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A Concise Introduction to Fuzzy Set

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Introduction

Mathematics has witnessed significant developments in representing information and knowledge, especially when dealing with imprecise or uncertain data.

In this context, Fuzzy Set Theory, introduced by Lotfi Zadeh in 1965, emerged as an extension of classical set theory concepts.

This theory aims to provide a mathematical model capable of describing phenomena characterized by vagueness and indeterminacy, which cannot be effectively represented using traditional mathematical approaches.

Fuzzy sets are considered powerful tools in various scientific and engineering applications, such as decision-making, artificial intelligence, and control systems, as they allow the representation of degrees of membership rather than limiting to full inclusion or exclusion.

In thus memory, we present A concise introduction to fuzzy set theory, highlighting its fundamental principles, the main operations on fuzzy sets, and some of its applications that demonstrate its importance in diverse fields.

CHAPTER 1

PRELIMINARIES

1.1 Introduction

In thus preliminaries we will give the various concepts in relation to set theory from the basic give, what is set

1.2 Definition of set

[1] **D. Dubois, H. Prade**
A set is a well defined collection of objects and these objects are termed as the members or elements of the set. But we need to elaborate on the term "well-defined" it means that each element bears certain characteristics with which it can be identified under a particular head.

1.3 Types of sets

[1] **D. Dubois, H. Prade**

1.3.1 Finite set

A set in countable from is a finite set and it means each element can be counted physically.

1.3.2 Infinite set

A set in uncountable form is an infinite set. The elements in the set cannot be counted.

1.3.3 Proper Subset

A set A is called proper subset of a set B, if each and every element of set A are contained in Set B and there exists at least one element in set B such that it is not an element of set A. Symbolically denoted as $A \subset B$

1.3.4 Subset

We can write this relationship as $A \subseteq B$ or $B \supseteq A$, i.e. it means if $x \in A$ and $A \subseteq B$ then $x \in B$

Properiets of Subsets

1. If set A is subset of set B then set B is called the super set of set A.
2. If set $A \subseteq$ set B and set $B \subseteq$ set A then set $A =$ set B.
3. If set $A \subseteq$ set B, set $B \subseteq$ set C then set $A \subseteq$ set C

1.3.5 Universal Set

Example 1.3.1. *If set* $A = \{1, 2, 3, 4\}$ *and set*
$B = \{5, 6, 7\}$ *then both sets A and B*
are subsets of the universal set of natural number.

1.3.6 Union of Sets

$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or } x \in \text{ both } A \text{ and } B\}$

Properties of Union of Sets

1. If set A and set B are two sets then $A \cup B$ is also a unique set
2. **Commutative Property**
   Union of set is commutative i.e. if set A and set B are two sets then
   $A \cup B = B \cup A$
3. **Union of sets is associative i.e. If set A, set B and set C are three sets then**
   $A \cup (B \cup C) = (A \cup B) \cup C$
4. **If set A is a set, then $A \cup \varnothing = A$ where $\varnothing$ is a null set**
5. **Union of sets idempotent**
   If set A is any set then $A \cup A = A$.
6. **If set A is a subset of universal Set U then $A \cup U = U$**
7. **If set A and set B are two sets such that $A \subseteq B$ then**
   $A \cup B = B$ **and if** $B \subseteq A$ **then** $A \cup B = A$

1.3.7 Intersection of Sets

$A \cap B = \{x : x \in A \text{ and } x \in B\}$
Otherwise if $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

Properties of Intersection of Sets

The following are the properties which hold with respect to intersection of sets

1. **Communicative Property**
   Intersection of sets is communicative i.e. if set A and Set B are two sets then
   $A \cap B = B \cap A$
2. **Associative Property**
   The intersection of sets are associative i.e. if set A, set B and set C are three sets then
   $(A \cap B) \cap C = A \cap (B \cap C)$.

