(\alpha), y_2(\alpha)), \alpha \in [0,1]$. (for sake of simplicity we write $y = (y_1, y_2)$). Therefore fuzzy number $y \in E$ can be considered as a continuous curve $\{(y_1(\alpha), y_2(\alpha)) : \alpha \in [0,1]\}$ in $\mathbb{R}^2$. For $t \in \mathbb{R}$, the membership function has the following form:

$$\mu_t(\xi) = \begin{cases} 1, & \xi = t, \\ 0, & \xi \neq t. \end{cases}$$

Then we have $t \in \mathbb{R} \subset E, [\mu_t]^\alpha = [t,t]$ for all $\alpha \in [0,1]$, therefore the parametric representation of $t \in \mathbb{R}$ is $t = (t,t), \alpha \in [0,1]$.
For $w = (u,v) \in C[0,1] \times C[0,1]$, define the norm

$$\|w\| = \max_{0 \le \alpha \le 1} \max\{|u(\alpha)|, |v(\alpha)|\}.$$

It is obvious that $C[0,1] \times C[0,1]$ is a Banach space.
For $y = (y_1, y_2), z = (z_1, z_2) \in E, k \in \mathbb{R}$, we have the following operations based on Zadeh's extension principle,
i) $y \oplus z = (y_1 + z_1, y_2 + z_2)$,
ii) $y \ominus z = (y_1 - z_2, y_2 - z_1)$,
iii) $k \otimes y = \begin{cases} (ky_1, ky_2), & k \geq 0, \\ (ky_2, ky_1), & k < 0. \end{cases}$

It is easy to see that E is not a linear space under these operations. In [8] the following operations were introduced.
For all $y, z \in E$, define $y - z = (y_1(\alpha) - z_1(\alpha), y_2(\alpha) - z_2(\alpha)), \alpha \in [0,1]$.

$$E - E := \{w : w = y - z, y, z \in E\}.$$

If the H-difference (Hukuhara difference) of y and z exists, then $y - z$ is the H-difference of y and z.

Remark 1.1. [2] *i)* $E - E$ *is a linear subspace of* $C[0,1] \times C[0,1]$*, and*
a) $y \oplus z = y + z$ *for all* $y, z \in E$.
b) $k \otimes y = k \cdot y$ *for all* $k \in [0, \infty)$ *for all* $y \in E$.
where "$+$", "$\cdot$" *are additive and product operations in linear space* $C[0,1] \times C[0,1]$.
ii) E *is a closed convex cone in Banach space* $C[0,1] \times C[0,1]$.

In this work, "$\oplus$", "$\ominus$", "$\otimes$" represent operations based on Zadeh's extension Principle, "$+$", "$-$", "$\cdot$" stand for operations based on linear space, if they agree, we use the later. We define a metric d on E by

$$d(y,z) = \sup_{0\le\alpha\le1} d_H([y]^\alpha,[z]^\alpha),$$

where d_H is the Hausdorff metric defined as

$$d_H([y]^\alpha,[z]^\alpha) = \max\{|y_1(\alpha)-z_1(\alpha)|,|y_2(\alpha)-z_2(\alpha)|\}.$$

It is well known that (E,d) is a complete metric space. We list some properties of the metric d:

$$d(y+w,z+w) = d(y,z), d(\lambda y,\lambda z) = |\lambda|d(y,z), \tag{1.1}$$
$$d(y,z) \le d(y,w)+d(w,z) \tag{1.2}$$
$$d(\lambda y,\gamma y) \le |\lambda-\gamma|d(y,\widehat{0}) \tag{1.3}$$

for all $y,z,w \in E$ and $\lambda,\gamma \in \mathbb{R}$.
Let $T \subset \mathbb{R}$ be an interval, $y : T \to E$ be a fuzzy function, and $t_0 \in T$. If for each $\varepsilon > 0$, there exists $\delta > 0$, such that

$$d(y(t),y(t_0)) < \varepsilon,$$

for all $t \in T$ with $|t-t_0| < \delta$, then y is said to be continuous at t_0. If y is continuous at each point of T, then y is said to be continuous on T. We denote by $C(T,E)$ the space of all continuous fuzzy functions on T.
Let $a > 0$ and $r \ge 0$. We need the following notion before proceeding further.

$$C_r([0,a],E) := \{y \in C((0,a],E) : \sup_{t\in[0,a]} d(t^r y(t),\widehat{0}) < \infty\},$$

and on this set we define the metric d_r by

$$d_r(y,z) := \sup_{t\in[0,a]} t^r d(y(t),z(t)).$$

Also, $d_r(y,\widehat{0})$ will be denoted by $\|y\|_r$. Clearly, $C([0,a],E) = C_0([0,a],E)$.

Theorem 1.2. *$(C_r([0,a],E),d_r)$ is a complete metric space.*

Proof. Let $\{y_n\}_{n=1}^\infty$ be a Cauchy sequence in $C_r([0,a],E)$. Then for each $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $d_r(y_n,y_m) < \varepsilon$ for all $n,m \ge M$. That is,

$$\sup_{t\in[0,a]} t^r d(y_n(t),y_m(t)) < \varepsilon, \quad \text{for all } n,m \ge M,$$

Let $z_n(t) := t^r y_n(t), n \ge 1$. Then

$$\sup_{t\in[0,a]} d(z_n(t), z_m(t)) < \varepsilon, \quad \text{for all } n, m \geq M.$$

This implies that $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the complete metric space $C([0,a],E)$. Therefore z_n converges uniformly to $z \in C([0,a],E)$.
Let $y(t) := t^{-r}z(t), t \in (0,a]$. Clearly $y \in C((0,a],E)$. we have

$$\begin{aligned} d_r(y_n, y) &= \sup_{t\in[0,a]} t^r d(y_n(t), y(t)) \\ &= \sup_{t\in[0,a]} d(t^r y_n(t), t^r y(t)) \\ &= \sup_{t\in[0,a]} d(z_n(t), z(t)) \to 0, \text{as } n \to \infty. \end{aligned}$$

Hence y_n converges to y. Since $\{y_n\} \subset C_r([0,a],E)$, we have

$$\sup_{t\in[0,a]} d(t^r y_n(t), \widehat{0}) < \infty, \quad \text{for all } n \geq 1. \tag{1.4}$$

Now using the inequality (1.2), we obtain

$$\begin{aligned} \sup_{t\in[0,a]} d(t^r y(t), \widehat{0}) &\leq \sup_{t\in[0,a]} [t^r d(y(t), y_M(t)) + d(t^r y_M(t), \widehat{0})] \\ &\leq \sup_{t\in[0,a]} t^r d(y(t), y_M(t)) + \sup_{t\in[0,a]} d(t^r y_M(t), \widehat{0}). \end{aligned}$$

Therefore by the convergence of y_n and from inequality (1.4), we get

$$\sup_{t\in[0,a]} d(t^r y(t), \widehat{0}) < \infty.$$

Hence $y \in C_r([0,a],E)$. Thus $C_r([0,a],E)$ is a complete metric space. □

Let $y : [a,b] \to E$ be a fuzzy function, $t_0 \in [a,b]$ and $\omega \in E$. If for each $\varepsilon > 0$, there exists $\delta > 0$, such that

$$d\left(\frac{y(t) - y(t_0)}{t - t_0}, \omega\right) < \varepsilon,$$

for all $t \in [a,b]$ with $|t - t_0| < \delta$, then y is said to be derivable at t_0. We denote $y'(t_0) = \omega$ or $\frac{d}{dt}y(t_0) = \omega$. If y is derivable at each point of $[a,b]$, then y is said to be derivable on $[a,b]$. Obviously, if $y : [a,b] \to E$ is derivable at t_0, then y is continuous at t_0.
We denote by $C_r^1[0,a]$ the space of functions $y(t)$ which are continuously derivable on $(0,a]$ and have the derivative $y'(t)$ of order 1 on $(0,a]$ such that $y'(t) \in C_r[0,a]$.

Proposition 1.1. [2] *Let* $y : [a, b] \to E$ *be derivable on* $[a, b]$ *and* $y(t) = (y_1(t, \alpha), y_2(t, \alpha)), t \in [a, b], \alpha \in [0, 1]$. *Then*

$$y'(t) = \left(\frac{d}{dt} y_1(t, \alpha), \frac{d}{dt} y_2(t, \alpha) \right), \quad \alpha \in [0, 1],$$

provided this equation defines a fuzzy number $y'(t) \in E$.

Remark 1.2. [2] *i) If* $y : [a, b] \to E$ *is derivable on* $[a, b]$*, then* y *is H-derivable (Hukuhara derivable) on* $[a, b]$*, and the H-derivative is the same as the derivative. That is to say on an interval derivability is equivalent to H-derivability.*
ii) If $y : [a, b] \to E$ *is Riemann integrable on* $[a, b]$*, then the parametric representation of its integral is given by*

$$\int_a^b y(t)dt = \left(\int_a^b y_1(t, \alpha)dt, \int_a^b y_2(t, \alpha)dt \right), \quad a, b \in T, \ \alpha \in [0, 1].$$

Also, we know that the fuzzy integral is a fuzzy number.

Lemma 1.1. *Let* $y : [a, b] \to E$ *and* $z : [a, b] \to E$ *be integrable on* $[a, b]$*. If the function* $g : [a, b] \to \mathbb{R}$ *defined by*

$$g(t) := d(y(t), z(t))$$

is Riemann integrable on $[a, b]$*. Then*

$$d \left(\int_a^b y(t)dt, \int_a^b z(t)dt \right) \leq \int_a^b d(y(t), z(t))dt.$$

Proof. It can be proved easily using the Riemann sum. □

The following results are given in [8].

Theorem 1.3. *Let* $f : [a, b] \to E$ *be continuous on* $[a, b]$*, then the fuzzy function* $F : [a, b] \to E$ *given by*

$$F(t) = \int_a^t f(s)ds, \ t \in [a, b],$$

is derivable on $[a, b]$ *and*

$$F'(t) = f(t), \ t \in [a, b].$$

Corollary 1.1. *Assume that* $f : [a, b] \to E$ *is continuously derivable on* $[a, b]$*, then*

$$\int_a^b f'(t)dt = f(b) - f(a),$$

where $f(b) - f(a)$ *is the H-difference of* $f(b)$ *and* $f(a)$.

