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الإهداء

{وَآخِرُ دَعْوَاهُم أَنِ الحَمْدُ للهِ رَبِّ العَالَمِين} سورة يونس -10

من قال أنا لها نالها وأنا لها وإن أبت رغماً عنها أتيتُ بها نلتها وعانقت اليوم مجداً عظيماً ، فعلتها بعد أن كانت مستحيلة ، كانت دروباً قاسية وطرقاً خسرت بها الكثير ولكني وصلت لحمد لله حباً وشكراً وإمتناناً ، الحمد لله الذي بفضله أدركت اسمى الغايات أنظر لنفسي ونجاحي كالذي ينظر إلى مُعجزته ، إلى الحلم الذي طال إنتظاره ، تحقق بفضل الله وأصبح واقعاً أفتخر به أهدي وبكل حب بحث تخرجي

إلى نفسي العظيمة القوية التي تحملت كل العثرات وأكملت رغم الصعوبات ها أنا اليوم أتوج لحظاتي الأخيرة في ذلك الطريق الذي كان يحمل في باطنه العثرات والأشواك ورغما عنها ظلت قدمي تخطو بكل صبر وطموح وعزيمه وتفائل وحسن ظن بالله وكم من الأيام مرت شعرت بثقلها و مرارتها ولكن لم تعيقني بل كانت ذكرى تمر لنيل الاحلام.

إلى ذالك الرجل العظيم الذي أحمل إسمه بكل فخريردد إسمي عاليا في عنان السماء حاملا شرف لقبك وبكل اعتزاز انا لهذا الرجل إبنة إلى من كلله الله بالهيبة والوقار إلى من غرس في روحي مكارم الأخلاق داعمي الأول في مسيرتي إلى من كان عمودي الفقري الذي ساندني بكل حب في ضعفي الذي أخرج أجمل مافي داخلي وشجعني دائما للوصول إلى طموحاتي إلى أول من إنتظر هذه اللحظة ليفتخر بي إلى سندي و حبيبي و رفيق عمري.

{والدي رحمه الله }

إلى التي تعجز الكلمات عن وصفها، لمن جعل الله الجنة تحت أقدامها الداعمة الأولى والأبدية، ملاكي الطاهرإلى من كان وجودها يمدني بالسعي دون الملل إلى التي ظلت دعواتها تضم إسمي دائما القلب الحنون معلمتي الأولى إلى الجنة التي أنجبتني و اليد الخفية التي أزالت عن طريقي الاشواك ومن تحملت كل لحظة ألم ممرت بها وساندتني عند ضعفي محبوبتي وملهمتي ممتنة إلى الله إصطفاك من البشر أما لي.

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إلى شريكة الصبا ورفيقة الخندق التي تقاوم الحياة بالضحك ، ملاكي الحارس التي كانت دوما موضع الإتكاء في عثرات حياتي و ربيع قلبي إلى التي لم تنجبها أمي إستودعتك الله الذي لا تضيع ودائعه

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إلى من لا يضاهيهما أحد في الكون، إلى من أمرنا الله ببرهما، إلى من بذل الكثير، وقدم ما لا يمكن أن يرد،

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شكرا لقلوبكم الطيبة، وابتسامتكم في كل وقت، كنتم الدفىء في أوقات التعب

إلى أصدقائي وزملائي

كنتم عونا وسندا لي في كل لحظة، لا تنسى الذكريات التي جمعتنا

Contents

INTRODUCTION

The objective of this work is to present some results from international scientific articles on Bullen and Milne inequalities. All integral inequalities are considered in the Lebesgue space L_p, where $p \geq 1$. The techniques used are the properties of integral calculus, integration by parts, convex function, k-Lipschitzian functions and bounded functions.

This memory is composed of four chapters.

In **the first chapter** (titled **Preliminary**), we provide notations and definitions for

1. L^p functions, k-Lipschitzian functions, bounded functions, Hölder's inequality.
2. Convex functions, specific cases of convexity: s-convex functions, P-convex functions and Godunova-levin functions.
3. h-convex function, specific cases of h-convexity.
4. B-function and special cases.

The second chapter is devoted to a result on Hermite-Hadamard inequality. We give the two versions on Hermite-Hadamard inequality via convex functions and Hermite-Hadamard inequality via h-convex functions.

In **the third chapter**, we present results related to Bullen inequality via h-convex functions, Hölder inequality, k-lipschitzian functions and bounded functions .

In **the fourth chapter**, we study results related to Milne inequality via h-convex functions, Hölder inequality, k-lipschitzian functions and bounded functions .

Chapter 1

Preliminary

1.1 Special sets of functions

1.1.1 L^p functions

Definition 1.1. *Let $\infty \leq a < b \leq +\infty$ and $1 \leq p < +\infty$ The space $L^p([a,b])$ is the classes of real functions f measurable such that*

$$\int_a^b \mid f(t) \mid^p dt < \infty.$$

The norme is defined by :

$$\|f\|_p = \left(\int_a^b \mid f(t) \mid^p dt\right)^{\frac{1}{p}}.$$

1.1.2 k-Lipschitzian functions

Definition 1.2. *Let $I \subseteq \mathbb{R}$, $k \in \mathbb{R}^+$ and $f : I \to \mathbb{R}$ be a real function. We say that f is k-Lipschitzian on I, if for all $x, y \in I$*

$$\mid f(x) - f(y) \mid \leq k \mid x - y \mid .$$

1.1.3 Bounded functions

Let $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ be a real function. We say that f is bounded on I, if there exist $m, M \in \mathbb{R}$ such that for all $x \in I$:

$$m \leq f(x) \leq M.$$

The caractiristic form of bounded function is: there exist $C > 0$ such that

$$\forall x \in I :\mid f(x) \mid \leq C.$$

1.2 Integral inequalities :

1.2.1 Hölder's inequality:

Let $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p([a,b])$, $g \in L^q([a,b])$ then

$$\int_a^b \mid f(x)g(x) \mid dx \leq \left(\int_a^b \mid f(x) \mid^p dx \right)^{\frac{1}{p}} \left(\int_a^b \mid g(x) \mid^q \right)^{\frac{1}{q}}.$$

Weighted Hölder's inequality (power mean inequality)

Let $p \geq 1$, $f \in L^p([a,b])$ and w be a weighted function (positive and measurable function) then

$$\int_a^b w(x) \mid f(x) \mid dx \leq \left(\int_a^b w(x)dx \right)^{1-\frac{1}{p}} \left(\int_a^b w(x) \mid f^p(x) \mid dx \right)^{\frac{1}{p}}.$$

Proof. Applying the Hölder's inequality for $p \geq 1$ gets

$$\begin{aligned} \int_a^b w(x) \mid f(x) \mid dx &= \int_a^b \left(w^{\frac{1}{q}}(x) w^{\frac{1}{p}}(x) \right) \mid f(x) \mid dx \\ &\leq \left(\int_a^b \left(w^{\frac{1}{q}}(x) \right)^q dx \right)^{\frac{1}{q}} \left(\int_a^b \left(w^{\frac{1}{p}}(x) \left| f(x) \right|^p dx \right) \right)^{\frac{1}{p}} \\ &= \left(\int_a^b w(x)dx \right)^{1-\frac{1}{p}} \left(\int_a^b w(x) \mid f(x) \mid^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

□

Integral by parts

Let $[a,b] \subseteq \mathbb{R}$ and u, v be a two function of class $\mathcal{C}^1([a,b])$:

$$\int_a^b u(t)v'(t)dt = [u(t)v(t)]_a^b - \int_a^b u'(t)v(t)dt.$$

Change of variables:

Let $[a,b] \subseteq \mathbb{R}$ and φ be a function of class $\mathcal{C}^1([a,b])$ and f a continuous function on $\varphi([a,b])$, then

$$\int_a^b f(\varphi(t))\varphi'(t)dt = \int_{\varphi(a)}^{\varphi(b)} f(x)dx.$$

If φ is bijective mapping: $[a,b] \to \varphi([a,b])$, then

$$\int_a^b f(x)dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\varphi'(t)dt.$$

1.3 Convex functions

Convexity theory offers powerful processes and notions for dealing with a wide range of pure and applied mathematics problems. Convex functions have been used in various mathematical disciplines, leading to new discovery of many inequalities in the literature.