3. **Idempotent Property**
   The intersection of sets is idempotent i.e. if set A is any set, then
   $A \cap A = A$

4. If set A is any set then $A \cap \varnothing = \varnothing, \varnothing$ is the null set

5. If set A is any set subset of an Universal set U then $A \cap U = A$

6. If A and B are disjoint sets then $A \cap B = \varnothing$

7. If set A and set B are two sets then $A \cap B \subseteq A$ and $A \cap B \subseteq B$ as
   $A \cap B$ contains only those elements which are in common A
   as well as in B. Therefore $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

1.3.8 Distributive Laws of Unions and Intersections

Proof. □

Let x be any element belonging to $A \cap (B \cup C)$, then,
$x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and $x \in (B \cup C)$
$\Leftrightarrow x \in A$ and $(x \in B$ or $x \in C)$
$\Leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$
$\Leftrightarrow x \in (A \cap B)$ or $x \in (A \cap C)$
$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$
Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. □

Let x be any element belonging to $A \cup (B \cap C)$, then,
$x \in A \cup (B \cap C) \Leftrightarrow x \in A$ or $x \in (B \cap C)$
$\Leftrightarrow x \in A$ or $(x \in B$ and $x \in C)$
$\Leftrightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$
$\Leftrightarrow x \in (A \cup B)$ and $x \in (A \cup C)$
$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$
Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

1.3.9 Complement of a Set

The complement of a set A is that set which contains all those elements of the universal set U which are not in A. The complement of set A is the set $U - A$ and is denoted by $A^c, A', \bar{A}$ or A. It can symbolically written as $A' = U - A = \{x : x \in U \text{ and } x \notin A\}$

Properties of the complement of Set

1. The intersection of a set A and its complement A' are disjoint sets i.e. $A \cap A'$ is a null set $\{\varnothing\}$
2. The union of set A and its complement is the universal set i.e. $A \cup A' = U$, the universal set.
3. Complement of complement a Set A is the set itself i.e. $(A')' = A$
4. If the set A is equal to the universal set U then $A' = \{\varnothing\}$.
5. If set A and set B are two sets then $A - B = A \cap B'$
6. If $A \subseteq B$ then $A \cup (B - A) = B$

1.3.10 Difference of Sets

Let set A and set B are two sets then,

$A - B = (x : x \in A \text{ and} x \notin B)$ similary

$B - A = (x : x \in B \text{ and } x \notin A)$.

Properties of Difference of Sets

1. $A - A = \varnothing$.
2. $A - \varnothing = A$.
3. $A - B, A \cap B$ and $B - A$ are mutually disjoint.
4. $(A - B) \cup A = A$

5. $(A - B) \cap B = \varnothing$

1.3.11 De Morgan's Laws

1^{st} Law

Let set A and set B are two sets then $(A \cup B)' = A' \cap B'$

Proof. $(A \cup B)' = A' \cap B'$
Let x be any element of $(A \cup B)'$,
Then $x \in (A \cup B)' \Leftrightarrow x \in U$ and $x \notin (A \cup B)$
$\Leftrightarrow x \in U$ and $(x \notin A$ or $x \notin B)$
$\Leftrightarrow (x \in U$ but $x \notin A)$ and $(x \in U$ but $x \notin B)$
$\Leftrightarrow x \in A'$ and $x \in B'$
$\Leftrightarrow x \in A' \cap B'$
Hence $(A \cup B)' = A' \cap B'$

□

2^{st} Law

Let set A and set B are two sets then $(A \cap B)' = A' \cup B'$

Proof. $(A \cap B)' = A' \cup B'$
To prove the above result, let x be any element belonging to $(A \cap B)'$ then
$x \in (A \cap B)' \Leftrightarrow x \notin (A \cap B)$
$\Leftrightarrow x \notin A$ or $x \notin B$
$\Leftrightarrow x \in A'$ or $x \in B'$
$\Leftrightarrow x \in A' \cup B'$.
Hence, $(A \cap B)' = A' \cup B'$ □

1.3.12 De Morgan's Laws on Difference of Sets

1. $A - (B \cup C) = (A - B) \cap (A - C)$

   Proof. Let x be any element such that
   $x \in A - (B \cup C) \Leftrightarrow x \in A$ and $x \notin (B \cup C)$

$\Leftrightarrow x \in A$ and $(x \notin B$ or $x \notin C)$
$\Leftrightarrow (x \in A$ but $x \notin B)$ and $(x \in A$ but $x \notin C)$
$\Leftrightarrow x \in (A - B)$ and $x \in (A - C)$
$\Leftrightarrow x \in (A - B) \cap (A - C)$
Hence $A - (B \cup C) = (A - B) \cap (A - C)$

□

2. $A - (B \cap C) = (A - B) \cup (A - C)$

Proof. Let x be any element such that
$x \in A - (B \cap C) \Leftrightarrow x \in A$ and $x \notin (B \cap C)$
$\Leftrightarrow x \in A$ and $(x \notin B$ and $x \notin C)$
$\Leftrightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$
$\Leftrightarrow x \in (A - B)$ or $x \in (A - C)$
$\Leftrightarrow x \in (A - B) \cup (A - C)$
Hence $A - (B \cap C) = (A - B) \cup (A - C)$. □