Theorem 1.4. *Let $\varphi : [a,b] \to \mathbb{R}$ be continuously derivable and $y : [a,b] \to E$ be continuously derivable. Then*

$$\int_a^b y(t)\varphi'(t)dt = [\varphi(t) \cdot y(t)]_a^b - \int_a^b \varphi(t) \cdot y'(t)dt.$$

Let y be a real valued function on $[0,a]$. The Riemann-Liouville fractional integral $I^q y$ of order $q > 0$ is defined by

$$I^q y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)ds, \quad 0 < t < a,$$

provided that the expression on the right hand side is defined.
The Riemann-Liouville fractional derivative $D^q y$ of y of order $0 < q < 1$ is defined by

$$D^q y(t) = \frac{d}{dt} I^{1-q} y(t), \qquad 0 < t < a,$$

provided the expression on right hand side is defined.

Lemma 1.2. *Let $q > 0$ and $y : [0,a] \to E$ be such that $y(t) = (y_1(t,\alpha), y_2(t,\alpha))$ for all $t \in [0,a]$ and $\alpha \in [0,1]$. Then the family of pairs*

$$F_\alpha := \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_1(s,\alpha)ds, \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_2(s,\alpha)ds \right), \alpha \in [0,1]$$

define a fuzzy number $u \in E$ such that $(u_1(\alpha), u_2(\alpha)) = F_\alpha$.

Proof. Fix $t \in [0,a]$, then by Theorem 1.1(i), $y_1(.,\alpha), y_2(.,\alpha) \in C([0,1],\mathbb{R})$, for all $\alpha \in [0,1]$. It is easy to see that $I^q y_1(t,\alpha)$ and $I^q y_2(t,\alpha)$ are continuous with respect to α. From Theorem 1.1(ii), we have $y_1(\alpha) \leq y_1(\beta)$ for $\alpha \leq \beta$, then

$$\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_1(s,\alpha)ds \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_1(s,\beta)ds,$$

therefore $I^q y_1$ is monotone increasing with respect to α, similarly, y_2 is monotone decreasing. $y \in E$ implies $y_1(1) = y_2(1)$ which gives

$$\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_1(s,1)ds = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_2(s,1)ds$$

Hence by Theorem 1.1 there exists a fuzzy number $u \in E$ such that $(u_1(\alpha), u_2(\alpha)) = F_\alpha$. □

Let $y \in C([0,a],E)$, where $y = (y_1, y_2)$, we define the fractional integral of order $q > 0$ of y by

$$I^q y(t) = \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_1(s,\alpha) ds, \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_2(s,\alpha) ds \right), \alpha \in [0,1]$$

Proposition 1.2. *Let $p, q > 0$ and $y \in C([0,a],E)$. Then*

$$I^p I^q y = I^{p+q} y$$

Proof. Similar to the proof of Lemma 4.1 in [4].

□

Let $y \in C([0,a],E)$, where $y = (y_1, y_2)$. We define the Riemann-Liouville fractional derivative of order $0 < q < 1$ of y by

$$D^q y(t) = \frac{d}{dt} I^{1-q} y(t)$$

The Riemann-Liouville derivative $D^q y(t)$ can be represented parametrically as follows

$$D^q y(t) = \left(\frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} y_1(s,\alpha) ds, \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} y_2(s,\alpha) ds \right)$$

,

where $\alpha \in [0,1]$.

Lemma 1.3. *Let $x : [0,a] \to E$ be a continuous function and $g : \mathbb{R}^2 \to \mathbb{R}$ be such that $g(t,s)$ is non negative and non decreasing with respect to t and continuous with respect to s and $\frac{\partial}{\partial t} g(t,s)$ is continuous with respect to t. Then the function $G : [0,a] \to E$ given by*

$$G(t) = \int_0^t g(t,s) x(s) ds, \quad t \in [0,a],$$

is derivable and

$$G'(t) = g(t,t)x(t) + \int_0^t \frac{\partial}{\partial t} g(t,s) x(s) ds, \quad t \in [0,a]. \tag{1.5}$$

Proof. For any $h > 0$ and by Proposition 3.5 in [2], we have

$$
\begin{aligned}
G(t+h) &= \int_0^{t+h} g(t+h,s)x(s)ds \\
&= \int_t^{t+h} g(t+h,s)x(s)ds + \int_0^t (g(t+h,s)-g(t,s))x(s)ds \\
&\quad + \int_0^t g(t,s)x(s)ds
\end{aligned}
$$

Now using Lemma 1.1, equation (1.1) and inequality (1.3), we have

$$
\begin{aligned}
& d\left(\frac{1}{h}\otimes (G(t+h)-G(t)), g(t,t)x(t) + \int_0^t \frac{\partial g(t,s)}{\partial t}x(s)ds\right) \\
\leq & d\left(\frac{1}{h}\int_0^t (g(t+h,s)-g(t,s))x(s)ds, \int_0^t \frac{\partial g(t,s)}{\partial t}x(s)ds\right) \\
& + d\left(\frac{1}{h}\int_t^{t+h} g(t+h,s)x(s)ds, g(t,t)x(t)\right) \\
\leq & \int_0^t d\left(\frac{g(t+h,s)-g(t,s)}{h}x(s), \frac{\partial g(t,s)}{\partial t}x(s)\right) ds \\
& + \frac{1}{h}\int_t^{t+h} d(g(t+h,s)x(s), g(t,t)x(t))ds \\
\leq & \int_0^t \left|\frac{g(t+h,s)-g(t,s)}{h} - \frac{\partial g(t,s)}{\partial t}\right| d(x(s),\widehat{0})ds \\
& + \frac{1}{h}\int_t^{t+h} [d(g(t+h,s)x(s), g(t,s)x(s)) + d(g(t,s)x(s), g(t,s)x(t)) \\
& + d(g(t,s)x(t), g(t,t)x(t))]ds \\
\leq & \int_0^t \left|\frac{g(t+h,s)-g(t,s)}{h} - \frac{\partial g(t,s)}{\partial t}\right| d(x(s),\widehat{0})ds \\
& + \frac{1}{h}\int_t^{t+h} (g(t+h,s)-g(t,s))d(x(s),\widehat{0})ds \\
& + \frac{1}{h}\int_t^{t+h} g(t,s)d(x(s),x(t))ds \\
& + \frac{1}{h}\int_t^{t+h} |g(t,s)-g(t,t)|d(x(t),\widehat{0})ds \to 0 \quad \text{as } h \downarrow 0.
\end{aligned}
$$

by the continuity of $x(t)$ and the results in analysis. Similarly, we have

$$d\left(\frac{1}{h}\otimes\left(G(t)-G(t-h),g(t,t)x(t)+\int_0^t\frac{\partial g(t,s)}{\partial t}x(s)ds\right)\to 0 \quad \text{as } h\downarrow 0.$$

Therefore $\lim_{h\to 0}\frac{1}{h}\otimes(G(t+h)-G(t))$ and $\lim_{h\to 0}\frac{1}{h}\otimes(G(t)-G(t-h))$ exist. It follows that $G'(t)$ exists and (1.5) holds.

□

Lemma 1.4. *Let $0<q<1$. Then the following assertions are true:*
a) If $y(t)\in C_{1-q}([0,a],E)$, then

$$D^qI^qy(t)=y(t) \text{ for all } t\in(0,a]$$

b) If $I^{1-q}y(t)\in C^1_{1-q}([0,a],E)$, then

$$I^qD^qy(t)=y(t)-\frac{t^{q-1}}{\Gamma(q)}I^{1-q}y(0) \text{ for all } t\in(0,a]$$

Proof. a) Using the definition of fractional derivative, Proposition 1.2 and Theorem 1.3, we obtain

$$\begin{aligned} D^qI^qy(t) &= \frac{d}{dt}I^{1-q}I^qy(t)=\frac{d}{dt}I^1y(t)\\ &= \frac{d}{dt}\int_0^t y(s)ds=y(t)\end{aligned}$$

b)By the definition of fractional integral and Lemma 1.3, we get

$$\begin{aligned} I^qD^qy(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}D^qy(s)ds\\ &= \frac{1}{\Gamma(q+1)}\frac{d}{dt}\int_0^t(t-s)^qD^qy(s)ds\end{aligned}$$

Now by Theorem 1.4, we have

$$\begin{aligned} \frac{1}{\Gamma(q+1)}\int_0^t(t-s)^qD^qy(s)ds &= \frac{1}{\Gamma(q+1)}\int_0^t(t-s)^q\frac{d}{ds}I^{1-q}y(s)ds\\ &= \frac{1}{\Gamma(q+1)}[(t-s)^qI^{1-q}y(s)\big|_0^t+q\int_0^t(t-s)^{q-1}I^{1-q}y(s)ds]\\ &= \frac{1}{\Gamma(q+1)}\left[-t^qI^{1-q}y(0)+q\Gamma(q)I^q\left(I^{1-q}y(t)\right)\right]\\ &= \frac{-t^q}{\Gamma(q+1)}I^{1-q}y(0)+I^1y(t)\end{aligned}$$

Hence

$$I^q D^q y(t) = y(t) - \frac{t^{q-1}}{\Gamma(q)} I^{1-q} y(0)$$

□

Lemma 1.5. *(i) If there exists* $\lim\limits_{t\to 0^+} t^{1-q}u(t) = v$ *then there also exists* $\lim\limits_{t\to 0^+} I^{1-q}u(t) = \Gamma(q)v$ *(ii) If there exists* $\lim\limits_{t\to 0^+} I^{1-q}u(t) = w$ *and if there exists* $\lim_{t\to 0^+} t^{1-q}u(t)$ *then* $\lim\limits_{t\to 0^+} t^{1-q}u(t) = \frac{w}{\Gamma(q)}$

Proof. (i)If there exists $\lim\limits_{t\to 0^+} t^{1-q}u(t) = v$ then, for each $\varepsilon > 0$, we can choose $\delta = \delta(\varepsilon) > 0$, such that

$$d(t^{1-q}u(t), v) < \frac{\varepsilon}{\Gamma(q)}$$

for $|t| < \delta$ Since $I^{1-q}t^{q-1} = \Gamma(q)$, we have

$$\begin{aligned} d(I^{1-q}u(t), \Gamma(q)v) &= d(l^{1-q}u(t), l^{1-q}t^{q-1}v) \\ &= \frac{1}{\Gamma(1-q)} d\left(\int_0^t (t-s)^{-q}u(s)\mathrm{d}s, \int_0^t (t-s)^{-q}s^{q-1}v\mathrm{d}s\right) \\ &\le \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q}s^{q-1} d\left(s^{1-q}u(s),\ v\right) ds < \varepsilon, \end{aligned}$$

which proves (i). Assertion (ii) is obvious. □

The following definitions are given.