Definition 1.3. *The function: $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said convex if for all $x, y \in \mathbb{I}$ and all $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$ we have:*

$$f(\alpha\, x + \beta\, y) \leq \alpha\, f(x) + \beta\, f(y). \tag{1.1}$$

f is said concave function if $(-f)$ is a convex function.

There is other type of formulations, such that for all $a, b \in \mathbb{I}$ and all $t \in [0,1]$

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b), \tag{1.2}$$

$$f\left(\frac{t}{2}a + \left(1 - \frac{t}{2}\right)b\right) \leq \frac{t}{2}f(a) + \left(1 - \frac{t}{2}\right)f(b), \tag{1.3}$$

and

$$f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right) \leq \left(\frac{1+t}{2}\right)f(a) + \left(\frac{1-t}{2}\right)f(b). \tag{1.4}$$

Graphical interpretation

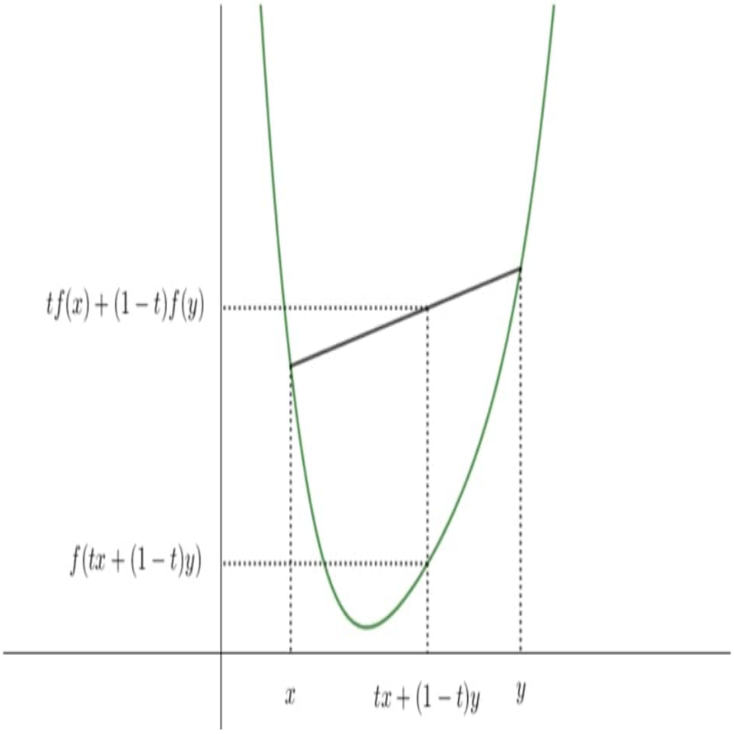

Figure 1.1: graph 1

For any two points $M(x, f(x))$ and $N(y, f(y))$ on the graph, the segment $[MN]$ appears on this graph.

1.4 Specific cases of convexity

1.4.1 s-convex functions

Definition 1.4. *[1] Let $s \in (0, 1]$. A function $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be s-convex in the second sense if:*

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b),$$

for all $a, b \in \mathbb{R}$ and $t \in [0, 1]$.

Definition 1.5. *Let $s \in (0, 1]$. A function $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be s-convex in the first sense if:*

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t^s) f(b),$$

for all $a, b \in \mathbb{R}$ *and* $t \in [0,1]$.

Remark 1.1. *For* $s = 1$*, the* s*-convexity becomes the classical convexity.*

1.4.2 P-functions

Definition 1.6. *[2, 3] We say that* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *is a* P*-functions, or that* f *belongs to the class* $P(I)$*, if for all* $a, b \in I$*,* $t \in [0,1]$*, we have*

$$f(ta + (1-t)b) \leq f(a) + f(b).$$

1.4.3 Godunova-levin functions

Definition 1.7. *[4] We say that* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *is a Godunova-Levin function, or that* f *belongs to the class* $\varrho(I)$ *if for all* $a, b \in I$ *and* $t \in (0,1)$*,*

$$f(ta + (1-t)b) \leq \frac{f(a)}{t} + \frac{f(b)}{1-t}.$$

1.5 h-convex functions

Definition 1.8. *[5] Let* $h : J \subset \mathbb{R} \longrightarrow \mathbb{R}$*,where* $(0.1) \subset J$ *be a positive function. We say that* $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ *is an* h*-convex function, if for all* $a, b \in I$ *and* $t \in [0,1]$ *we have*

$$f(ta + (1-t)b) \leq h(t)f(a) + h(1-t)f(b). \tag{1.5}$$

If the inequality (1.5) *is reversed, then* f *is called* h*-concave function .*

1.5.1 Special cases of h-convexity

1. If $h(t) = t$, the definition reduces to classical convexity .
$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b).$$
2. If $h(t) = t^s$ where $s \in (0,1]$, the definition reduces to s-convex functions in the second sense.
$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b).$$
3. If $h(t) = \frac{1}{t}$, the definition reduces to Godunova-Levin function with $t \in (0,1)$.
$$f(ta + (1-t)b) \leq \frac{f(a)}{t} + \frac{f(b)}{1-t}.$$
4. If $h(t) = 1$, the definition reduces to class P-functions.
$$f(ta + (1-t)b) \leq f(a) + f(b).$$

1.6 B-function

Recently, in [6]-[7] the authors presented a new class of functions called B-function defined as:

Definition 1.9. *Let $g : [0, \infty) \to \mathbb{R}$ be a non-negative function. The function g is called a B-function if*

$$g(x-a) + g(b-x) \leq 2g\left(\frac{a+b}{2}\right) \tag{1.6}$$

where $a < x < b$ with $a, b \in [0, \infty)$.

If the inequality (1.6) is reversed, g is called A-function, or that g belongs to the class $A(a, b)$.
If we have equality in (1.6), g is called AB-function, or that g belongs to the class $AB(a, b)$.

Proposition 1.1. *Let $h : (0, 1) \to \mathbb{R}$ be a positive function. The function h is a B-function, if only if we have: for all $\lambda \in (0, 1)$*

$$h(\lambda) + h(1-\lambda) \leq 2\,h\left(\frac{1}{2}\right). \tag{1.7}$$

Remark 1.2. *There is other type of formulations, such that for all $t \in [0, 1]$*

$$h\left(\frac{t}{2}\right) + h\left(1 - \frac{t}{2}\right) \leq 2h\left(\frac{1}{2}\right), \tag{1.8}$$

and

$$h\left(\frac{1+t}{2}\right) + h\left(\frac{1-t}{2}\right) \leq 2h\left(\frac{1}{2}\right). \tag{1.9}$$

1.6.1 Special cases

1 If $h(t) = t$, then h is AB-function, B-function and A-function.

2 If $h(t) = 1$, then h is AB-function, B-function and A-function.

3 If $h(t) = t^s$ with $s \in (0, 1]$, then h is B-function.

4 $h(t) = \frac{1}{t}$ with $t \in (0, 1)$, then h is A-function.

Proof. 1. Taking $h(t) = t$ in the inequality (1.7) gets

$$t + (1-t) = 1 \text{ (} h \text{ is } AB\text{-function)} \;\Rightarrow\; t + (1-t) \leq 1 \text{ (} h \text{ is } B\text{-function)}.$$

$$t + (1-t) = 1 \text{ (} h \text{ is } AB\text{-function)} \;\Rightarrow\; t + (1-t) \geq 1 \text{ (} h \text{ is } A\text{-function)}.$$

2. applying $h(t) = 1$ in the inequality (1.7) become

$$1 + 1 = 2 \text{ (} h \text{ is } AB\text{-function)} \;\Rightarrow\; 1 + 1 \leq 2 \text{ (} h \text{ is } B\text{-function)}.$$

$$1 + 1 = 2 \text{ (} h \text{ is } AB\text{-function)} \;\Rightarrow\; 1 + 1 \geq 2 \text{ (} h \text{ is } A\text{-function)}.$$

3. Putting $h(t) = t^s$ for $s \in (0, 1]$ in the inequality (1.7) yields

$$t^s + (1-t)^s \leq 2\left(\frac{1}{2}\right)^s.$$

To proof the above result, we need the following famous inequality:
let $a, b > 0$, $p > 0$,

$$\min(1, 2^{p-1})\left(a^p + b^p\right) \leq (a+b)^p \leq \max(1, 2^{p-1})\left(a^p + b^p\right). \tag{1.10}$$

The above inequalities (1.10) can be reformulated as follows:

$$\min(1, 2^{1-p})(a+b)^p \leq a^p + b^p \leq \max(1, 2^{1-p})(a+b)^p. \tag{1.11}$$

For $0 < p \leq 1$, we have

$$a^p + b^p \leq 2^{1-p}(a+b)^p,$$

taking $a = t$, $b = 1-t$ and $p = s$ gives

$$t^s + (1-t)^s \leq 2^{1-s},$$

thus

$$t^s + (1-t)^s \leq 2\left(\frac{1}{2}\right)^s.$$

We deduce that $h(t) = t^s$ is a B-function.