1.3.13 Some Important results on Difference, Union and Intersection

1. If set A and set B are two sets then $A \cup B = (A - B) \cup B$

2. If set A and set B are two sets then $A \cap (B - A) = \{\varnothing\}$

3. If set A and set B are two sets then $(A - B) \cap B = \{\varnothing\}$

4. If set A and set B are two sets then $A - (A - B) = A \cap B$

5. If set A and set B are two sets then
$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

6. If set A, set B and set C are three sets then
$(A \cap B) - C = (A - C) \cap (B - C)$.

7. If set A, set B and set C are three sets then
$A \cap (B - C) = (A \cap B) - (A \cap C)$.

8. If set A and set B are two sets then $(A - B) = A \cap B'$

9. If set A and set B are two sets then $(A - B) = B' - A'$

10. If set A, set B and set C are three sets then
$A \cup (B - C) \neq (A \cup B) - (A \cup C)$

1.3.14 Symmetric Difference of Two Sets

Symmetric difference of set A and set B is symbolically represented as
$A \Delta B = (A - B) \cup (B - A)$

Some Properties of Symmetric Difference

1. Commutative Property $A \Delta B = B \Delta A$.
2. Associative Property $(A \Delta B) \Delta C = A \Delta (B \Delta C)$.
3. $A \Delta A = \{\varnothing\}$.
4. $A \Delta \varnothing = A$
5. $A \Delta (A \cap B) = A - B$.
6. $(A \Delta B) \cup (A \cap B) = A \cup B$.
7. $A \Delta B = (A - B) \cup (B - A)$
8. $A \Delta B = (A \cup B) - (A \cap B)$.
9. $A \Delta B = \{\varnothing\} \Leftrightarrow A = B$.
10. $(A \Delta B) \cup (A \cap B) = A \cup B$.

CHAPTER 2

FUZZY SET THEORY

2.1 The Crisp Sets

Definition 2.1.1. *[6]* ***Al-Saphory, R. (2010)***
Let χ be a nonempty set, called the ***universe set****,*
consisting of all the possible elements of concern in a particular context.
Each of these elements is called a ***member****, or an* ***element****, of χ .*

$x \in \chi$ *means* x *is an element of* χ.

$x \notin \chi$ *means* x *is not an element of* χ.

Definition 2.1.2. *[6]* ***Al-Saphory, R. (2010)***
A union of several (finite or infinite) members of χ
is called a ***subset of*** χ*, which is denoted by* $A \subset \chi$.

There are two cases of subset:

1. ***Proper subset*** $(A \subset \chi)$ *means* $\exists\ x \in \chi$ *but* $x \notin A$

2. ***Subset*** $(A \subseteq \chi)$ *means* $A \subset \chi$ *or* $\chi \subset A$

Remark 2.1.1. *[6] **Al-Saphory, R. (2010)***
*Two sets A and B is **equal** if $A \subset B$ and $B \subset A$. Thus subset ($A \subseteq \chi$) means $A \subset \chi$ or $A = \chi$*

Definition 2.1.3. *The empty set is denoted by $\emptyset$.*

Example 2.1.1. *Let $I\!R^2$ be the universe set and $I\!R$ is a subset of $I\!R^2$ ($I\!R \subset I\!R^2$).*

Set - Theoretic Opeations

[6] **Al-Saphory, R. (2010)**

Let A, B be two subsets of the universe χ

1. **The Difference of two subsets:**

$$A - B := \{x \in \chi | x \in A \,\text{but}\, x \notin B\}$$

2. **The Complement of a subset:**

$$A^c := \chi - A := \{x \in \chi | x \notin A\}$$

Remarks 2.1.1.

(a) $(A^c)^c = A$

(b) $\chi^c = \emptyset$

(c) $\emptyset^c = \chi$

3. **The Union of two subsets:**

$$A \cup B := \{x | x \in A \,\text{or}\, x \in B\} = B \cup A$$

Remarks 2.1.2.

(a) $A \cup \chi = \chi$.

(b) $A \cup \emptyset = A$.

(c) $(A \cup A^c = \chi$.

4. **The Intersection of two subsets:**

$$A \cap B := \{x | x \in A \, \text{and} \; x \in B\} = B \cap A.$$

Remarks 2.1.3.

(a) $A \cap \chi = A$.

(b) $A \cap \emptyset = \emptyset$.

(c) $A \cap A^c = \emptyset$.