Definition 1.1. [4] *"A fuzzy number is a fuzzy set* P *if for its membership function* $\mu_P : \mathbb{R} \to [0,1]$ *the following holds:*
(i) P *is normal. i.e, there exists a real member* q_0 *such that* $\mu_P(q_0) = 1$.
(ii) P *is fuzzy convex. i.e,*
for two arbitrary real numbers q_1, q_2 *and* $l \in [0,1]$ *we have,*

$$\mu_P(lq_1 + (1-l)q_2) \ge Min\{\mu_P(q_1), \mu_P(q_2)\}.$$

(iii) P *is upper semi-continuous.*
(iv) The closure of $\mathrm{Supp}(P) = \{q \in \mathbb{R} : \mu_P(q) > 0\}$ *is compact."*

The Supp represents the support set of the fuzzy set P *and is defined as in above.*

Definition 1.2. [4] *"The parametric form of a fuzzy number P is given by $q_P = [P_l(q), P_u(q)]$ for any $0 \leq q \leq 1$, iff,*
(i) $P_l(q) \leq P_u(q)$.
(ii) $P_l(q)$ increases with q and is left continuous function on $[0,1]$ and right continuous on 0 with respect to q.
(iii) $P_u(q)$ decreases with q and is left continuous function on $[0,1]$ and right continuous on 0 with respect to q.
(iv) $q_P = [P_l(q), P_u(q)]$ is a compact interval for any $0 \leq q \leq 1$."

Definition 1.3. [4] *"A singleton fuzzy number is a real number a, if $q_a = [a_l(q), a_u(q)] = [a, a]$ i.e, the membership function at a is 1 and at other values is zero."*

For example, $\mathbf{0}$ denotes the singleton fuzzy zero with,

$$\mu_{\mathbf{0}}(q) = \begin{cases} 1, & q = 0 \\ 0, & otherwise \end{cases}.$$

Definition 1.4. [4] *"Let P and Q be two fuzzy numbers in parametric form then the addition R of P and Q is given by*

$$P \oplus Q = R,$$
$$q_R = [R_l(q), R_u(q)] = q_P + q_Q = [P_l(q), P_u(q)] + [Q_l(q), Q_u(q)],$$

where,

$$R_l(q) = P_l(q) + Q_l(q), R_u(q) = P_u(q) + Q_u(q).\text{''}$$

Definition 1.5. [4] *"Let P and Q be two fuzzy numbers in parametric form then the generalized Hukuhara difference of P and Q is given by*

$$P \ominus_g Q = R \Leftrightarrow \begin{cases} (i) P = Q \oplus R \\ or \\ (ii) Q = P \oplus (-1)R.\text{''} \end{cases}$$

Definition 1.6. [4] *"Let P and Q be two fuzzy numbers in parametric form then the multiplication of P and Q is given by*

$$P \odot Q = R, q_R = [R_l(q), R_u(q)] = q_P \times q_Q = [P_l(q), P_u(q)] \times [Q_l(q), Q_u(q)],$$

where,

$$R_l(q) = \min\{P_l(q) \times Q_l(q), P_l(q) \times Q_u(q), P_u(q) \times Q_l(q), P_u(q) \times Q_u(q)\},$$

$R_u(q) = \max\{P_l(q) \times Q_l(q), P_l(q) \times Q_u(q), P_u(q) \times Q_l(q), P_u(q) \times Q_u(q)\}.$

This is also valid if one of P and Q is a real number.

Definition 1.7. [4] *"The Hausdorff distance $D_H : \mathbb{F}_{\mathbb{R}} \times \mathbb{F}_{\mathbb{R}} \to \mathbb{E}$, between two fuzzy numbers P and Q is given by*

$$D_H(P, Q) = \sup_{q\in[0,1]} \max\{|P_l(q) - Q_l(q)|, |P_u(q) - Q_u(q)|\}."$$

Following are some properties satisfied by D_H .Here $P, Q, R, S \in \mathbb{F}_{\mathbb{R}}$ and $k \in \mathbb{R}$

(i) $D_H(P \oplus R, Q \oplus R) = D_H(P, Q)$.

(ii) $D_H(k \odot P, k \odot Q) = |k| D_H(P, Q)$.

(iii) $D_H(P \oplus Q, R \oplus S) \leq D_H(P, R) + D_H(Q, S)$.

(iv) $(\mathbb{F}_{\mathbb{R}}, D_H)$ *is a complete metric space.*

(v) $D_H(P \ominus_g Q, R \ominus_g S) \leq D_H(P, R) + D_H(Q, S)$.

(vi) $D_H(P \ominus_g Q, \mathbf{0}) = D_H(P, Q)$

Last two properties are very useful as they relate the Hausdorff distance to the generalized Hukuhara difference.

Definition 1.8. [4] *"The generalized Hukuhara derivative of a fuzzy number valued function $f : [0, T] \to \mathbb{F}_{\mathbb{R}}$ at $t_0 \in [0, T]$ is given by :*

$$f'(t_0) = \lim_{h\to 0} \frac{f(t_0 + h) \ominus_g f(t_0)}{h},$$

provided that the difference $f(t_0 + h) \ominus_g f(t_0)$ and the limit exists then the function f is called gH -differentiable.

The level-wise form of gH -differentiable function I is given in following two cases:
CASEI: $f'(t, r) = |f_l'(t, r), f_u'(t, r)|$ *if f is $i - gH$ differentiable at t*
CASEII: $f'(t, r) = [f_u'(t, r), f_l'(t, r)]$ *if f is $ii - gH$ differentiable at t*

Definition 1.9. [12] *"The fractional integration of $f \in L_{1,loc}([t_0, t], \mathbb{R})$ of order $\alpha > 0$ in Riemann Liouville sense is given by*

$$_{t_0}I_t^{-\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} y(s) ds."$$

Definition 1.10. [12] *The fractional derivative of $f \in L_{1,loc}([t_0, t], \mathbb{R})$ of order $0 \le \alpha < 1$ ir Riemann-Liouville sense is given by*

$$ {}_{t_0}^{RL}D_t^{\alpha} f(t) = \frac{d}{dt}({}_{t_0}I_t^{-(1-\alpha)} f(t)) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-\alpha} f(s) ds." $$

Definition 1.11. [12] *"The fractional derivative of a differentiable function f ,such that $f' \in L_{1,loc}([t_0, t], \mathbb{R})$,of order $0 < \alpha < 1$ in Caputo sense is given by*

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) =_{t_0} I_t^{-(1-\alpha)} \frac{d}{dt} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} f'(s) ds." $$

Definition 1.12. [12] *"The Mittag-Leffler function with two parameter for $z \in \mathbb{C}$ is given by*

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha, \beta > 0)" $$

Definition 1.13. [4] *"The fuzzy fractional Caputo derivative of order $0 < \alpha < 1$ of a fuzzy number valued function f , such that f is gH differentiable and $f' \in L_{1,loc}([t_0, t], \mathbb{F}_{\mathbb{R}})$, is defined as:*

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \odot \int_{t_0}^{t} (t-s)^{-\alpha} \odot f'(s) ds." $$

Definition 1.14. [4] *"Let f be a fuzzy number valued function with parametric form $f(t,r) = [f_l(t,r), f_u(t,r)]$, then length of f is defined as*

$$ length(f(t,r)) = f_u(t,r) - f_l(t,r)." $$

Note: The length function is monotonically increasing if the function x is $i-gH$ differentiable and is monotonically decreasing if it is $ii-gH$ differentiable

Lemma 1.6. [4] *"Let $f(t)$ be a i or $ii-gH$ differentiable fuzzy function. Then. $f(t)$ is the solution of (3) iff, $f(t)$ is the solution of the following integral equation:*

$$ f(t) \ominus_g x_0 = \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-s)^{\alpha-1} \odot f(s, f(s)) ds." $$

Definition 1.15. [11] *"Let $g : X \to X$ and (X, d) a metric space then y is a contraction if there exists a fixed constant $l < 1$ such that*

$$ d(g(a), g(b)) \le l d(a,b), \forall a, b \in X." $$

Theorem 1.5. [11] *"Each contraction map $g : X \to X$ on a complete metric space (X, d) has a unique fixed point."*

Definition 1.16. *Let $C([t_0, T], \mathbb{F}_{\mathbb{R}})$ be the set of all continuous functions from $[t_0, T]$ to $\mathbb{F}_{\mathbb{R}}$. Define*

$$H(f, n) = \sup_{t \in [t_0, T]} D_H(f(t), n(t)).$$

Then $(C([t_0, T], \mathbb{F}_{\mathbb{R}}), H)$ is a complete metric space.

Theorem 1.6. [11] *"Let $g : X \to X$ and (X, d) a complete metric space with the property that for some positive integer n, g^n is contraction on X Then, y has a unique fixed point."*

Proof. Proof Let U be the fixed point of g^n ,then

$$g^n(a) = a \implies g^{n+1}(a) = g(a) \implies g^n(g(a)) = g(a)$$

This means that $g(a)$ is also a fixed point of g^n .But the fixed point is unique, hence $g(a) = a$ that is a is a unique fixed point of g □

Definition 1.17. [9] *Let X be a nonempty set,then $\widetilde{A} = \{(u_{\widetilde{A}(x)}, x) | x \in X\}$ is called fuzzy set on X. Here $u_{\widetilde{A}}(x)$ is the number specified on [0, 1] and called the membership degree of a point X to a set $\widetilde{A}$ that is*

$$u_{\widetilde{A}}(x) : X \to [0, 1],$$

$$x \to u_{\widetilde{A}}(x).$$

We denote by $F(X)$ the collection of all fuzzy subsets of X

We identify a fuzzy set with its membership function. Other notations that can be used are the following: $u_{\widetilde{A}}(x) = \widetilde{A}(x)$

Let us denote by R_F the space of fuzzy numbers.

For $r \in (0, 1]$, we denote

$$[u]_r = \{x \in \mathbb{R} : u(x) \geqslant r\}$$

and

$$[u]_0 = \{x \in \mathbb{R} : u(x) \geqslant 0\}.$$

Thus u_r is called the r-level set of the fuzzy number u .The 1-level set is called the core of the fuzzy number, while the 0-level set is called the support of the fuzzy number.

Lemma 1.7. [10] *If $u \in \mathbb{R}_F$ is a fuzzy number and u_r is its level-sets, then:*

(1) u_r is a closed interval $u_r = [\underline{u}_r, \overline{u}_r]$, for any $r \in [0, 1]$;

(2) functions $\underline{u}_r, \overline{u}_r : [0, 1] \to \mathbb{R}$;

(3) $\underline{u}_r = \underline{u} \in \mathbb{R}$ is a bounded, nondecreasing, left-continuous function in (0, 1] and right-continuous at 0;

(4) $\overline{u}_r = \overline{u} \in \mathbb{R}$ is a bounded, nonincreasing, left-continuous function in (0, 1] and right-continuous. at 0;

(5) $\underline{u} \leqslant \overline{u}$.