4. Setting $h(t) = \frac{1}{t}$ with $t \in (0,1)$ in the reverse inequality (1.7) gives

$$\frac{1}{t} + \frac{1}{1-t} \geq 4.$$

For $t \in (0,1)$, we have

$$\begin{aligned} \frac{1}{t} + \frac{1}{1-t} - 4 &= \frac{1 - 4t + 4t^2}{t(1-t)} \\ &= \frac{(2t-1)^2}{t(1-t)} \geq 0, \end{aligned}$$

then

$$\frac{1}{t} + \frac{1}{1-t} \geq 4.$$

It results that h is A-function.

□

Chapter 2

Hermite-Hadamard inequality

The Hermite-Hadamard inequality is a significant mathematical theorem that determines the integral average value of a convex function over a designated interval. The Hermite-Hadamard inequality (H.H in abbreviated form) is employed in research pertaining to numerical integration, optimization, and it play an essential role in convexity theory.

This inequality gives for convex function, an estimate of the integral mean $\frac{1}{b-a}\int_a^b f(x)dx$ in an interval $[a, b]$ in relation to the average image $f(\frac{a+b}{2})$ and the average of the images $\frac{f(a)+f(b)}{2}$.

Regarding Hermite–Hadamard inequalities, please refer to the following references: [6],[7], [8],[9],[10], [11],[12] and [13].

2.1 Hermite-Hadamard inequality via convex function

In the following, $a, b \in \mathbb{R}$ with $a < b$.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function ,then the following inequality is verified*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2.1}$$

Proof. Let $x \in [a, b]$, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{a}{2}+\frac{b}{2}+\frac{x}{2}-\frac{x}{2}\right) \\ &= f\left(\frac{1}{2}(a+b-x)+\frac{1}{2}x\right), \end{aligned}$$

since $x \in [a,b]$, we get

$$\begin{aligned} a \leq x \leq b &\Rightarrow -b \leq -x \leq -a \\ &\Rightarrow a+b-b \leq a+b-x \leq a+b-a \\ &\Rightarrow a \leq a+b-x \leq b \\ &\Rightarrow (a+b-x) \in [a,b]. \end{aligned}$$

It is assumed that f is a convex function on $[a,b]$, we deduce

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}(a+b-x)+\frac{1}{2}x\right) \\ &\leq \frac{1}{2}f(a+b-x)+\frac{1}{2}f(x), \end{aligned}$$

so we have

$$2f\left(\frac{a+b}{2}\right) \leq f(a+b-x)+f(x). \tag{2.2}$$

Given that f is a convex function, putting $x = ta+(1-t)b$ for $t \in [0,1]$ gives

$$\begin{aligned} f(a+b-x)+f(x) &= f(a+b-ta-(1-t)b)+f(ta+(1-t)b) \\ &= f((1-t)a+tb)+f(ta+(1-t)b) \\ &\leq (1-t)f(a)+tf(b)+tf(a)+(1-t)f(b) \\ &= f(a)+f(b), \end{aligned}$$

hence

$$f(a+b-x)+f(x) \leq f(a)+f(b). \tag{2.3}$$

From (2.2) and (2.3), we deduce

$$2f\left(\frac{a+b}{2}\right) \leq f(a+b-x)+f(x) \leq f(a)+f(b). \tag{2.4}$$

Integrating over $x \in [a,b]$ the above inequality (2.4) yeilds

$$2(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(a+b-x)dx+\int_a^b f(x)dx \leq (b-a)(f(a)+f(b)). \tag{2.5}$$

Changing the variable $s = (a + b - x)$, we have

$$\begin{aligned}\int_a^b f(a+b-x)dx &= \int_b^a f(s)(-ds)\\ &= \int_a^b f(s)ds\\ &= \int_a^b f(x)dx,\end{aligned}$$

we obtain

$$\int_a^b f(a+b-x)dx = \int_a^b f(x)dx. \tag{2.6}$$

Applying the equality (2.6) in the inequality (2.5) gives

$$2(b-a)f\left(\frac{a+b}{2}\right) \leq 2\int_a^b f(x)dx \leq (b-a)(f(a)+f(b)),$$

one devide by $2(b-a)$, we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

wich gives the desired result. □

2.2 Hermite-Hadamard inequality via h-convex functions

This section presents a generalization of the previously mentioned Hermite-Hadamard inequality using the h-convex functions. In the following $a, b \in \mathbb{R}$ with $a < b$.

Theorem 2.2. *Let h be a B-function and $f : [a, b] \to \mathbb{R}$ be an h-convex function, then the following inequalities hold*

$$\frac{f\left(\frac{a+b}{2}\right)}{2h\left(\frac{1}{2}\right)} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq h\left(\frac{1}{2}\right)[f(a)+f(b)]. \tag{2.7}$$

Proof. Let $x \in [a, b]$, we have

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &= f\left(\frac{a}{2}+\frac{b}{2}+\frac{x}{2}-\frac{x}{2}\right)\\ &= f\left(\frac{1}{2}(a+b-x)+\frac{1}{2}x\right),\end{aligned}$$

since $x \in [a,b]$, $(a+b-x) \in [a,b]$ and f is an h-convex function on $[a,b]$, we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}(a+b-x)+\frac{1}{2}x\right) \\ &= h\left(\frac{1}{2}\right)f(a+b-x)+h\left(\frac{1}{2}\right)f(x) \\ &\leq h\left(\frac{1}{2}\right)\left[f(a+b-x)+f(x)\right], \end{aligned}$$

since h is a positive function, then

$$\frac{f\left(\frac{a+b}{2}\right)}{h\left(\frac{1}{2}\right)} \leq f(a+b-x)+f(x). \tag{2.8}$$

Given f is an h-convex function, setting $x = ta+(1-t)b$ for $t \in [0,1]$ gets

$$\begin{aligned} f(a+b-x)+f(x) &= f(a+b-ta-(1-t)b)+f(ta+(1-t)b) \\ &= f((1-t)a+tb)+f(ta+(1-t)b) \\ &\leq h(1-t)f(a)+h(t)f(b)+h(t)f(a)+h(1-t)f(b) \\ &= [h(1-t)+h(t)]f(a)+[h(1-t)+h(t)]f(b) \\ &= [h(1-t)+h(t)](f(a)+f(b)). \end{aligned}$$

Assuming that h is a B-function, we obtain

$$[h(1-t)+h(t)](f(a)+f(b)) \leq 2h\left(\frac{1}{2}\right)[f(a)+f(b)],$$

thus

$$f(a+b-x)+f(x) \leq 2h\left(\frac{1}{2}\right)[f(a)+f(b)]. \tag{2.9}$$

From (2.8) and (2.9), we result

$$\frac{f\left(\frac{a+b}{2}\right)}{h\left(\frac{1}{2}\right)} \leq f(a+b-x)+f(x) \leq 2h\left(\frac{1}{2}\right)[f(a)+f(b)]. \tag{2.10}$$

Integrating the above inequality (2.10) over $x \in [a,b]$ results

$$(b-a)\frac{f\left(\frac{a+b}{2}\right)}{h\left(\frac{1}{2}\right)} \leq \int_a^b f(a+b-x)dx + \int_a^b f(x)dx \leq (b-a)2h\left(\frac{1}{2}\right)[f(a)+f(b)],$$

by using the equality (2.6), we obtain

$$(b-a)\frac{f\left(\frac{a+b}{2}\right)}{h\left(\frac{1}{2}\right)} \leq 2\int_a^b f(x)dx \leq (b-a)2h\left(\frac{1}{2}\right)[f(a)+f(b)],$$

multiplaying the previously inequality by $\frac{1}{2(b-a)}$ yields

$$\frac{f\left(\frac{a+b}{2}\right)}{2h\left(\frac{1}{2}\right)} \leq \frac{1}{(b-a)}\int_a^b f(x)dx \leq h\left(\frac{1}{2}\right)[f(a)+f(b)].$$

wich gives the needed consequence (2.7). □

Now, we will look at some particular cases of Hermite-Hadamard inequality over convexity.