Definition 2.1.4. *[6]* ***Al-Saphory, R. (2010)***
Two subsets A and B are said to be ***disjoint***
if $A \cap B = \emptyset$.

Definition 2.1.5. *[6]* ***Al-Saphory, R. (2010)***
Let A be a set, ***a partion of*** *A wich is denoted by* $\pi(A)$ *is*

$$\pi(A) := \{A_i | i \in I; \;\; A_i \subseteq A\}$$

satisfied

(a) $A_i \neq \emptyset$ *for all* $i \in I$.

(b) $A_i \cap A_j = \emptyset$ *for all* $i \neq j$.

(c) $\cup A_i = A$.

If condition (b) *does not satisfy, then* $\pi(A)$ *becomes a* ***cover or covering of the set*** A**.**

5. **The Multiplication of a real number** r **and a subset** A **of** $\mathbb{R}$**:**

$$rA := \{ra|a \in A\}.$$

2.1.1 Properties of Classical Set Operations

[6] **Al-Saphory, R. (2010)**

Involutive law

$$(A^c)^c = A$$

Commutative law

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associative law

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cup A = A$$
$$A \cap A = A$$
$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$
$$A \cup (A^c \cap B) = A \cup B$$
$$A \cap (A^c \cup B) = A \cap B$$
$$A \cup \chi = \chi$$
$$A \cap \emptyset = \emptyset$$
$$A \cup \emptyset = A$$
$$A \cap \chi = A$$
$$A^c \cap A = \emptyset$$
$$A^c \cup A = \chi$$

DeMorgan's law

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Definition 2.1.6. *[6]* ***Al-Saphory, R. (2010)***
The number of elements in a set A is denoted by the ***cardinality*** $| A |$.

Definition 2.1.7. *[6]* ***Al-Saphory, R. (2010)***
A power set $P(A)$ *is a family set containing the subsets of set A. Therefore the number of elements in the power set* $P(A)$ *is represented by*

$$| P(A) |= 2^{|A|}.$$

Example 2.1.2. *If* $A = \{a, b, c\}$, *then*

$$| A |= 3$$

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$
$$| P(A) |= 2^3 = 8.$$

Definition 2.1.8. *[6]* ***Al-Saphory, R. (2010)***
A subset $A \subseteq I\!R^n$ *that is said to be* ***convex*** *if for each* $x, y \in A$,

$$\lambda x + (1 - \lambda)y \in A, \text{ for each } \lambda \in [0, 1]$$

i.e every point on the line connected between two points $x, y \in A$ *is also in* A.

Definition 2.1.9. *[6]* ***Al-Saphory, R. (2010)***
Let χ *be the universe.* ***Membership*** *in a crisp subset* A *of* χ *is often viewed as a characteristic function,*

$$\mu_A : \chi \to \{0, 1\}$$

defined as

$$\mu_A(x) = \begin{cases} 1, x \in A; \\ 0, x \notin A. \end{cases}$$

Remark 2.1.2. *[6]* ***Al-Saphory, R. (2010)***
By using the membership function, the union and the intersection of two sets A, B *will be, For all* $x \in \chi$

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

2.2 Definitions

Definition 2.2.1. *[6]* ***Al-Saphory, R. (2010)***
Let χ be the universal set, ***The fuzzy set*** *A* ***in*** *χ is a set of ordered pairs;*

$$A := \{(x, \mu_A(x)) : x \in \chi\}$$

where,

$$\mu_A : \chi \to [0, 1]$$

is called ***the membership function****, and each $x \in \chi$, the value of $\mu_A(x)$ is called* ***the grade of memberrship of*** *x* ***in*** *A.*

Example 2.2.1.

$$\mu_A(x) = \begin{cases} 1, h \textit{ is full member of } A(h \geq 190) \\ (0,1), h \textit{ is partial member of } A(170 < h < 190) \\ 0, h \textit{ is not member of } A(h \leq 170) \end{cases}$$

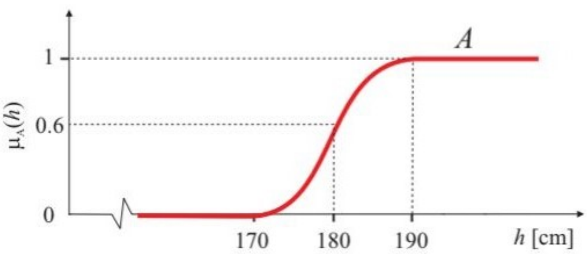

Figure 2.1: The fuzzy set

Notation 2.2.1. *[6]* ***Al-Saphory, R. (2010)***

1. When χ is a finite set $\{x_1, x_2, ..., x_n\}$, a fuzzy set A on χ is expressed as
   $A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + ... + \mu_A(x_n)/x_n = \sum_{i=1}^{n} \mu_A(x_i)/x_i$
   where the term $\mu_A(x_i)$, $i = 1, ..., n$ signifies that μ_i is the grade of membership of x_i in A and the plus sign represents the union.