According to the Lemma 1.7, we can denote zero fuzzy number by $[\widetilde{0}]_r = \widetilde{0}_r = [0, 0]$, for any $r \in [0, 1]$

Definition 1.18. [9] *Let $u, v \in \mathbb{R}_F \in [0, 1]$*
(1) if $\underline{u}_r \leqslant \underline{\nu}_r$ and $\overline{u}_r \leqslant \overline{v}_r$, then $u \leqslant v$
(2) if $u_r + v_r = w_r$, then $u \oplus \nu = w$

The Hukuhara difference (H-difference $\ominus_H$) is defined by $u \ominus_H v = w \Longleftrightarrow u = w$, being $+$ the standard fuzzy addition. If $u \ominus_H v$ exists, its r-cuts are $[u \oplus_H v]_r = [\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r]$. It is easy to verify that $u \ominus_H u = 0$ for any fuzzy numbers u ,but as we have earlier discussed $u - u \neq 0$

Definition 1.19. [13] *Let $u, v \in \mathbb{R}_F$. The generalized Hukuhara differential is defined as follows*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (1)\ u = v \oplus w, & or \\ (2)\ v = u \ominus w. \end{cases}$$

Proposition 1.3. *For any $u, v \in \mathbb{R}_F$ we have*

$$[u \ominus_{gH} v]_r = [min\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}, max\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}].$$

Definition 1.20. *Le t $D_\infty : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \longrightarrow \mathbb{R}_+ \bigcup\{0\}$,*

$$D(u, v) = \sup_{0 \leqslant r \leqslant 1} max\{|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\} = \sup_{0 \leqslant r \leqslant 1} d_H([u]_r, [v]_r),$$

where d_H is the classical Hausdorff Pompeiu distance between real intervals, then D_∞ is called the Hausdorff distance between fuzzy numbers.

Lemma 1.8. [14] *$(\mathbb{R}_F, D_\infty)$ is a complete metric space.*

Let us denote $(X, N, *)$ as a fuzzy metric space and $\|u\|_{\mathcal{F}} = D_\infty(u, 0)$ as the norm of a fuzzy number $\|u\|_{\mathcal{F}}$ has the following properties:

(1) $\|u\|_{\mathcal{F}} = 0 \Longleftrightarrow u = 0$;

(2) $\|\lambda \cdot u\|_{\mathcal{F}} = |\lambda| \|u\|_{\mathcal{F}}, \forall \lambda \in \mathbb{R}, u \in \mathbb{R}_{\mathcal{F}}$;

(3) $\|u + v\|_{\mathcal{F}} \leqslant \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, \forall u, v \in \mathbb{R}_{\mathcal{F}}$;

(4) $|\|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}}| \leqslant D_\infty(u, v), \forall u, v \in \mathbb{R}_{\mathcal{F}}$;

(5) $D(a \cdot u, b \cdot u) = |b - a| \cdot \|u\|_{\mathcal{F}}, \forall u \in \mathbb{R}_{\mathcal{F}}$;

(6) $D_\infty(u, v) = \|u \ominus_{gH} v\|_{\mathcal{F}}, \forall u, v \in \mathbb{R}_{\mathcal{F}}$.

Lemma 1.9. *Let $(X, N, *)$ be a complete fuzzy metric space and. $T : X \to X$ a fuzzy compression mapping, then T has a unique fixed point.*

Lemma 1.10. *Let $(X, N, *)$ be a complete fuzzy metric space and $T : X \to X$ If T^n is a fuzzy compression mapping $T : X \to X$ is called fuzzy power compression mapping), then T has a unique fixed point*

Definition 1.21. [15] *Let $f : (a, b) \to \mathbb{R}_F, x_0 \in (a, b)$ then the fuzzy. gH - derivative of a function f at x_0 is defined*

$$D_{gH} f(x_0) = \lim_{h \to 0} \frac{1}{h} \cdot [f(x_0 + h) \ominus_{gH} f(x_0)].$$

If $D_{gH} f(x_0) \in \mathbb{R}_F$ exists, then f is called gH-differentiable at x_0

Definition 1.22. [15] *Let $f \in C^1_{gH}((a, b), \mathbb{R}_{\mathcal{F}})$ with $[f(x)]_r = [\underline{f}_r, \overline{f}_r]$for all $x \in (a, b)$, $r \in [0, 1]$. We call*

(1) f is (i)-gH differentiable at $x_0 \in (a, b)$ if $[D_{gH} f(x_0)]_r = [(\underline{f}_r)'(to), (\overline{f}_r)'(x_0)]$ (denoted by $D^i_{gH} f(x_0)$)

(2) f is (ii)-gH differentiable at $x_0 \in (a, b)$ if $[D_{gH} f(t_0)]_r = [(\overline{f}_r)'(to), (\underline{f}_r)'(x_0)]$ (denoted by $D^{ii}_{gH} f(x_0)$).

.

Definition 1.23. [3, 15] *Let $\widetilde{f} : [a, b] \to \mathbb{R}_{\mathcal{F}}$ $[\widetilde{f}(x)]_r = [\underline{f}_r, \overline{f}_r]$ for all $x \in [a, b], r \in [0, 1]$ where $\underline{f}_r$, $\overline{f}_r$ are measurable and Lebesgue integrable on $[a, b]$, then the Choquet integral of $\widetilde{f}$ (is denoted by $\int_{(a,b)} f(x)dx$) can be defined as*

$$\left[\int_{(a,b)} f(x)dx \right]_\alpha = \left(\int_a^b \underline{f}_\alpha(x)dx, \int_a^b \overline{f}_\alpha(t)dt \right),$$

for all $r \in [0, 1]$

Definition 1.24. [15] *Let* $0 < \lambda < 1$ *. The fuzzy fractional differential equation* (5) *is equivalent to one of the following integral equations:*

$$\widetilde{u}(t) = \widetilde{u}_0 \oplus \frac{1}{\Gamma(\lambda)} \int_0^t \frac{\widetilde{f}(s, \widetilde{u}(s))}{(t-s)^{1-\lambda}} ds, \ t \in [a, b],$$

if $\widetilde{u}$ *is (i)-gH differentiable, and*

$$\widetilde{u}(t) = \widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)} \int_0^t \frac{\widetilde{f}(s, \widetilde{u}(s))}{(t-s)^{1-\lambda}} ds, \ t \in [a, b],$$

if u *is (ii)-gH differentiable, provided that the Hukuhara difference exists.*

Definition 1.25. *The solution of the initial value problem* (5) *is a fuzzy numerical function* u *that satisfies* (5). *We say that* $\widetilde{u}$ *is (i)-solution if the solution of the initial value problem* (5) *is Caputo (i)-gH differentiable; we say that* $\widetilde{u}$ *is (ii)-solution if the solution of the initial value problem* (5) *is Caputo (ii)-gH differentiable.*

Definition 1.26. *Gamma function* $\Gamma(\cdot)$ *is defined:*

$$\Gamma(s) = \left[\int_{[0,\infty)} \widetilde{x}^{s-1} e^{-\overline{x}} d\widetilde{x} \right]_\alpha = \left[\int_{[0,\infty)} \underline{x}_\alpha^{s-1} e^{-\underline{x}} d\underline{x}, \int_{[0,\infty)} \overline{x}_\alpha^{s-1} e^{-\overline{x}} d\overline{x} \right],$$

where $s > 0$

Definition 1.27. *Beta function* $B(\cdot, \cdot)$ *is defined*

$$B(p,q) = \left[\int_{[0,1]} \widetilde{x}^{p-1} (1 \ominus \widetilde{x})^{q-1} d\widetilde{x} \right]_\alpha = \left[\int_{[0,1]} \underline{x}_\alpha^{p-1} (1 - \underline{x}_\alpha)^{q-1} d\underline{x}, \int_{[0,1]} \overline{x}_\alpha^{p-1} (1 - \overline{x}_\alpha)^{q-1} d\overline{x} \right],$$

where $p, q > 0$

Proposition 1.4. *The relationship of Gamma function* $\Gamma(\cdot)$ *and Beta function* $B(\cdot, \cdot)$ *is as follows:*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad \alpha\Gamma(\alpha) = \Gamma(\alpha + 1).$$

Chapter 2

Fuzzy Fractional Differential Equation with an Initial Condition

2.1 Introduction

In this chapter, we establishes the equivalence between a fuzzy fractional differential equation with an initial condition and a fuzzy integral equation. Consider the following fuzzy fractional differential equation and the initial condition:

$$\begin{cases} D^q y(t) = f(t, y(t)) \\ \lim_{t\to 0^+} t^{1-q} y(t) = y_0 \in E \end{cases}$$

we provides examples of fuzzy fractional differential equations with specific initial conditions.

2.2 Existence and Uniqueness

$$D^q y(t) = f(t, y(t)) \tag{2.1}$$

where $0 < q < 1$, and $f : [0, a] \times E \to E$ is continuous on $(0, a] \times E$.
A fuzzy function $y : (0, a] \to E$ is a solution of fuzzy fractional differential equation (2.1) if it is continuous on $(0, a]$ and

$$D^q y(t) = f(t, y(t))$$

for all $t \in (0, a]$. We can associate with the fuzzy fractional differential equation with the following initial condition

$$\lim_{t\to 0^+} t^{1-q} y(t) = y_0 \in E \tag{2.2}$$

Remark 2.1. *Let* $0 < q < 1$ *and* $y(t) \in C_{1-q}((0,a], E)$*. Using a similar proof as in Lemma 1.5, we have*
a) If

$$\lim_{t \to 0^+} t^{1-q} y(t) = b \in E$$

then

$$I^{1-q} y\left(0^+\right) := \lim_{t \to 0^+} I^{1-q} y(t) = b\Gamma(q)$$

b) If

$$\lim_{t \to 0^+} I^{1-q} y(t) = c \in E$$

and if there exists the limit $\lim_{t \to 0^+} t^{1-q} y(t)$*, then*

$$\lim_{t \to 0^+} t^{1-q} y(t) = \frac{c}{\Gamma(q)}$$

Lemma 2.1. *Let* $0 < q < 1, K > 0$*, and* $a > 0$*. Define*

$$G = \left\{(t, y) \in [0,a] \times E : y \in E \text{ for } t = 0 \text{ and } d\left(t^{1-q} y, y_0\right) < K \text{ else } \right\}$$

and assume that the function $f : G \to E$ *is a continuous and bounded in* G *and there exists a constant* $A > 0$ *such that,*