1. Putting $h(t) = t$ in the Theorem 2.2 gives the following Corollary.

   Corollary 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function, then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

2. Setting $h(t) = 1$ in the Theorem 2.2 gets the next Corollary.

   Corollary 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a P-functions, then the next inequalities hold*

$$\frac{f\left(\frac{a+b}{2}\right)}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq [f(a)+f(b)].$$

3. Taking $h(t) = t^s$ in the Theorem 2.2 yields the next Corollary.

   Corollary 2.3. *Let $s \in (0, 1]$ and $f : [a, b] \to \mathbb{R}$ be a s-convex function, then*

$$\frac{f\left(\frac{a+b}{2}\right)}{2^{1-s}} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \left(\frac{1}{2^s}\right)[f(a)+f(b)].$$

Chapter 3

Bullen inequality

3.1 Introduction

Bullen's inequality originates from work by Peter Southcott Bullen (1928-2024), a Brithish mathematician .

Explorations in numerical integration and the definition of error bounds are critical in mathematical literature. Researchers have thoroughly investigated error boundaries for functions with variable differentiability, from once to many times. The Bullen-type inequality is a significant mathematical tool for integral estimate. The well-known Hermite-Hadamard inequality is stated as follows, for convex function f:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{3.1}$$

In [14], Bullen improved the right side of (3.1) by the following inequality, which is known as Bullen's inequality:

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}.$$

The estimation of Bullen is as follows.

$$\frac{1}{b-a}\int_a^b f(t)dt \approx \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right].$$

For more details regarding Bullen inequality, please consult the following references: [15], [16],[17], [18] and [19].

3.2 Basic identity

Let $I^\circ \subset \mathbb{R}$ be an open interval, with a and b as real numbers such that $[a,b] \subset I^\circ$.

Lemma 3.1. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a,b] \subset I^\circ$. If $f' \in L^1([a,b])$, then*

$$\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt =$$

$$\frac{b-a}{8}\int_0^1 (2t-1)\left[f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) - f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right]dt. \tag{3.2}$$

Proof. Let $f : I^\circ \to \mathbb{R}$ be a differentiable function on I° with $[a,b] \subset I^\circ$ and $f' \in L[a,b]$. Putting

$$I_1 = \int_0^1 (2t-1) f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)dt,$$

and

$$I_2 = \int_0^1 (2t-1) f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)dt.$$

By using the integration by parts, taking

$$\begin{cases} U(t) & = 2t-1 \\ V'(t) & = f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right), \end{cases}$$

then

$$\begin{cases} U'(t) = 2 \\ V(t) = \frac{2}{b-a} f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right). \end{cases}$$

We get

$$\begin{aligned}\int_0^1 U(t)V'(t)dt &= [U(t)V(t)]_0^1 - \int_0^1 U'(t)V(t)dt \\ &= \frac{2}{b-a}\left[(2t-1)f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}b\right)\right)\right]_0^1 \\ &\quad -\frac{4}{b-a}\int_0^1 f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)dt \\ &= \frac{2}{b-a}\left[f(b) + f\left(\frac{a+b}{2}\right)\right] - \frac{4}{b-a}\int_0^1 f\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)dt.\end{aligned}$$

We use the following change of variable $x = \frac{1-t}{2}a + \frac{1+t}{2}b = \frac{b-a}{2}t + \frac{a+b}{2}$, then

$$dx = \tfrac{b-a}{2}dt \qquad \Rightarrow \qquad dt = \tfrac{2}{b-a}dx$$

$$t = 0 \Rightarrow x = \tfrac{a+b}{2} \quad \text{and} \quad t = 1 \Rightarrow x = b,$$

therefore

$$I_1 = \frac{2}{b-a}\left[f(b) + f\left(\frac{a+b}{2}\right)\right] - \frac{8}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(x)dx. \tag{3.3}$$

Similarly, for the second integral

$$I_2 = \int_0^1 (2t-1)f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)dt,$$

we use

$$\begin{cases} U(t) &= 2t-1 \\ V'(t) &= f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right), \end{cases}$$

then

$$\begin{cases} U'(t) &= 2 \\ V(t) &= -\frac{2}{b-a}f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right), \end{cases}$$

hence

$$I_2 = \int_0^1 (2t-1) f'\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right) dt$$

$$= -\frac{2}{b-a}\left[(2t-1) f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right)\right]_0^1$$

$$+\tfrac{4}{b-a}\int_0^1 f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right) dt$$

$$= -\frac{2}{b-a}\left[f(a) + f\left(\frac{a+b}{2}\right)\right] + \frac{4}{b-a}\int_0^1 f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right) dt.$$

We use the following change of variable $x = \left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b = -\left(\frac{b-a}{2}\right) t + \frac{a+b}{2}$, then

$$dx = -\tfrac{b-a}{2} dt \qquad \Rightarrow \qquad dt = -\tfrac{2}{b-a} dx$$

$$t = 0 \Rightarrow x = \tfrac{a+b}{2} \quad \text{and} \quad t = 1 \Rightarrow x = a,$$

then

$$I_2 = -\frac{2}{b-a}\left[f(a) + f\left(\frac{a+b}{2}\right)\right] + \frac{8}{(b-a)^2}\int_a^{\frac{a+b}{2}} f(x) dx. \tag{3.4}$$

Consequently, from the equalities (3.3) and (3.4), the following equality holds

$$\begin{aligned} I_1 - I_2 &= \frac{2}{b-a}\left[f(b) + f\left(\frac{a+b}{2}\right) + f(a) + f\left(\frac{a+b}{2}\right)\right] \\ &\quad -\frac{8}{(b-a)^2}\left[\int_{\frac{a+b}{2}}^b f(t)dt + \int_a^{\frac{a+b}{2}} f(t)dt\right] \\ &= \frac{2}{b-a}\left[f(a) + f(b) + 2f\left(\frac{a+b}{2}\right)\right] - \frac{8}{(b-a)^2}\int_a^b f(t)dt \\ &= \frac{4}{b-a}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{8}{(b-a)^2}\int_a^b f(t)dt, \end{aligned}$$

multiplying the above equality by $\frac{b-a}{8}$ gives

$$\tfrac{1}{2}\left[\tfrac{f(a)+f(b)}{2} + f\left(\tfrac{a+b}{2}\right)\right] - \tfrac{1}{b-a}\int_a^b f(t)dt = \frac{b-a}{8}\left[I_1 - I_2\right].$$

Thus, we obtain the required equality (3.2). □

3.3 Bullen inequality via h-convex function

Theorem 3.1. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a,b] \subset I^\circ$, and $f' \in L^1([a,b])$. If $|\, f' \,|$ is an h-convex function, then the following inequality holds.*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{8}\,h\left(\frac{1}{2}\right)\left[|f'(a)|+|f'(b)|\right]. \tag{3.5}$$

Proof. Using the absolute value on the identity (3.2), we result

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&=\left|\frac{b-a}{8}\int_0^1 (2t-1)\left[f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right.\right. \\
&\qquad \left.\left.-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt\right| \\
&\leq \frac{b-a}{8}\int_0^1 |2t-1|\left[\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right.\right. \\
&\qquad \left.\left.-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|\right]dt \\
&\leq \frac{b-a}{8}\int_0^1 |2t-1|\left[\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|\right. \\
&\qquad \left.+\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|\right]dt.
\end{aligned}$$

Since $\mid f' \mid = g$ is an h-convex function, we deduce

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{8}\int_0^1 |2t-1|\left[g\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)+g\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt$$

$$\leq \frac{b-a}{8}\int_0^1 |2t-1|\left[h\left(\frac{1-t}{2}\right)g(a)+h\left(\frac{1+t}{2}\right)g(b)\right.$$

$$\left.+h\left(\frac{1+t}{2}\right)g(a)+h\left(\frac{1-t}{2}\right)g(b)\right]dt$$

$$= \frac{b-a}{8}\int_0^1 |2t-1|\left[\left(h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right)g(a)\right.$$

$$\left.+\left(h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right)g(b)\right]dt$$

$$= \frac{b-a}{8}\int_0^1 |2t-1|\left[\left(h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right)(g(a)+g(b))\right]dt$$

$$= \frac{b-a}{8}\int_0^1 |2t-1|\left[\left(h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right)(|f'(a)|+|f'(b)|)\right]dt.$$