2. When χ is not finite, we write,

$$A = \int_x \mu_A(x)/x$$

Definition 2.2.2. *[6]* ***Al-Saphory, R. (2010)***
Let $x \in \chi$, then x is called
Not include *in the fuzzy set if $\mu_A(x) = 0$.*
Partial include *if $0 < \mu_A(x) < 1$.*
Full include *if $\mu_A(x) = 1$.*

Definition 2.2.3. *[6]* ***Al-Saphory, R. (2010)***
A fuzzy set is ***empty*** *if and only if its membership function is zero on χ.*

Definition 2.2.4. *[6]* ***Al-Saphory, R. (2010)***
Two fuzzy sets A and B are ***equal****, written as $A = B$, if and only if $\mu_A(x) = \mu_B(x)$ for all x in χ.*

Example 2.2.2.

1. A realtor wants to classify the house he offers to his clients.One indicator of comfort of these houses is the number of bedrooms in it. Let $\chi = \{1, 2, 3, 4, ..., 10\}$ be the set of available types of houses described by x = number of bedrooms in a house. Then the fuzzy set "comfortable type of house for a four-person family" may be described as

$$A = \{(1,0,2),(2,0,5),(3,0,8),(4,1),(5,0,7),(6,0,3)\}$$

or

$$A = 0,2/1 + 0,5/2 + 0,8/3 + 1/4 + 0,7/5 + 0,3/6$$

2. the universe set χ is the set of people. B fuzzy subset **YOUNG** is also defined, which answers the question "to what degree is person x young?" To each person in the universe set, we have to assign a degree of membership in the fuzzy subset **YOUNG**. The easiest way to do this is with a membership function based on the person's age.

$$\mu_B(x) = \begin{cases} 1, \quad age(x) \leq 20 \\ (30 - age(x))/10, 20 \leq age(x) \leq 30 \\ 0, \quad age(x) > 30 \end{cases}$$

Thus $B = \int_B \mu_B(x)/x$

3. $A =$ "real numbers close to 10”

$$A = \{(x, \mu_A(x)) : \mu_A(x) = (1 + (x - 10^2))^{-1}\}$$

Thus

$$A = \int_A (1 + (x - 10)^2)^{-1}/x$$

2.3 Expanding Concepts of Fuzzy set

Definition 2.3.1. *[6]* ***Al-Saphory, R. (2010)***
The support of a fuzzy set *A, $supp(A)$, is the crisp set of all $x \in \chi$ such that $\mu_A(x) > 0$. i.e.*

$$supp(A) := \{x \in \chi : \mu_A(x) > 0\}.$$

Example 2.3.1. *Let* $\chi := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ *be the universe set, and* $A = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6$*, then the support of* $supp(A) = \{1, 2, 3, 4, 5, 6\}$.
The elements $\{7, 8, 9, 10\}$ *are not part of the support of* A.

Definition 2.3.2. *[6]* ***Al-Saphory, R. (2010)***
A fuzzy subset A *of the universal set* χ *is called* ***normal*** *if there exists an* $x \in \chi$ *such that* $\mu_A(x) = 1$*. Otherwise* A *is* ***subnormal****.*

Example 2.3.2. *Let* $A = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6$.
Since $\mu_A(4) = 1$*, then this fuzzy set is normal.*
While the fuzzy set $A = 0.2/1 + 0.5/2 + 0.8/3 + 0.7/5 + 0.3/6$ *is subnormal.*

Definition 2.3.3. *[6]* ***Al-Saphory, R. (2010)***
The maximum value of the membership is called ***height***

Example 2.3.3. *Let* $A = 0.2/1 + 0.5/2 + 0.8/3 + 0.3/6$*. the height of this fuzzy set is* 0.8.