$$d(f(t,u), f(t,v)) \leq A d(u,v)$$

for all $(t,u), (t,v) \in G$*. If* $y(t) \in C((0,a], E)$*, then* $y(t)$ *satisfies the relations* (2.1) *and* (2.2) *if and only if* $y(t)$ *satisfies the integral equation*

$$y(t) = y_0 t^{q-1} + I^q f(t, y(t)) \tag{2.3}$$

Proof. Suppose $y(t) \in C((0,a], E)$ satisfy (2.1), (2.2), we define $u(t) := f(t, y(t))$. By assumption, u is a continuous function and

$$u(t) = f(t, y(t)) = D^q y(t) = \frac{d}{dt}\left(I^{1-q} y\right)(t)$$

Thus $\frac{d}{dt}\left(I^{1-q} y\right)(t) \in C_{1-q}([0,a], E)$. Therefore $I^{1-q} y(t) \in C^1_{1-q}([0,a], E)$. Applying I^q to both sides of (2.1) and using Lemma 1.4 (b), we have

$$y(t) - \frac{t^{q-1}}{\Gamma(q)} I^{1-q} y(0) = I^q f(t, y(t))$$

Therefore by Lemma 2.0,

$$y(t) = y_0 t^{q-1} + I^q f(t, y(t))$$

Suppose that $y \in C((0,a], E)$ satisfy (2.3). Applying D^q to both sides of (2.3) and then using Lemma 1.4(a), we obtain

$$\begin{aligned} D^q y(t) &= D^q \left(y_0 t^{q-1}\right) + D^q I^q f(t, y(t)) \\ &= f(t, y(t)) \end{aligned}$$

The following theorem help us to prove the next result. □

Theorem 2.1. [2] *Let (U, d) be non empty complete metric space, and let $\beta_n \geq 0$ for all $n \in \{0, 1, 2, \ldots\}$ be such that $\sum_{n=0}^{\infty} \beta_n$ converges. Moreover, let the mapping $T : U \to U$ satisfy the inequality*

$$d\left(T^n u, T^n v\right) \leq \beta_n d(u, v)$$

for all $n \in \mathbb{N}$ and for all $u, v \in U$. Then the operator T has a unique fixed point $u^* \in U$. Furthermore, for any $u_0 \in U$, the sequence $\{T^n u_0\}_{n=1}^{\infty}$ converges to the above fixed point u^*.

Theorem 2.2. *Let $0 < q < 1, K > 0$ and $a^* > 0$. Define*

$$G = \left\{(t, y) \in [0, a^*] \times E : y \in E \text{ for } t = 0 \text{ and } d\left(t^{1-q} y, y_0\right) < K\right\}$$

and assume that the function $f : G \to E$ is a continuous and bounded in G and there exists a constant $A > 0$ such that,

$$d(f(t, u), f(t, v)) \leq A d(u, v) \tag{2.4}$$

for all $(t, u), (t, v) \in G$. Then there exists a unique solution $y(t) \in C((0, a], E)$ to the Cauchy problem (2.1) and (2.2), where

$$a := \min \left\{ a^*, \widetilde{a}, \left(\frac{\Gamma(q+1)K}{M} \right) \right\}$$

with $M := \sup_{(t,y) \in G} d(f(t, y), \widehat{0})$ and $\widetilde{a}$ being a positive number such that

$$\widetilde{a} < \left(\frac{\Gamma(2q)}{\Gamma(q) A} \right)^{\frac{1}{q}}$$

Proof. Define the set

$$U := \{y \in C((0,a],E) : \sup_{t\in[0,a]} d\left(t^{1-q}y, y_0\right) \leq K\}$$

U is a closed subset of the complete metric space $C_{1-q}([0,a],E)$. Therefore U is a complete metric space. We define the operator $T : U \to C_{1-q}([0,a],E)$ by

$$Ty(t) := y_0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,y(s))ds$$

In order to prove the desired result, it is sufficient to prove that the operator T has a unique fixed point. Note that for $y \in U, Ty$ is also a continuous function on $(0,a]$. Moreover,

$$\begin{aligned}
d\left(t^{1-q}Ty(t), y_0\right) &\leq d\left(\frac{t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,y(s))ds, \widehat{0}\right) \\
&\leq \frac{t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1} d(f(s,y(s)), \widehat{0})ds \\
&\leq \frac{t^{1-q}M}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \\
&\leq \frac{aM}{\Gamma(q+1)} \leq K
\end{aligned}$$

for $t \in (0,a]$. This shows that the operator T maps the set U into itself. Now we show by induction that for $y, z \in U$,

$$\|T^n y - T^n z\|_{1-q} \leq \left(\frac{Aa^q\Gamma(q)}{\Gamma(2q)}\right)^n \|y-z\|_{1-q} \tag{2.5}$$

For $n = 0$, this statement is trivially true. Suppose that (2.5) is true for $n \geq 1$. Then from inequality (2.4), we have

$$
\begin{aligned}
&\left\|T^{n+1}y - T^{n+1}z\right\|_{1-q} \\
&= \sup_{t\in[0,a]} t^{1-q} d\left(\left(T^{n+1}y(t) - T^{n+1}z(t)\right), \widehat{0}\right) \\
&= \sup_{t\in[0,a]} t^{1-q} d\left(\left(TT^{n}y(t) - TT^{n}z(t)\right), \widehat{0}\right) \\
&= \sup_{t\in[0,a]} \frac{t^{1-q}}{\Gamma(q)} d\left(\int_0^t (t-s)^{q-1}\left(f\left(s, T^n y(s)\right) - f\left(s, T^n z(s)\right)\right) ds, \widehat{0}\right) \\
&\leq \sup_{t\in[0,a]} \frac{t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1} d\left(f\left(s, T^n y(s)\right) - f\left(s, T^n z(s)\right)\right), \widehat{0}\Big) ds \\
&\leq \frac{A}{\Gamma(q)} \sup_{t\in[0,a]} t^{1-q} \int_0^t (t-s)^{q-1} d\left(T^n y(s) - T^n z(s)\right), \widehat{0}\Big) ds \\
&\leq \frac{A}{\Gamma(q)} \sup_{t\in[0,a]} t^{1-q} \int_0^t (t-s)^{q-1} s^{q-1} s^{1-q} d\left(T^n y(s) - T^n z(s)\right), \widehat{0}\Big) ds \\
&\leq \frac{A}{\Gamma(q)} \left\|T^n y - T^n z\right\|_{1-q} \sup_{t\in[0,a]} t^{1-q} \int_0^t (t-s)^{q-1} s^{q-1} ds \\
&= \left(\frac{A a^q \Gamma(q)}{\Gamma(2q)}\right) \left\|T^n y - T^n z\right\|_{1-q}.
\end{aligned}
$$

Now using induction we get (2.5). Therefore we can apply Theorem 1.5 with $\beta_n = \left(\frac{Aa^q\Gamma(q)}{\Gamma(2q)}\right)^n$. It remains to show that the series $\sum_{n=0}^{\infty} \beta_n$ is convergent. Since $a \leq \widetilde{a}$ and the definition of $\widetilde{a}$ implies that $\left(\frac{Aa^q\Gamma(q)}{\Gamma(2q)}\right) < 1$. Thus by Theorem 1.5, there exists a unique solution of the integral equation (2.3). Then using Lemma 1.5 yields the existence and uniqueness of the Cauchy problem (2.1) and (2.2). □

2.3 Examples

Example 2.1.

$$
\begin{cases} D^q y(t) = \lambda y(t) + b(t) \\ \lim\limits_{t\to 0^+} t^{1-q} y(t) = (1|3|4) \end{cases}
$$

where $t \in [0,a], 0 < q \leq 1, \lambda \geq 0,$ *and* $\lim\limits_{t\to 0^+} t^{1-q} y(t) = (1|3|4) \in E$ *is a fuzzy triangular number, that is,* $\lim\limits_{t\to 0^+} t^{1-q} y(t) = (2\alpha + 1, 4 - \alpha)$ *for* $\alpha \in (0,1]$. *If we put* $y(t) = (y_1(t,\alpha), y_2(t,\alpha))$, *then* $D^q y(t) = (D^q y_1(t,\alpha), D^q y_2(t,\alpha))$. *Thus we have*

$$\begin{cases} D^q y_1(t,\alpha) = \lambda y_1(t,\alpha) + b_1(t,\alpha) \\ \lim\limits_{t\to 0^+} t^{1-q} y_1(t,\alpha) = 2\alpha + 1 \end{cases} \tag{2.6}$$

and

$$\begin{cases} D^q y_2(t,\alpha) = \lambda y_2(t,\alpha) + b_2(t,\alpha) \\ \lim\limits_{t\to 0^+} t^{1-q} y_2(t,\alpha) = 4 - \alpha \end{cases} \tag{2.7}$$

solution of (2.6) *and* (2.7) *are given by (see [6])*

$$y_1(t,\alpha) = \frac{2\alpha+1}{\Gamma(q)} t^{q-1} E_{q,q}\left(\lambda t^q\right) + \int_0^t (t-s)^{q-1} E_{q,q}\left(\lambda (t-s)^q\right) b_1(s,\alpha) ds$$

and

$$y_2(t,\alpha) = \frac{4-\alpha}{\Gamma(q)} t^{q-1} E_{q,q}\left(\lambda t^q\right) + \int_0^t (t-s)^{q-1} E_{q,q}\left(\lambda (t-s)^q\right) b_2(s,\alpha) ds$$

where

$$E_{q,q}\left(\lambda t^q\right) = \sum_{n=0}^{\infty} \frac{\left(\lambda t^q\right)^n}{\Gamma(q(n+1))}$$

Example 2.2.

$$\begin{cases} D^q y(t) = -\lambda y(t) + b(t), \quad \lambda \geq 0 \\ \lim\limits_{t\to 0^+} t^{1-q} y(t) = (1|3|4). \end{cases} \tag{2.8}$$

We obtain the following system

$$\begin{aligned} D^q y_1(t,\alpha) &= -\lambda y_2(t,\alpha) + b_1(t,\alpha), \quad \lim_{t\to 0^+} t^{1-q} y_1(t,\alpha) = 2\alpha + 1 \\ D^q y_2(t,\alpha) &= -\lambda y_1(t,\alpha) + b_2(t,\alpha), \quad \lim_{t\to 0^+} t^{1-q} y_2(t,\alpha) = 4 - \alpha \end{aligned}$$

or

$$\begin{gathered} D^q z(t) = A z(t) + B(t) \\ \lim_{t\to 0^+} t^{1-q} z(t) = c \end{gathered} \tag{2.9}$$

where

$$z(t)=\begin{bmatrix} y_1(t,\alpha)\\ \\ y_2(t,\alpha)\end{bmatrix},\ A=\begin{bmatrix} 0 & -\lambda\\ \\ -\lambda & 0\end{bmatrix},\ B(t)=\begin{bmatrix} b_1(t,\alpha)\\ \\ b_2(t,\alpha)\end{bmatrix}\ \ c=\begin{bmatrix} 2\alpha+1\\ \\ 4-\alpha\end{bmatrix}.$$