Given that h is a B-function, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{8}\int_0^1 |2t-1|\left[\left(h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right)(|f'(a)|+|f'(b)|)\right]dt$$

$$\leq \frac{b-a}{8}\int_0^1 |2t-1|\, 2h\left(\frac{1}{2}\right)(|f'(a)|+|f'(b)|)\, dt$$

$$= \frac{b-a}{4}h\left(\frac{1}{2}\right)[|f'(a)|+|f'(b)|]\int_0^1 |2t-1|\, dt.$$

As

$$| 2t - 1 |= \begin{bmatrix} 2t-1 & , & t \in [\frac{1}{2}, 1] \\ \\ 1-2t & , & t \in [0, \frac{1}{2}], \end{bmatrix} \tag{3.6}$$

then

$$\begin{aligned} \int_0^1 | 2t-1 | \, dt &= \int_0^{\frac{1}{2}} (1-2t)dt + \int_{\frac{1}{2}}^1 (2t-1)dt \\ &= \left[t - t^2\right]_0^{\frac{1}{2}} + \left[t^2 - t\right]_{\frac{1}{2}}^1 \\ &= \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}. \end{aligned}$$

Hence

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{8} h\left(\frac{1}{2}\right) \left[|f'(a)| + |f'(b)|\right].$$

This gives as the desired result. □

Now, we present some special cases over the convexity.

1. Putting $h(t) = t$ in the Theorem 3.1 gives the following Corollary.

   Corollary 3.1. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a, b] \subset I^\circ$, and $f' \in L^1([a, b])$. If $| f' |$ is a convex function, then the next inequality holds.*

$$\begin{aligned} &\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\leq \frac{b-a}{16} \left[|f'(a)| + |f'(b)|\right]. \end{aligned} \tag{3.7}$$

2. Setting $h(t) = 1$ in the Theorem 3.1 gets the next Corollary.

   Corollary 3.2. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a, b] \subset I^\circ$, and $f' \in L^1([a, b])$. If $| f' |$ is P-functions, then*

$$\begin{aligned} &\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\leq \frac{b-a}{8} \left[|f'(a)| + |f'(b)|\right]. \end{aligned} \tag{3.8}$$

3. Taking $h(t) = t^s$ in the Theorem 3.1 yields the next Corollary.

Corollary 3.3. *Let* $f : I^\circ \to \mathbb{R}$ *be a differentiable function, with* $[a, b] \subset I^\circ$, *and* $f' \in L^1([a, b])$. *If* $| f' |$ *is an s-convex function, then the following inequality holds.*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{b-a}{8 \cdot 2^s}\left[|f'(a)|+|f'(b)|\right].
\end{aligned} \tag{3.9}$$

3.4 Bullen inequality through Hölder inequality

Theorem 3.2. *Let* $p, q > 1$ *where* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f : I^\circ \to \mathbb{R}$ *be a differentiable function, with* $[a, b] \subset I^\circ$, *and* $f' \in L^p([a, b])$. *If* $| f' |^p$ *is h-convex function then the next inequality holds.*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{b-a}{4}\left(\int_0^1 |2t-1|^q\,dt\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}.
\end{aligned} \tag{3.10}$$

Proof. Using the absolute value on the identity (3.2), we result

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$=\left|\frac{b-a}{8}\int_0^1 (2t-1)\left[f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right.\right.$$

$$\left.\left.-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt\right|$$

$$\leq \frac{b-a}{8}\int_0^1 |2t-1|\left[\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|\right.$$

$$\left.+\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|\right]dt$$

$$=\frac{b-a}{8}\int_0^1 |2t-1|\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|dt$$

$$+\frac{b-a}{8}\int_0^1 |2t-1|\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|dt.$$

Using Hölder's inequality gives

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{8}\left(\int_0^1 |2t-1|^q\,dt\right)^{\frac{1}{q}}\cdot\left(\int_0^1\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}$$

$$+\frac{b-a}{8}\left(\int_0^1 |2t-1|^q\,dt\right)^{\frac{1}{q}}\cdot\left(\int_0^1\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}$$

$$=\frac{b-a}{8}\left(\int_0^1 |2t-1|^q\,dt\right)^{\frac{1}{q}}\left[\left(\int_0^1\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\int_0^1\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}\right].$$

Since, for $a, b > 0$, $\alpha > 0$

$$\min\left(1, 2^{1-\alpha}\right)(a+b)^{\alpha} \leq a^{\alpha} + b^{\alpha} \leq \max\left(1, 2^{1-\alpha}\right)(a+b)^{\alpha},$$

by taking

$$a = \left(\int_0^1 \left|f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right|^p dt\right)$$

$$b = \left(\int_0^1 \left|f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|^p dt\right),$$

then for $\alpha = \frac{1}{p}$, we get

$$a^{\frac{1}{p}} + b^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(a+b)^{\frac{1}{p}}.$$

Therefore

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{8}\left(\int_0^1 |2t-1|^q dt\right)^{\frac{1}{q}} 2^{1-\frac{1}{p}}$$

$$\times\left\{\int_0^1 \left[\left|f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right|^p + \left|f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|^p\right] dt\right\}^{\frac{1}{p}}.$$

Given that $| f' |^p$ is an h-convex function, we result

$$\left|f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right|^p \leq h\left(\frac{1-t}{2}\right)|f'(a)|^p + h\left(\frac{1+t}{2}\right)|f'(b)|^p,$$

and

$$\left|f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|^p \leq h\left(\frac{1+t}{2}\right)|f'(a)|^p + h\left(\frac{1-t}{2}\right)|f'(b)|^p,$$

then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{8}\left(\int_0^1 |2t-1|^q dt\right)^{\frac{1}{q}} 2^{1-\frac{1}{p}}$$

$$\times\left\{\int_0^1 \left[h\left(\frac{1+t}{2}\right) + h\left(\frac{1-t}{2}\right)\right]\left[|f'(a)|^p + |f'(b)|^p\right] dt\right\}^{\frac{1}{p}}.$$

Assumption h is a B-function, we obtain

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{8}\left(\int_0^1 |2t-1|^q\,dt\right)^{\frac{1}{q}} 2^{1-\frac{1}{p}}\left\{\int_0^1 2h\left(\frac{1}{2}\right)\left[|f'(a)|^p+|f'(b)|^p\right]dt\right\}^{\frac{1}{p}}$$

$$= \frac{b-a}{8}\left(\int_0^1 |2t-1|^q\,dt\right)^{\frac{1}{q}} 2^{1-\frac{1}{p}}\,2^{\frac{1}{p}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}$$

$$= \frac{b-a}{4}\left(\int_0^1 |2t-1|^q\,dt\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}.$$

□

Now, we derive some specific cases over the convexity.

1. Putting $h(t)=t$ in the Theorem 3.2 gives the following Corollary.

   Corollary 3.4. *Let $p,q>1$ where $\frac{1}{p}+\frac{1}{q}=1$ and $f: I^\circ \to \mathbb{R}$ be a differentiable function, with $[a,b]\subset I^\circ$, and $f'\in L^p([a,b])$. If $\mid f' \mid^p$ is a convex function then the next inequality holds.*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}\left(\int_0^1 |2t-1|^q\,dt\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}. \tag{3.11}$$

2. Setting $h(t)=1$ in the Theorem 3.2 gets the next Corollary.

   Corollary 3.5. *Let $p,q>1$ where $\frac{1}{p}+\frac{1}{q}=1$ and $f: I^\circ \to \mathbb{R}$ be a differentiable function, with $[a,b]\subset I^\circ$, and $f'\in L^p([a,b])$. If $\mid f' \mid^p$ is P-functions then the following inequality holds.*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}\left(\int_0^1 |2t-1|^q\,dt\right)^{\frac{1}{q}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}. \tag{3.12}$$

3. Taking $h(t) = t^s$ in the Theorem 3.2 yields the next Corollary.

Corollary 3.6. *Let $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a, b] \subset I^\circ$, and $f' \in L^p([a, b])$. If $\mid f' \mid^p$ is s-convex function then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq \frac{b-a}{4}\left(\int_0^1 |2t-1|^q\,dt\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}. \tag{3.13}$$