Definition 2.3.4. *[6]* ***Al-Saphory, R. (2010)***
The (crisp) set of elements that belong to the fuzzy set A *at least to the degree* α *is called* ***the*** α***-cut****:*

$$A_\alpha := \{x \in \chi : \mu_A(x) \geq \alpha\}$$

and

$$\tilde{A}_\alpha := \{x \in \chi : \mu_A(x) > \alpha\}$$

is called ***strong*** α***-cut****.*

Example 2.3.4. $A_\alpha = \{x|\mu_A(x) > \alpha\}$ *or* $\tilde{A}_\alpha = \{x|\mu_A(x) \geq \alpha\}$

A_α *is an ordinary set*

Example 2.3.5. *Let* $A = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6$*, then list possible* α*-cut sets:*

$$A_{0.2} = \{1, 2, 3, 4, 5, 6\}$$

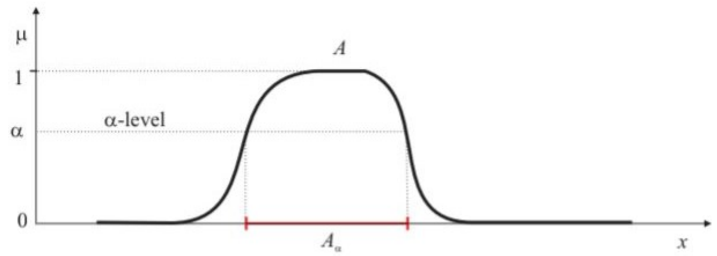

Figure 2.2: α-cut of a Fuzzy set

$$A_{0.5} = \{2, 3, 4, 5\}$$

$$A_{0.8} = \{3, 4\}$$

$$A_1 = \{4\}$$

Definition 2.3.5. *[6]* ***Al-Saphory, R. (2010)***
*The value α which explicity shows the value of the membership function, is in the range of $[0, 1]$. The **level set** is obtained by the α's. That is,* $\Lambda_A := \{\alpha : \mu_A(x) = \alpha, \alpha \geq 0, x \in \chi\}$

Example 2.3.6. *Let* $A = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6$, *then the level set is*

$$\Lambda_A := \{0.2, 0.3, 0.5, 0.7, 0.8, 1\}$$

Example 2.3.7. *Consider a universal set χ which is defined on the age domain.* $\chi := \{5, 15, 25, 35, 45, 55, 65, 75, 85\}$

age(element)	*infant*	*young*	*adult*	*senior*
5	0	0	0	0
15	0	0.2	0.1	0
25	0	1	0.9	0
35	0	0.8	1	0
45	0	0.4	1	0.1
55	0	0.1	1	0.2
65	0	0	1	0.6
75	0	0	1	1
85	0	0	1	1

2.4 Basic Set-Theoretic Operations for Fuzzy Sets

Definition 2.4.1. *[6] **Al-Saphory, R. (2010)***
*Let χ be the universe set, a fuzzy set A is a **subset** of a fuzzy set B, if and only if $\mu_A(x) \leq \mu_B(x)$ for all $x \in \chi$, which is denoted by $A \subseteq B$.*

Definition 2.4.2. *[6] **Al-Saphory, R. (2010)***
*Let χ be the universe set, a fuzzy set A is a **proper subset** of a fuzzy set B, if and only if $\mu_A(x) < \mu_B(x)$ for all $x \in \chi$,*
which is denoted by $A \subset B$.

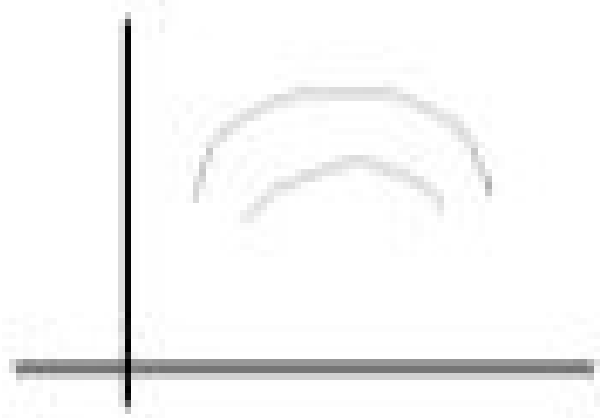

Figure 2.3: fuzzy subset

Remarks 2.4.1. *[6] **Al-Saphory, R. (2010)***

1. Every fuzzy subset is included in itself.
2. Empty fuzzy subset is included in every fuzzy subset.

Definition 2.4.3. *[6] **Al-Saphory, R. (2010)***
***The complement of a fuzzy set** A is denoted by $\tilde{A}$ and is defined by*