Using the same method as in [6], we obtain the solution of (2.9). *It is given by*

$$z(t)=t^{q-1}E_{q,q}\left(At^q\right)c+\int_0^t (t-s)^{q-1}E_{q,q}\left(A(t-s)^q\right)B(s)ds$$

where

$$E_{q,q}\left(At^q\right)=\sum_{i=0}^{\infty}\frac{\left(At^q\right)^i}{\Gamma(q(i+1))}=\begin{bmatrix} W_1(t) & 0\\ \\ 0 & W_1(t)\end{bmatrix}+\begin{bmatrix} 0 & -W_2(t)\\ \\ -W_2(t) & 0\end{bmatrix},$$

and

$$W_1(t)=\sum_{k=0}^{\infty}\frac{\left(\lambda t^q\right)^{2k}}{\Gamma(q(2k+1))},\quad W_2(t)=\sum_{k=0}^{\infty}\frac{\left(\lambda t^q\right)^{2k+1}}{\Gamma(2q(k+1))}$$

Let

$$S_1(t)=\sum_{k=0}^{\infty}\frac{\lambda^{2k}t^{(2k+1)q-1}}{\Gamma(q(2k+1))},\quad S_2(t)=\sum_{k=0}^{\infty}\frac{\lambda^{2k+1}t^{2(k+1)q-1}}{\Gamma(2q(k+1))}$$

Then

$$t^{q-1}E_{q,q}\left(At^q\right)c=\begin{bmatrix} S_1(t) & -S_2(t)\\ \\ -S_2(t) & S_1(t)\end{bmatrix}\begin{bmatrix} 2\alpha+1\\ \\ 4-\alpha\end{bmatrix}=\begin{bmatrix} U_1(t,\alpha)\\ \\ U_2(t,\alpha)\end{bmatrix}$$

where

$$U_1(t,\alpha)=S_1(t)(2\alpha+1)-S_2(t)(4-\alpha),\quad U_2(t,\alpha)=S_1(t)(4-\alpha)-S_2(t)(2\alpha+1)$$

if we take

$$V_1(t,s,\alpha)=S_1(t-s)b_1(s,\alpha)-S_2(t-s)b_2(s,\alpha)$$
$$V_2(t,s,\alpha)=S_1(t-s)b_2(s,\alpha)-S_2(t-s)b_1(s,\alpha))$$

then we get

$$\int_0^t (t-s)^{q-1} E_{q,q}\left(A(t-s)^q\right) B(s)ds = \begin{bmatrix} \int_0^t V_1(t,s,\alpha)ds \\ \\ \int_0^t V_2(t,s,\alpha)ds \end{bmatrix}$$

Then we obtain

$$y_1(t,\alpha) = U_1(t,\alpha) + \int_0^t V_1(t,s,\alpha)ds$$
$$y_2(t,\alpha) = U_2(t,\alpha) + \int_0^t V_2(t,s,\alpha)ds$$

It easy to see that $(y_1(t,\alpha), y_2(t,\alpha))$ *define a fuzzy number, and therefore it is the solution of the fuzzy fractional differential equation* (2.7).

Chapter 3

Existence and uniqueness results on the solutions of the Fuzzy Fractional Differential Equation

3.1 Introduction

In this chapter. we give existence and uniqueness results on the solutions of the fuzzy fractional differential equation using Banach fixed point Theorem (Contraction principle).

$$ {}_{t_0}^{C}D_t^{\alpha} x(t) = f(t, x(t)), \quad x(t_0) = x_0 \in \mathbb{R} \tag{3.1} $$

3.2 Main Result

Theorem 3.1. *Let f be a continuous function in system* (3.1) *such that for any $m, n \in C([t_0, T], B(x_0, \eta))$ and $L > 0$*

$$ D_H(f(t, n(t)), f(t, m(t))) \leq L D_H(n(t), m(t)), \tag{3.2} $$

then the IVP (3) *has a unique solution*

Proof. We'll exploit the Banach contraction principle to prove this result. Define, $T : C([t_0, T], \mathbb{F}_{\mathbb{R}}) \to C([t_0, T], \mathbb{F}_{\mathbb{R}})$ as

$$ Tx(t) = x_0 \oplus \frac{1}{\Gamma(q)} \odot \int_{t_0}^{t} (t - s)^{q-1} \odot f(s, x(s)) ds. $$

Since $t \in [t_0, T]$, the right hand side is a continuous fuzzy number valued function on $[t_0, T]$ and hence is well defined. Now, consider the following

$$D_H(Tx(t), Ty(t)) = D_H(x_0 \oplus \frac{1}{\Gamma(q)} \odot \int_{t_0}^{t} (t-s)^{q-1} \odot f(s, x(s))ds,$$
$$x_0 \oplus \frac{1}{\Gamma(q)} \odot \int_{t_0}^{t} (t-s)^{q-1} \odot f(s, y(s))ds),$$

using properties of the Hausdorff distance and assumptions on f

$$\leq \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} D_H((s, x(s)), f(s, y(s)))ds$$
$$\leq \frac{L}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} D_H(x(s), y(s))ds,$$

taking supremum over $[t_0, T]$ both the sides

$$\sup_{[t_0,T]} D_H(Tx(t), Ty(t)) \leq \frac{L}{\Gamma(q)} \int_{t_0}^{t} \sup_{s\in[t_0,T]} (t-s)^{q-1} D_H(x(s), y(s))ds$$
$$\leq \frac{L}{\Gamma(q)} H(x,y) \left[\int_{t_0}^{t} (t-s)^{q-1} ds \right]$$

$$\begin{aligned} H(Tx, Ty) &\leq \frac{L}{\Gamma(q+1)} H(x,y)(t-t_0)^q \\ &\leq \frac{L}{\Gamma(q+1)} H(x,y)(T-t_0)^q. \end{aligned} \tag{3.3}$$

Now,

$$D_H(T^2x(t), T^2y(t)) = D_H(T(Tx(t)), T(Ty(t)))$$
$$\leq \frac{L}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} D_H(Tx(t), Ty(t)),$$

taking supremum over $[t_0, T]$ both the sides

$$\sup_{[t_0,T]} D_H(T^2x(t), T^2y(t)) \leq \frac{L}{\Gamma(q)} \frac{L}{\Gamma(q+1)} \int_{t_0}^{t} (t-s)^{q-1}(s-t_0)^q H(x,t)ds$$
$$= \frac{L^2}{\Gamma(q+1)} H(x,y) \Big(\frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}(s-t_0)^q ds \Big)$$
$$= \frac{L^2}{\Gamma(q+1)} H(x,y) \frac{\Gamma(q+1)}{\Gamma(q+1+q)} (t-t_0)^{q+q}$$
$$= \frac{L^2}{\Gamma(2q+1)} (t-t_0)^{2q} H(x,y)$$

$$H(T^2x, T^2y) \leq \frac{L^2}{\Gamma(2q+1)}(T-t_0)^{2q}H(x,y).$$

Inductively,

$$H(T^nx, T^ny) \leq \frac{L^n}{\Gamma(nq+1)}(T-t_0)^{nq}H(x,y).$$

But since $\frac{L^n(T-t_0)^{nq}}{\Gamma(nq+1)} \to 0$ as $n \to \infty$ $\exists n \in \mathbb{N}$. sufficiently large such that $\frac{L^n(T-t_0)^{nq}}{\Gamma(nq+1)} < 1$. This implies that T^n is a contraction

Now, Theorem 1.6 implies that T has a fixed point which assures the existence of a unique solution of (3) by Theorem 1.5

□

3.3 Example

Example 3.1. *We are considering an example which satisfies the conditions of our Theorem 3.1 and works as an illustration of the Theorem 3.1. For* $t \in [0, T]$ *we have:*

$$ {}_0^C D_t^q x(t) = \mathbf{k} \odot x(t) \ominus_g x^2(t), \quad x(0) = [0.1 \;\; 0.5 \;\; 1] \in \mathbb{F}_{\mathbb{R}}, \tag{3.4}$$

here, k *is a singleton fuzzy number. Since*

$$f(t, x(t)) = \mathbf{k} \odot x(t) \ominus_g x^2(t), \quad x(t) \in B([0.1 \;\; 0.5 \;\; 1], \eta)$$

is a continuous function with fuzzy ball $B([0.10.51], \eta)$ *as its co-domain and* $[0, T]$ *as its domain, where* $\eta > 0$ *is the radius of the fuzzy ball* B *with centre* $[0.10.51]$*, then it can be shown that the natural map produced by* (3.4) *is a contraction and hence it has a unique solution.*

Chapter 4

Application of power compression mapping principle in fractional ordinary differential equations in fuzzy space

4.1 Introduction

In this chapter, by employing the principle of fuzzy compression mapping and Choquet integral of fuzzy numerical functions, we will establish the existence and uniqueness of solutions to initial value problems for fuzzy ordinary differential equations.

4.2 Existence and Uniqueness

Next, we prove the existence and uniqueness of the initial value problem of Equation (5) in the fuzzy metric space by employing the principle of fuzzy compression mapping and Choquet integral of fuzzy numerical functions.