3.5 Bullen inequality with k-lipschizian functions

Theorem 3.3. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a, b] \subset I^\circ$ and $f' \in L^1([a, b])$. If f' is a k-Lipchizian, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{k(b-a)^2}{32}.$$

Proof. Using the absulute value of the identity (3.2) gets

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$=\left|\frac{b-a}{8}\int_0^1 (2t-1)\left[f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right.\right.$$

$$\left.\left.-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt\right|$$

$$\leq \frac{b-a}{8}\int_0^1 |2t-1|$$

$$\times\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|dt,$$

since f' is a k-Lipchizian functions then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{k(b-a)}{8}\int_0^1 |2t-1|$$

$$\times\left|\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b-\left(\frac{1+t}{2}\right)a-\left(\frac{1-t}{2}\right)b\right|dt$$

$$=\frac{k(b-a)}{8}\int_0^1 |2t-1|\,|(b-a)t|\,dt$$

$$=\frac{k(b-a)^2}{8}\int_0^1 t\,|2t-1|\,dt$$

$$=\frac{k\,(b-a)^2}{32}.$$

Because

$$|\ 2t-1\ | \quad = \quad \begin{cases} 2t-1, & t\in\left[\frac{1}{2},1\right] \\ \\ -2t+1, & t\in\left[0,\frac{1}{2}\right], \end{cases} \tag{3.14}$$

then

$$\int_0^1 t\,|(2t-1)|\,dt \quad = \quad \int_0^{\frac{1}{2}}(-2t^2+t)dt+\int_{\frac{1}{2}}^1(2t^2-t)dt$$

$$= \quad \left[-\frac{2}{3}t^3+\frac{1}{2}t^2\right]_0^{\frac{1}{2}}+\left[\frac{2}{3}t^3-\frac{1}{2}t^2\right]_{\frac{1}{2}}^1=\frac{1}{4}.$$

□

3.6 Bullen inequality involving bounded functions

Theorem 3.4. *Let $-\infty<m<M<+\infty$, $f\in C^1([a,b])$.*

If $m\leq f'(t)\leq M$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{(b-a)(M-m)}{16}.$$

Proof. Using the absulute value of identity (3.2) gives

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$=\frac{b-a}{8}\left|\int_0^1 (2t-1)\right.$$

$$\left.\times\left[f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt\right|$$

$$\leq\frac{b-a}{8}\int_0^1 |2t-1|$$

$$\times\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|dt$$

$$=\frac{b-a}{8}\int_0^1 |2t-1|\times\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-\frac{M+m}{2}\right.$$

$$\left.-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)+\frac{M+m}{2}\right|dt$$

$$=\frac{b-a}{8}\int_0^1 |2t-1|\times\left|\left[f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-\frac{M+m}{2}\right]\right.$$

$$\left.-\left[f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)-\frac{M+m}{2}\right]\right|dt$$

$$\leq\frac{b-a}{8}\int_0^1 |2t-1|\left[\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-\frac{M+m}{2}\right|\right.$$

$$\left.+\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)-\frac{M+m}{2}\right|\right]dt.$$

For all $t\in[0,1]$, we have

$$m\leq f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\leq M,$$

then

$$m-\frac{M+m}{2}\leq f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-\frac{M+m}{2}\leq M-\frac{M+m}{2},$$

hence

$$-\frac{M-m}{2} \leq f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) - \frac{M+m}{2} \leq \frac{M-m}{2},$$

witch give

$$\left|f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) - \frac{M+m}{2}\right| \leq \frac{M-m}{2}.$$

Similary

$$\left|f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right) - \frac{M+m}{2}\right| \leq \frac{M-m}{2}.$$

Consequently, using (3.6) gives

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{b-a}{8}\int_0^1 |2t-1| \left[\frac{M-m}{2} + \frac{M-m}{2}\right] dt \\
&= \frac{(b-a)(M-m)}{8}\int_0^1 |2t-1|\, dt \\
&= \frac{(b-a)(M-m)}{16}.
\end{aligned}$$

□

Chapter 4

Milne inequality

4.1 Introduction

Milne's inequality originates from work by Edward Arthur Milne (1896-1950), a Brithish astrophysicist and mathematician. The Milne's estimation is presnted as follows.

$$\frac{1}{b-a}\int_a^b f(t)dt \approx \frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right].$$

For additional details on Milne's inequality, refer to [20], [21], [22], and [23].

4.2 Basic identity

Let $I^\circ \subset \mathbb{R}$ be an open interval, with a and b as real numbers such that $[a,b] \subset I^\circ$.

Lemma 4.1. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a,b] \subset I^\circ$. If $f' \in L^1([a,b])$, then*

$$\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt =$$

$$\frac{b-a}{4}\int_0^1 \left(t+\frac{1}{3}\right)\left[f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) - f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right] dt. \tag{4.1}$$

Proof. Let $f : I^\circ \to \mathbb{R}$ be a differentiable function on I° with $[a,b] \subset I^\circ$ and $f' \in L[a,b]$. taking

$$J_1 = \int_0^1 \left(t+\frac{1}{3}\right) f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) dt,$$

and

$$J_2 = \int_0^1 \left(t + \frac{1}{3}\right) f'\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right) dt.$$

Applying the integration by parts, we deduce

$$\begin{cases} \mu(t) & = t + \frac{1}{3} \\ \\ \eta'(t) & = f'\left(\left(\frac{1-t}{2}\right) a + \left(\frac{1+t}{2}\right) b\right), \end{cases}$$

then

$$\begin{cases} \mu'(t) = 1 \\ \\ \eta(t) = \dfrac{2}{b-a} f\left(\left(\dfrac{1-t}{2}\right) a + \left(\dfrac{1+t}{2}\right) b\right). \end{cases}$$

We get

$$\int_0^1 \mu(t)\eta'(t)dt = [\mu(t)\eta(t)]_0^1 - \int_0^1 \mu'(t)\eta(t)dt$$

$$= \frac{2}{b-a}\left[\left(t + \frac{1}{3}\right) f\left(\left(\frac{1-t}{2}\right) a + \left(\frac{1+t}{2} b\right)\right)\right]_0^1$$

$$- \frac{2}{b-a}\int_0^1 f\left(\left(\frac{1-t}{2}\right) a + \left(\frac{1+t}{2}\right) b\right) dt$$

$$= \frac{2}{b-a}\left[\frac{4}{3} f(b) - \frac{1}{3} f\left(\frac{a+b}{2}\right)\right] - \frac{2}{b-a}\int_0^1 f\left(\left(\frac{1-t}{2}\right) a + \left(\frac{1+t}{2}\right) b\right) dt.$$

Utilizing the change of variable $x = \frac{1-t}{2}a + \frac{1+t}{2}b = \frac{b-a}{2}t + \frac{a+b}{2}$ yields

$$dx = \tfrac{b-a}{2}dt \qquad \Rightarrow \qquad dt = \tfrac{2}{b-a}dx$$

$$t = 0 \Rightarrow x = \tfrac{a+b}{2} \quad \text{and} \quad t = 1 \Rightarrow x = b,$$

therefore

$$J_1 = \frac{2}{b-a}\left[\frac{4}{3} f(b) - \frac{1}{3} f\left(\frac{a+b}{2}\right)\right] - \frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(x)dx. \qquad (4.2)$$

Similarly, for the second integral

$$J_2 = \int_0^1 \left(t + \frac{1}{3}\right) f'\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right) dt,$$

we use

$$\begin{cases} \mu(t) & = t + \frac{1}{3} \\ \eta'(t) & = f'\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right), \end{cases}$$

and

$$\begin{cases} \mu'(t) & = 1 \\ \eta(t) & = -\frac{2}{b-a} f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right), \end{cases}$$

then

$$\begin{aligned} J_2 &= \int_0^1 \left(t + \frac{1}{3}\right) f'\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right) dt \\ &= -\frac{2}{b-a}\left[\left(t+\frac{1}{3}\right) f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right)\right]_0^1 \\ &\quad + \frac{2}{b-a}\int_0^1 f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right) dt \\ &= -\frac{2}{b-a}\left[\frac{4}{3} f(a) - \frac{1}{3} f\left(\frac{a+b}{2}\right)\right] + \frac{2}{b-a}\int_0^1 f\left(\left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b\right) dt. \end{aligned}$$