$$\mu_{\tilde{A}}(x) = 1 - \mu_A(x)$$

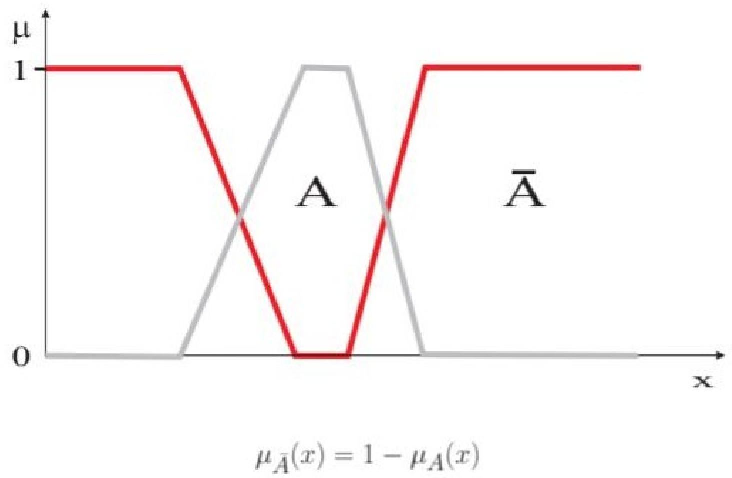

Figure 2.4: The comlement of a fuzzy subset

Example 2.4.1. *Let* $A = 0/3 + 0.4/7 + 1/8$, *then*

$$\mu_{\tilde{A}}(3) = 1 - \mu_A(3) = 1 - 0 = 1$$

$$\mu_{\tilde{A}}(7) = 1 - \mu_A(7) = 1 - 0.4 = 0.6$$

$$\mu_{\tilde{A}}(8) = 1 - \mu_A(8) = 1 - 1 = 0$$

thus, $\tilde{A} = 1/3 + 0.6/7 + 0/8$

Definition 2.4.4. *[6]* ***Al-Saphory, R. (2010)***
The union of two fuzzy sets A ***and*** B *with*
respective membership functions $\mu_A(x)$ *and* $\mu_B(x)$ *is a fuzzy set* C, *written as* $C = A \cup B$, *whose membership function is related to those of* A *and* B *by*

$$\mu_C(x) := \max\{\mu_A(x), \mu_B(x)\}$$

Example 2.4.2. *Let* $A := \{(5, 1), (15, 0.9), (25, 0.1), \}$ *and*
$B := \{(5, 0.1), (10, 0.7), (25, 0.8)\}$, *then*

$$\mu_C(5) = \max\{\mu_A(5), \mu_B(5)\} = \max\{1, 0.1\} = 1$$

$$\mu_C(10) = \max\{\mu_A(10), \mu_B(10)\} = \max\{0, 0.7\} = 0.7$$

$$\mu_C(15) = \max\{\mu_A(15), \mu_B(15)\} = \max\{0.9, 0\} = 0.9$$

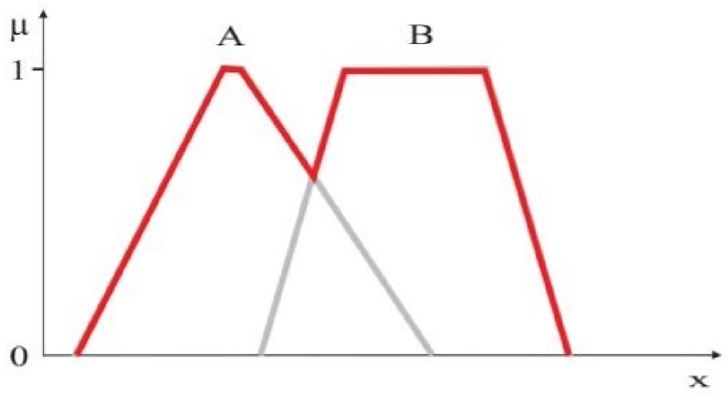

Figure 2.5: The union of a fuzzy subset

$$\mu_C(25) = \max\{\mu_A(25), \mu_B(25)\} = \max\{0.1, 0.8\} = 0.8$$

Thus, $C = \{(5,1),(10,0.7),(15,0.9),(25,0.8)\}$

Proposition 2.4.1. *[6]* ***Al-Saphory, R. (2010)***
Let A, B, C be fuzzy sets, then

$$A \cup (B \cup C) = (A \cup B) \cup C$$

Proof. □

Without lose of the generality, we can assume that

$$\mu_A(x) \le \mu_B(x) \le \mu_C(x) \text{ for all } x \in \chi$$

then

$$\begin{aligned}\mu_{A\cup(B\cup C)}(x) &= \max\{\mu_A(x), \max\{\mu_B(x), \mu_C(x)\}\} \\ &= \max\{\mu_A(x), \mu_C(x)\} \\ &= \mu_C(x) \end{aligned} \tag{2.4.1}$$