By employing the principle of fuzzy compression mapping and Choquet integral of fuzzy numerical functions, if u is (i)-gH differentiable, that is,

$$\widetilde{u}(t) = \widetilde{u}_0 \oplus \frac{1}{\Gamma(\lambda)} \int_0^t \frac{\widetilde{f}(s, \widetilde{u}(s))}{(t-s)^{1-\lambda}} ds,\ t \in [a, b],$$

we can come to the following results: Theorems 4.1 and 4.2

Theorem 4.1. *Let the nonlinear fuzzy number value function.* $\widetilde{f}(t, \widetilde{u}(t))$ *be continuous over the rectangular region* $S = \{(t, ||\widetilde{u}||)|0 \leqslant t \leqslant a, |\widetilde{u} \ominus \widetilde{u}_0| \leqslant b\}$ *and satisfy* *the Lispchitz condition about u, Then the initial value problem of fuzzy fractional differential equation* (5) *has a unique (i)-solution on interval* $I = [0, h]$, *where*

$$h = \min\Big\{a, (\frac{b\Gamma(\lambda+1)}{M})^{\frac{1}{\lambda}}\Big\}, \quad M = \max_{(t,\|\widetilde{u}\|)\in S} |\widetilde{f}(t,\widetilde{u}(t)) \ominus \widetilde{0}|. \qquad (4.1)$$

Proof. If u is (i)-gH differentiable, we can define the operator $A : C(I, \mathbb{R}_{\mathcal{F}}) \to C(I, \mathbb{R}_{\mathcal{F}})$ as follows:

$$A\widetilde{u}(t) = \widetilde{u}_0 \oplus \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} \widetilde{f}(s, \widetilde{u}(s))ds. \qquad (4.2)$$

It is shown by Definition 1.24 that the solution of the initial value problem of fractional differential equation (5) is equivalent to the fixed point of operator A defined by (4.2). The fuzzy power compression mapping fixed point theorem is used to find the fixed point of operator A. If we define

$$D = \{\widetilde{u} \in C(I, R_F) : |\widetilde{u}(t) \ominus \widetilde{u}_0| \leqslant b, t \in I\},$$

then D is a bounded fuzzy convex closed set in $C(I, \mathcal{T})$

First of all, we can prove $A : D \to D$. In fact, for $\forall u \in D$ and $t \in I$, according to Proposition 1.4 and (4.2), it can be obtained by calculation that

$$|A\underline{u}(t) - \underline{u}_0| = \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s, \underline{u}(s))ds$$
$$\leqslant \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} |f(s, \underline{u}(s)) \ominus 0|ds$$
$$\leqslant \frac{M}{\Gamma(\lambda)} \frac{t^\lambda}{\lambda} = \frac{Mt^\lambda}{\Gamma(\lambda+1)} \leqslant b.$$

We can obtain in the same way $|A\overline{u}(t) - \overline{u}_0| \leqslant b$, so we get $|Au(t) \ominus u_0| \leqslant b$ by Definition 1.18, furthermore $Au(D) \in D$,that is $A : D \to D$

Additionally, we can prove that $A : D \to D$ is a power compression mapping.In fact, for $\forall u_1, u_1 \in D$, according to Proposition 1.4, (4.2), and Lispchitz continuity of nonlinear fuzzy number value function $f(t, u(t))$ about the second argument,it can be obtained by calculation that

$$|A\underline{u}_2(t) - A\underline{u}_1(t)| \leqslant \frac{1}{\Gamma(\lambda)} \int_0^t |(t-s)^{\lambda-1}(f(s, \underline{u}_2(s)) - f(s, \underline{u}_1(s)))|ds$$
$$\leqslant \frac{L}{\Gamma(\lambda)} \frac{t^\lambda}{\lambda} \|\underline{u}_2 - \underline{u}_1\| = \frac{Lt^\lambda}{\Gamma(\lambda+1)} \|\underline{u}_2 - \underline{u}_1\|.$$

We can obtain in the same way

$$|A\overline{u}_2(t) - A\overline{u}_1(t)| \leqslant \frac{Lt^{\lambda}}{\Gamma(\lambda+1)}\|\overline{u}_2 - \overline{u}_1\|.$$

By Definition 2.2, we get

$$|A\widetilde{u}_2(t) \ominus A\widetilde{u}_1(t)| \leqslant \frac{Lt^{\lambda}}{\Gamma(\lambda+1)}D_C(u_2, u_1). \tag{4.3}$$

We can obtain, according to Proposition 1.4, (4.2), (4.3), and Lispchitz continuity of nonlinear fuzzy number value function $f(t, u(t))$, with respect to the second argument,

$$\begin{aligned}
&|A^2\underline{u}_2(t) - A^2\underline{u}_1(t)| \\
=&|\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}(f(s, A\underline{u}_2(s)) - f(s, A\underline{u}_1(s)))ds| \\
\leqslant&\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}|(A\underline{u}_2(s) - A\underline{u}_1(s)|ds \\
\leqslant&\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}\frac{Ls^{\lambda}}{\Gamma(\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|ds \\
=&\frac{L^2}{\Gamma(\lambda)\Gamma(\lambda+1)}\int_0^t (t-s)^{\lambda-1}s^{\lambda}ds\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^2}{\Gamma(\lambda)\Gamma(\lambda+1)}\int_0^t (t-st)^{\lambda-1}s^{\lambda}t^{\lambda}tds\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^2t^{2\lambda}}{\Gamma(\lambda)\Gamma(\lambda+1)}\int_0^t (1-s)^{\lambda-1}s^{\lambda}ds\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^2t^{2\lambda}}{\Gamma(\lambda)\Gamma(\lambda+1)}B(\lambda+1, \lambda)\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^2t^{2\lambda}}{\Gamma(\lambda)\Gamma(\lambda+1)}\frac{\Gamma(\lambda)\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^2t^{2\lambda}}{\Gamma(2\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|.
\end{aligned} \tag{4.4}$$

Suppose that it holds for $n = k - 1$,and we get

$$|A^{k-1}\underline{u}_2(t) - A^{k-1}\underline{u}_1(t)| \leqslant \frac{(Lt^{\lambda})^{k-1}}{\Gamma((k-1)\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|. \tag{4.5}$$

When $n = k$, according to Proposition 1.4, (4.2), (4.5), and Lispchitz continuity of nonlinear fuzzy number value function $f(t, u(t))$ about the second argument, we have

$$
\begin{aligned}
&|A^k\underline{u}_2(t) - A^k\underline{u}_1(t)| \\
=&|\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}(f(s, A^{k-1}\underline{u}_2(s)) - f(s, A^{k-1}\underline{u}_1(s)))ds| \\
\leqslant&\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}|(A^{k-1}\underline{u}_2(s) - A^{k-1}\underline{u}_1(s)|ds \\
\leqslant&\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}\frac{(Ls^\lambda)^{k-1}}{\Gamma((k-1)\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|ds \\
=&\frac{L^k}{\Gamma(\lambda)\Gamma((k-1)\lambda+1)}\int_0^t (t-s)^{\lambda-1}s^{(k-1)\lambda}ds\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^k t^{k\lambda}}{\Gamma(\lambda)\Gamma((k-1)\lambda+1)}B((k-1)\lambda+1, \lambda)\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^k t^{k\lambda}}{\Gamma(\lambda)\Gamma((k-1)\lambda+1)}\frac{\Gamma((k-1)\lambda+1)\Gamma(\lambda)}{\Gamma(k\lambda+1)}\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^k t^{k\lambda}}{\Gamma(k\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|.
\end{aligned} \tag{4.6}
$$

By mathematical induction, for every positive integer n and $t \in I$,we have

$$|A^n\underline{u}_2(t) - A^n\underline{u}_1(t)| \leqslant \frac{(Lh^\lambda)^n}{\Gamma(n\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|.$$

We can obtain in the same way:

$$|A^n\overline{u}_2(t) - A^n\overline{u}_1(t)| \leqslant \frac{(Lh^\lambda)^n}{\Gamma(n\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|.$$

By Definition 1.18,we get

$$|A^n u_2(t) \ominus A^n u_1(t)| \leqslant \frac{(Lh^\lambda)^n}{\Gamma(n\lambda+1)}D_C(u_2, u_1),$$

which means that

$$D_C(A^n u_2, A^n u_1) \leqslant \frac{(Lh^\lambda)^n}{\Gamma(n\lambda+1)}D_C(u_2, u_1). \tag{4.7}$$

We have,by the Stirling formula,

$$\Gamma(n\lambda+1) = \sqrt{2\pi n\lambda}(\frac{n\lambda}{e})^{n\lambda}e^{\frac{\theta}{12n\lambda}}, 0 < \theta < 1,$$

then

$$\frac{(Lh^{\lambda})^n}{\Gamma(n\lambda+1)} = \frac{(Lh^{\lambda})^n}{\sqrt{2\pi n\lambda}(\frac{n\lambda}{e})^{n\lambda}e^{\frac{\theta}{12n\lambda}}} \to 0\ (n \to \infty).$$

So,there exists a sufficiently large integer n_0 such that

$$\frac{(Lh^{\lambda})^{n_0}}{\Gamma(n_0\lambda+1)} < 1. \tag{4.8}$$

By combining (4.7) and (4.8), it can be obtained

$$D_C(A^{n_0}u_2, A^{n_0}u_1) < D_C(\widetilde{u}_2, \widetilde{u}_1).$$

That is. A^{n_0} is a fuzzy compression operator, so A is a fuzzy power compression operator. Therefore according to Lemma 1.10, operator A has a unique fixed point $\widetilde{u} \in D$ .This fixed point is the unique (i)-solution of the initial value problem of fuzzy fractional differential equation (5) in the interval $I = [0, h]$ □

Theorem 4.2. *If all the assumptions of Theorem 4.1 are satisfied, then the initial value problem of fuzzy fractional differential equation* (5) *has a unique (i)-solution on* $I^{'} = [0, h^{'}]$ *where*

$$h^{'} < \min\left\{a, (\frac{b\Gamma(\lambda+1)}{M})^{\frac{1}{\lambda}}, (\frac{\Gamma(\lambda+1)}{L})^{\frac{1}{\lambda}}\right\}, \quad M = \max_{(t,\|\widetilde{u}\|)\in S} |\widetilde{f}(t, \widetilde{u}(t)) \ominus \widetilde{0}|.$$

Proof. If u is (i)-gH differentiable, we can define the operator $A : C(I, \mathbb{R}_{\mathcal{F}}) \to C(I, \mathbb{R}_{\mathcal{F}})$ as follows:

$$A\widetilde{u}(t) = \widetilde{u}_0 \oplus \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} \widetilde{f}(s, \widetilde{u}(s))ds.$$

It is shown by Definition 1.24 that the solution of the initial value problem of fuzzy fractional differential equation (5) is equivalent to the fixed point of operator A defined by (4.9). In the following, we find the fixed point of operator A by use of the compression mapping fixed point theorem. If we define $D' = \{u \in C(I^{'}, \mathbb{R}_{\mathcal{F}}) : |\widetilde{u}(t) \ominus \widetilde{u}_0| \leqslant b, t \in I^{'}\}$,then $D^{'}$ is a bounded fuzzy convex closed set in $C(I^{'}, \mathbb{R}_{\{})$. To begin, we know from the proof of Theorem 4.1 that $A : D^{'} \to D^{'}$. Now we prove that $A : D^{'} \to D^{'}$ is a fuzzy compression operator. For every $u_1, u_2 \in D^{'}$, we get

$$
\begin{aligned}
&|A\underline{u}_2(t) - A\underline{u}_1(t)| \\
=&|\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}(f(s,\underline{u}_2(s)) - f(s,\underline{u}_1(s)))ds| \\
\leqslant&\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}|(f(s,\underline{u}_2(s)) - f(s,\underline{u}_1(s)))|ds \\
\leqslant&\frac{L}{\Gamma(\lambda)}\frac{t^\lambda}{\lambda}\|\underline{u}_2 - \underline{u}_1\| \\
<&\frac{L(h^{'})\lambda}{\Gamma(\lambda+1)}\|\underline{u}_2 - \underline{u}_1\| < \|\underline{u}_2 - \underline{u}_1\|.
\end{aligned} \tag{4.9}
$$

We can obtain in the same way:

$$|A\overline{u}_2(t) - A\overline{u}_1(t)| < \|\overline{u}_2 - \overline{u}_1\|.$$

Therefore,by Definition 1.18, we get

$$D_C(A\widetilde{u}_2, A\widetilde{u}_1) < D_C(\widetilde{u}_2, \widetilde{u}_1).$$

That is, $A : D^{'} \to D^{'}$ is a fuzzy compression operator. Therefore, according to Lemma 1.9, operator A has a unique fixed point $u \in D^{'}$, and u is the unique (i)-solution of the initial value problem of fuzzy fractional differential equation (5) in the interval $I^{'} = [0, h^{'}]$ □

If u is (ii)-gH differentiable, that is,

$$\widetilde{u}(t) = \widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)}\int_0^t \frac{\widetilde{f}(s,\widetilde{u}(s))}{(t-s)^{1-\lambda}}ds,\ t \in [a,b],$$

we can come to the following conclusion: Theorems 4.3 and 4.4.