Changing the variable $x = \left(\frac{1+t}{2}\right) a + \left(\frac{1-t}{2}\right) b = -\left(\frac{b-a}{2}\right) t + \frac{a+b}{2}$ gives

$$dx = -\tfrac{b-a}{2} dt \qquad \Rightarrow \qquad dt = -\tfrac{2}{b-a} dx$$

$$t = 0 \Rightarrow x = \tfrac{a+b}{2} \quad \text{and} \quad t = 1 \Rightarrow x = a,$$

then

$$J_2 = -\frac{2}{b-a}\left[\frac{4}{3} f(a) - \frac{1}{3} f\left(\frac{a+b}{2}\right)\right] - \frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}} f(x) dx. \qquad (4.3)$$

Consequently, from the equalities (4.2) and (4.3), the following equality holds

$$\begin{aligned}
J_1 - J_2 &= \frac{2}{b-a}\left[\frac{4}{3}f(b) - \frac{1}{3}f\left(\frac{a+b}{2}\right) + \frac{4}{3}f(a) - \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] \\
&\quad - \frac{4}{(b-a)^2}\left[\int_a^b f(t)dt\right] \\
&= \frac{2}{b-a}\left[\frac{4}{3}f(a) + \frac{4}{3}f(b) - \frac{2}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{4}{(b-a)^2}\int_a^b f(t)dt \\
&= \frac{4}{b-a}\left[\frac{2}{3}f\,(a) + \frac{2}{3}f(b) - \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{4}{(b-a)^2}\int_a^b f(t)dt,
\end{aligned}$$

multiplying the above equality by $\frac{b-a}{4}$ gives

$$\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt = \frac{b-a}{4}\left[J_1 - J_2\right].$$

Hence, we get the desired equality (4.1). □

4.3 Milne inequality using h-convex functions

Theorem 4.1. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a,b] \subset I^\circ$, and $f' \in L^1([a,b])$. If $|\, f'\,|$ is an h-convex function, then the following inequality holds.*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{5(b-a)}{12}\, h\left(\frac{1}{2}\right)\left[|f'(a)| + |f'(b)|\right].
\end{aligned} \tag{4.4}$$

Proof. Using the absolute value on the identity (4.1), we obtain

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$=\left|\frac{b-a}{4}\int_0^1\left(t+\frac{1}{3}\right)\left[f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right.\right.$$

$$\left.\left.-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt\right|$$

$$\leq\frac{b-a}{4}\int_0^1\left|t+\frac{1}{3}\right|\left[\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right.\right.$$

$$\left.\left.-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|\right]dt$$

$$\leq\frac{b-a}{4}\int_0^1\left|t+\frac{1}{3}\right|\left[\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|\right.$$

$$\left.+\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|\right]dt.$$

Given that $|f'|$ is an h-convex function, we result

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{4}\int_0^1\left|t+\frac{1}{3}\right|\left[\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|\right.$$

$$\left.+\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|\right]dt$$

$$\leq \frac{b-a}{4}\int_0^1\left|t+\frac{1}{3}\right|\left[h\left(\frac{1-t}{2}\right)|f'(a)|+h\left(\frac{1+t}{2}\right)|f'(b)|\right.$$

$$\left.+h\left(\frac{1+t}{2}\right)|f'(a)|+h\left(\frac{1-t}{2}\right)|f'(b)|\right]dt$$

$$=\frac{b-a}{4}\int_0^1\left|t+\frac{1}{3}\right|\left[\left(h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right)|f'(a)|\right.$$

$$\left.+\left(h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right)|f'(b)|\right]dt$$

$$=\frac{b-a}{4}\int_0^1\left|t+\frac{1}{3}\right|\left[\left(h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right)(|f'(a)|+|f'(b)|)\right]dt.$$

Since h is a B-function, we get

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{4}\int_0^1\left|t+\frac{1}{3}\right|\left[\left(h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right)(|f'(a)|+|f'(b)|)\right]dt$$

$$\leq \frac{b-a}{4}\int_0^1\left|t+\frac{1}{3}\right|2h\left(\frac{1}{2}\right)(|f'(a)|+|f'(b)|)\,dt$$

$$=\frac{b-a}{2}h\left(\frac{1}{2}\right)[|f'(a)|+|f'(b)|]\int_0^1\left|t+\frac{1}{3}\right|dt.$$

We have

$$\int_0^1 \left|t+\frac{1}{3}\right| dt = \int_0^1 \left(t+\frac{1}{3}\right) dt = \left[\frac{t^2}{2}+\frac{1}{3}t\right]_0^1 = \frac{5}{6}. \tag{4.5}$$

Hence

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{5(b-a)}{12} h\left(\frac{1}{2}\right)\left[|f'(a)|+|f'(b)|\right].$$

This gives as the desired result. □

Now, we present some special cases over the convexity.

1. Putting $h(t)=t$ in the Theorem 4.1 gives the following Corollary.

   Corollary 4.1. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a,b] \subset I^\circ$, and $f' \in L^1([a,b])$. If $\mid f' \mid$ is a convex function, then the next inequality holds.*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\leq \frac{5(b-a)}{24}\left[|f'(a)|+|f'(b)|\right].\end{aligned} \tag{4.6}$$

2. Setting $h(t)=1$ in the Theorem 4.1 gets the next Corollary.

   Corollary 4.2. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a,b] \subset I^\circ$, and $f' \in L^1([a,b])$. If $\mid f' \mid$ is P-functions, then the following inequality holds.*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\leq \frac{5(b-a)}{12}\left[|f'(a)|+|f'(b)|\right].\end{aligned} \tag{4.7}$$

3. Taking $h(t)=t^s$ in the Theorem 4.1 yields the next Corollary.

   Corollary 4.3. *Let $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a,b] \subset I^\circ$, and $f' \in L^1([a,b])$. If $\mid f' \mid$ is an s-convex function, then*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\leq \frac{5(b-a)}{12\cdot 2^s}\left[|f'(a)|+|f'(b)|\right].\end{aligned} \tag{4.8}$$

4.4 Milne inequality via Hölder inequality

Theorem 4.2. *Let $p, q > 1$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a, b] \subset I^\circ$, and $f' \in L^p([a, b])$. If $| f' |^p$ is h-convex function then the next inequality holds.*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leq \frac{b-a}{4}\left(\int_0^1 \left|t + \frac{1}{3}\right|^q dt\right)^{\frac{1}{q}} \left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}} \left[|f'(a)|^p + |f'(b)|^p\right]^{\frac{1}{p}}.
\end{aligned} \tag{4.9}$$

Proof. Using the absolute value on the identity (4.1), we result

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \\
&= \left|\frac{b-a}{4}\int_0^1 \left(t + \frac{1}{3}\right)\left[f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right.\right. \\
&\quad \left.\left. - f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right]dt\right| \\
&\leq \frac{b-a}{4}\int_0^1 \left|t + \frac{1}{3}\right|\left[\left|f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right|\right. \\
&\quad \left. + \left|f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|\right]dt \\
&= \frac{b-a}{4}\int_0^1 \left|t + \frac{1}{3}\right|\left|f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right)\right|dt \\
&\quad + \frac{b-a}{4}\int_0^1 \left|t + \frac{1}{3}\right|\left|f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|dt.
\end{aligned}$$

Using Hölder's inequality gives

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{4}\left(\int_0^1\left|t+\frac{1}{3}\right|^q dt\right)^{\frac{1}{q}}\cdot\left(\int_0^1\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}$$

$$+\frac{b-a}{4}\left(\int_0^1\left|t+\frac{1}{3}\right|^q dt\right)^{\frac{1}{q}}\cdot\left(\int_0^1\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}$$

$$=\frac{b-a}{4}\left(\int_0^1\left|t+\frac{1}{3}\right|^q dt\right)^{\frac{1}{q}}\left[\left(\int_0^1\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\int_0^1\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|^p dt\right)^{\frac{1}{p}}\right].$$

Since, for $A,B>0$, $\alpha>0$

$$\min\left(1,2^{1-\alpha}\right)(A+B)^\alpha\leq A^\alpha+B^\alpha\leq\max\left(1,2^{1-\alpha}\right)(A+B)^\alpha,$$

by taking

$$A=\left(\int_0^1\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|^p dt\right)$$

$$B=\left(\int_0^1\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|^p dt\right),$$

and for $\alpha=\frac{1}{p}$, we get

$$A^{\frac{1}{p}}+B^{\frac{1}{p}}\leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}.$$

Therefore

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq\frac{b-a}{4}\left(\int_0^1\left|t+\frac{1}{3}\right|^q dt\right)^{\frac{1}{q}}2^{1-\frac{1}{p}}$$

$$\times\left\{\int_0^1\left[\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|^p+\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|^p\right]dt\right\}^{\frac{1}{p}}.$$

Given that $\mid f' \mid^p$ is an h-convex function, we result

$$\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right|^p \leq h\left(\frac{1-t}{2}\right)|f'(a)|^p + h\left(\frac{1+t}{2}\right)|f'(b)|^p,$$

and

$$\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|^p \leq h\left(\frac{1+t}{2}\right)|f'(a)|^p + h\left(\frac{1-t}{2}\right)|f'(b)|^p,$$

then

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{4}\left(\int_0^1\left|t+\frac{1}{3}\right|^q dt\right)^{\frac{1}{q}} 2^{1-\frac{1}{p}}$$

$$\times\left\{\int_0^1\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]\left[|f'(a)|^p+|f'(b)|^p\right]dt\right\}^{\frac{1}{p}}.$$

Assumption h is a B-function, we obtain

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{4}\left(\int_0^1\left|t+\frac{1}{3}\right|^q dt\right)^{\frac{1}{q}} 2^{1-\frac{1}{p}}\left\{\int_0^1 2h\left(\frac{1}{2}\right)\left[|f'(a)|^p+|f'(b)|^p\right]dt\right\}^{\frac{1}{p}}$$

$$= \frac{b-a}{4}\left(\int_0^1\left|t+\frac{1}{3}\right|^q dt\right)^{\frac{1}{q}} 2^{1-\frac{1}{p}}\cdot 2^{\frac{1}{p}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}$$

$$= \frac{b-a}{2}\left(\int_0^1\left|t+\frac{1}{3}\right|^q dt\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}.$$

□

Now, we drive some specific cases over the convexity.

1. Putting $h(t) = t$ in the Theorem 4.2 gives the following Corollary.

   Corollary 4.4. *Let $p, q > 1$ where $\frac{1}{p}+\frac{1}{q} = 1$ and $f : I^\circ \to \mathbb{R}$ be a differentiable function, with $[a, b] \subset I^\circ$, and $f' \in L^p([a, b])$. If $\mid f' \mid^p$ is a convex* *function then the next inequality holds.*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\le\frac{b-a}{2}\left(\int_0^1\left|t+\frac{1}{3}\right|^q dt\right)^{\frac{1}{q}}\left[\frac{|f'(a)|^p+|f'(b)|^p}{2}\right]^{\frac{1}{p}}.\end{aligned}\tag{4.10}$$

2. Setting $h(t)=1$ in the Theorem 4.2 gets the next Corollary.

Corollary 4.5. *Let $p,q>1$ where $\frac{1}{p}+\frac{1}{q}=1$ and $f:I^\circ\to\mathbb{R}$ be a differentiable function, with $[a,b]\subset I^\circ$, and $f'\in L^p([a,b])$. If $\mid f'\mid^p$ is P-functions then*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\le\frac{b-a}{2}\left(\int_0^1\left|t+\frac{1}{3}\right|^q dt\right)^{\frac{1}{q}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}.\end{aligned}\tag{4.11}$$

3. Taking $h(t)=t^s$ in the Theorem 4.2 yields the next Corollary.

Corollary 4.6. *Let $p,q>1$ where $\frac{1}{p}+\frac{1}{q}=1$ and $f:I^\circ\to\mathbb{R}$ be a differentiable function, with $[a,b]\subset I^\circ$, and $f'\in L^p([a,b])$. If $\mid f'\mid^p$ is s-convex function then the following inequality holds.*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\\&\le\frac{b-a}{2}\left(\int_0^1\left|t+\frac{1}{3}\right|^q dt\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left[|f'(a)|^p+|f'(b)|^p\right]^{\frac{1}{p}}.\end{aligned}\tag{4.12}$$

4.5 Milne inequality by k-lipschizian functions

Theorem 4.3. *Let $f:I^\circ\subset\mathbb{R}$ be a differentiable function, with $[a,b]\subset I^\circ$ and $f'\in L^1([a,b])$. If f' is a k-Lipchizian, then*

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{k(b-a)^2}{8}.$$

Proof. Using the absulute value of the identity (4.1) gets

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$=\left|\frac{b-a}{4}\int_0^1\left(t+\frac{1}{3}\right)\left[f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\right.\right.$$

$$\left.\left.-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right]dt\right|$$

$$\leq\frac{b-a}{4}\int_0^1\left|t+\frac{1}{3}\right|$$

$$\times\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)\right|dt,$$

since f' is a k-Lipchizian, then

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq\frac{k(b-a)}{4}\int_0^1\left|t+\frac{1}{3}\right|$$

$$\times\left|\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b-\left(\frac{1+t}{2}\right)a-\left(\frac{1-t}{2}\right)b\right|dt$$

$$=\frac{k(b-a)}{4}\int_0^1\left|t+\frac{1}{3}\right|\left|(b-a)t\right|dt$$

$$=\frac{k(b-a)^2}{4}\int_0^1 t\left|t+\frac{1}{3}\right|dt$$

$$=\frac{k(b-a)^2}{4}\int_0^1 t\left(t+\frac{1}{3}\right)dt=\frac{k\left(b-a\right)^2}{8}.$$

Because

$$\int_0^1 t\left(t+\frac{1}{3}\right)dt \;=\; \int_0^1\left(t^2+\frac{1}{3}t\right)dt=\left[\frac{t^3}{3}+\frac{1}{6}t^2\right]_0^1=\frac{1}{2}.$$

□

4.6 Milne inequality with bounded functions

Theorem 4.4. *Let* $-\infty < m < M < +\infty$ *and* $f \in C^1([a,b])$. *If* $m \leq f'(t) \leq M$, *then*

$$\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{5(b-a)(M-m)}{24}.$$

Proof. Using the absulute value of identity
Gives:

$$\left|\frac{1}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$= \frac{b-a}{4}\left|\int_0^1 \left(t+\frac{1}{3}\right)\right.$$

$$\left.\times\left[f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) - f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right]dt\right|$$

$$\leq \frac{b-a}{4}\int_0^1 \left|t+\frac{1}{3}\right|$$

$$\times\left|f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) - f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right)\right|dt$$

$$= \frac{b-a}{4}\int_0^1 \left|t+\frac{1}{3}\right| \times \left|f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) - \frac{M+m}{2}\right.$$

$$\left. - f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right) + \frac{M+m}{2}\right|dt$$

$$= \frac{b-a}{4}\int_0^1 \left|t+\frac{1}{3}\right| \times \left|\left[f'\left(\left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b\right) - \frac{M+m}{2}\right]\right.$$

$$\left. - \left[f'\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b\right) - \frac{M+m}{2}\right]\right|dt,$$

then

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq \frac{b-a}{4}\int_0^1\left(t+\frac{1}{3}\right)\left[\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-\frac{M+m}{2}\right|\right.$$

$$\left.+\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)-\frac{M+m}{2}\right|\right]dt.$$

For all $t\in[0,1]$, we have

$$m\leq f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)\leq M,$$

hence

$$m-\frac{M+m}{2}\leq f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-\frac{M+m}{2}\leq M-\frac{M+m}{2},$$

thus

$$-\frac{M-m}{2}\leq f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-\frac{M+m}{2}\leq\frac{M-m}{2},$$

witch give

$$\left|f'\left(\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b\right)-\frac{M+m}{2}\right|\leq\frac{M-m}{2}.$$

In the same way

$$\left|f'\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b\right)-\frac{M+m}{2}\right|\leq\frac{M-m}{2}.$$

Therefore, applying (4.5) gives

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leq\frac{b-a}{4}\int_0^1\left(t+\frac{1}{3}\right)\left[\frac{M-m}{2}+\frac{M-m}{2}\right]dt$$

$$=\frac{(b-a)(M-m)}{4}\int_0^1\left(t+\frac{1}{3}\right)dt$$

$$=\frac{5(b-a)(M-m)}{24}.$$

□

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