Theorem 4.3. *Suppose $C([0,a],\mathbb{R}_{\mathcal{F}}) \neq \emptyset$, and for any $\widetilde{u} \in C([0,a],\mathbb{R}_{\mathcal{F}})$ïijŇ $\widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)}\int_0^t \frac{\widetilde{f}(s,\widetilde{u}(s))}{(t-s)^{1-\lambda}}ds$ exists for all $t \in [0,a]$. If all the conditions in Theorem 4.1 are satisfied, then the initial value problem* (5) *has a unique (ii)-solution*

Proof. If u is (ii)-gH differentiable, we can define the operator $A : C(I,\mathbb{R}_{\mathcal{F}}) \to C(I,\mathbb{R}_{\mathcal{F}})$ as follows

$$A\widetilde{u}(t) = \widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}\widetilde{f}(s,\widetilde{u}(s))ds,$$

and the remaining proof is similar to Theorem 4.1. □

Theorem 4.4. *Suppose $C([0,a],\mathbb{R}_{\mathcal{F}}) \neq \emptyset$ and for any $u \in C([0,a],\mathbb{R}_{\mathcal{F}})$ $\widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)}\int_0^t \frac{\widetilde{f}(s,\widetilde{u}(s))}{(t-s)^{1-\lambda}}ds$ exists for all $t \in [0,a]$. If all the conditions in Theorem 4.2 are satisfied, then the initial value problem* (5) *ha a unique (ii)-solution*

Proof. If u is (ii)-gH differentiable, we can define the operator $A : C(I,\mathbb{R}_{\mathcal{F}}) \to C(I,\mathbb{R}_{\mathcal{F}})$ as follows:

$$A\widetilde{u}(t) = \widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}\widetilde{f}(s,\widetilde{u}(s))ds,$$

and the remaining proof is similar to Theorem 4.2. □

4.3 Examples

The existence and uniqueness of the solution to the initial value problem of the fractional-order differential equation is an important mathematical concept. Studying the fractional-order differential equation initial value problem can help us understand the actual situation better and solve the problem accurately. However for this kind of problem, the theory of fuzzy metric space is still lacking, and the lack of theory greatly limits the practical application of fractional-order ordinary differential equations In the face of some complex environments, the usual practice is to combine the solution of the system to minimize the complexity of the initial value problem to reduce the complexity of the solution. However,this operation will lead to the loss of a lot of information, so the accuracy and effectiveness of the actual problem cannot be guaranteed. Therefore, we study the existence and uniqueness of initial value problems of fractional ordinary differential equations in fuzzy metric space. By the following example,to show the validity of the derived results,an appropriate example and applications are discussed in this section

Example 4.1. *Consider the initial problem of fractional fuzzy differential equations as follows.*

$$\begin{cases} {}^cD_t^{\frac{1}{2}}\widetilde{u}(t) = \frac{1}{1\oplus|u(t)|} \odot \cos t, t > 0, \\ \widetilde{u}(0) = \widetilde{u}_0 \in \mathbb{R}_{\mathcal{F}}. \end{cases} \tag{4.10}$$

The conditions corresponding to Theorem 4.1 yield the following data information: $\lambda = \frac{1}{2}, \widetilde{f}(t,\widetilde{u}(t)) = \frac{1}{1\oplus|u(t)|} \odot \cos t, a = 1$ and $b = 1$ .Set $u_0 = (1,2,3) \in \mathbb{R}_{,\mathcal{F}}$,then $\alpha = 0$ level set $(u_0)_0 = [1,3]$ of symmetric triangular fuzzy number u_0 .It can be obtained that the construction of fuzzy-valued function $f(t,\widetilde{u}(t))$ is continuous over the rectangular region $S = \{(t,||\widetilde{u}||)|0 \leqslant t \leqslant 1, |\widetilde{u} \ominus \widetilde{u}_0| \leqslant 1\}$ .Let's verify that the function $f(t,\widetilde{u}(t))$ satisfies the Lispchitz condition about u on the rectangular *region* $S = \{(t, \|\widetilde{u}\|)|0 \leqslant t \leqslant 1, |\widetilde{u} \ominus \widetilde{u}_0| \leqslant 1\}$ *as follows. By Definition 1.23, we can obtain*

$$\begin{aligned}
&\|f(t, \underline{u}_2(t)) - f(t, \underline{u}_1(t))\| \\
=&\|\frac{1}{1+|\underline{u}_2(t)|}\cos t - \frac{1}{1+|\underline{u}_1(t)|}\cos t\| \\
=&\|\frac{1}{1+|\underline{u}_2(t)|} - \frac{1}{1+|\underline{u}_1(t)|}\| \cdot |\cos t| \\
=&\|\frac{|\underline{u}_2(t)| - |\underline{u}_1(t)|}{(1+|\underline{u}_1(t)|)\cdot(1+|\underline{u}_2(t)|)}\| \cdot |\cos t| \\
\leqslant&\|\underline{u}_2 - \underline{u}_1\| \cdot |\cos t|.
\end{aligned} \tag{4.11}$$

$$\|f(t, \overline{u}_2(t)) - f(t, \overline{u}_1(t))\| \leqslant \|\overline{u}_2 - \overline{u}_1\| \cdot |\cos t|. \tag{4.12}$$

Thus, the initial value problem of above fuzzy fractional differential equation (4.10) *has a unique (i)-solution on the fuzzy interval* $I = [0, h]$ *where*

$$h = \min\left\{1, (\frac{\Gamma(\frac{1}{2}+1)}{M})^2\right\} = 0.785, \quad M = \max_{(t,u)\in S} |\widetilde{f}(t, \widetilde{u}(t)) \ominus \widetilde{0}| = 2. \tag{4.13}$$

Example 4.2. *Consider the initial problem of fractional fuzzy differential equations as follows*

$$\begin{cases} {}^cD_t^{\lambda}\widetilde{u}(t) = 2\sin\mu(A) \oplus e^{tA}, 0 < \lambda < 1, t > 0, \\ \widetilde{u}(0) = \widetilde{0} \in \mathbb{R}_{\mathcal{F}}, \end{cases} \tag{4.14}$$

where $\widetilde{f}(t, \widetilde{u}(t)) = 2\sin\mu(A) \oplus e^{tA}$ *and* $A =$ *(1,2,3)* $\in \mathbb{R}_{,\mathcal{F}}$ *is a symmetric triangular fuzzy number Clearly, it can be obtained that the construction of fuzzy-valued function* $\widetilde{f}(t, \widetilde{u}(t))$ *is continuous over the rectangular region* $S = \{(t, \|\bar{u}\|)|0 \leqslant t \leqslant 1, |\widetilde{u} \ominus \widetilde{u}_0| \leqslant 1\}$ *, Let's verify that the function* $f(t, \widetilde{u}(t))$ *satisfies the Lispchitz condition about* u *with* $L = 2$ *on the rectangular region* $S = \{(t, \|\bar{u}\|)|0 \leqslant t \leqslant 1, |\widetilde{u} \ominus \widetilde{u}_0| \leqslant 1\}$ *as follows. By definition 1.23, we can obtain*

$$\begin{aligned}
&\|f(t, \underline{u}_2(t)) - f(t, \underline{u}_1(t))\| \\
=&\|(2\sin\underline{u}_2(A) + e^{tA}) - (2sin\underline{u}_1(A) + e^{tA})\| \\
\leqslant&2\|\underline{u}_2 - \underline{u}_1\|.
\end{aligned} \tag{4.15}$$

We can get in the same way

$$\|f(t, \overline{u}_2(t)) - f(t, \overline{u}_1(t))\| \leqslant 2\|\overline{u}_2 - \overline{u}_1\|. \tag{4.16}$$

Thus equation (4.15) *has a unique solution according to the Theorem 4.1.*

As can be seen from the above example,we do not need to idealize the actual information. We can directly draw the desired conclusion, reducing the error value of the research on the actual problem.

Chapter 5

Conclusion

In this dissertation, we proved the existence and uniqueness of the solution of fuzzy fractional differential equations. In the first work the study established the equivalence between fuzzy fractional differential equations (FFDEs) and their integral forms, enabling the application of fixed point theorems for proving existence and uniqueness.

In the second work under the suitable restrictions on the function f we provide a result that tells when to expect a unique solution for the fuzzy fractional initial value problem (3.1) using a variant 1.6 of the Banach contraction principle stated in theorem 1.5. An example to support the theory is presented.

In the third work, the initial value problem to a class of fractional fuzzy differential equations is studied. We obtain new existence and uniqueness of solutions for initial value problems of fractional ordinary differential equations in fuzzy metric space by means of the fuzzy power compression mapping principle and the related properties of the Gamma function under the assumption that the nonlinear functions satisfy the Lispchitz condition. It is proved that the existence interval of the solution is larger than that of directly using the Banach compression mapping theorem. However, due to the lack of research on this kind of problem in fuzzy metric space, the calculation involved is relatively difficult, and few practical application examples can be found. In the future, we will continue to promote this work, strive to combine theoretical research with practice, and make breakthroughs in numerical simulation.

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