On the other hand,

$$\begin{aligned}\mu_{(A\cup B)\cup C}(x) &= \max\{\max\{\mu_A(x), \mu_B(x)\}, \mu_C(x)\} \\ &= \max\{\mu_B(x), \mu_C(x)\} \\ &= \mu_C(x) \end{aligned} \tag{2.4.2}$$

Thus from (2.4.1) and (2.4.2) we get,

$$A \cup (B \cup C) = (A \cup B) \cup C$$

Theorem 2.4.1. *If D is any fuzzy set contains both A and B, then it also contains the union of A and B.*

Proof. □

Let $x \in \chi$, and $C = A \cup B$.

the $A \subseteq D$ and $B \subseteq D$ for all $x \in \chi$

the $\mu_D(x) \geq \mu_A(x)$ and $\mu_D(x) \geq \mu_B(x)$ for all $x \in \chi$

the $\mu_D(x) \geq \max\{\mu_A(x), \mu_B(x)\} = \mu_C(x)$ for all $x \in \chi$

the $C = A \cup B \subseteq D$.

Remark 2.4.1. *[6]* ***Al-Saphory, R. (2010)***
The union of A and B is the smallest fuzzy set containing both A and B.

Definition 2.4.5. *[6]* ***Al-Saphory, R. (2010)***
The intersection of two fuzzy sets A and B
with respective membership functions $\mu_A(x)$ and $\mu_B(x)$ is a fuzzy set C, written as $C = A \cap B$, whose membership function is related to those of A and B by

$$\mu_C(x) := \min\{\mu_A(x), \mu_B(x)\}$$

Remark 2.4.2. *[6]* ***Al-Saphory, R. (2010)***
the intersection of A and B is the largest fuzzy set which is contained in both A and B.

Theorem 2.4.2. *[6]* ***Al-Saphory, R. (2010)***
(De Morgan's laws). Let A and B be two fuzzy sets, then

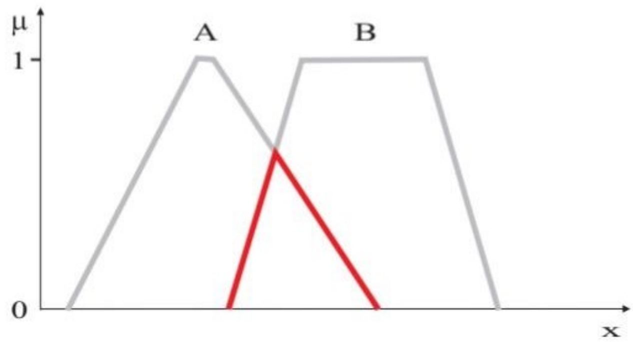

Figure 2.6: The intersection of a fuzzy subset
2

1. $(\widetilde{A \cup B}) = \tilde{A} \cap \tilde{B}$
2. $(\widetilde{A \cap B}) = \tilde{A} \cup \tilde{B}$

Proof. □

1. Without lose of the generality, assume that

$$\mu_A(x) < \mu_B(x) \text{ for all } x \in \chi$$

the $1 - \mu_A(x) > 1 - \mu_B(x)$

Thus

$$\begin{aligned} \mu_{\widetilde{A \cup B}}(x) = & \quad 1 - \mu_{A \cup B}(x) \\ = & \quad 1 - \max\{\mu_A(x), \mu_B(x)\} \\ = & \quad 1 - \mu_B(x) \end{aligned} \tag{2.4.3}$$

On the other hand,

$$\begin{aligned} \mu_{\tilde{A} \cap \tilde{B}}(x) = & \quad \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \\ = & \quad \min\{1 - \mu_A(x), 1 - \mu_B(x)\} \\ = & \quad 1 - \mu_B(x) \end{aligned} \tag{2.4.4}$$

Hence form (2.4.3) and (2.4.4) we get

$$\mu_{\widetilde{A \cup B}}(x) = \mu_{\tilde{A}}(x) \cap \mu_{\tilde{B}}(x)$$

Therefore, $(\widetilde{A \cup B}) = \tilde{A} \cap \tilde{B}$.

Theorem 2.4.3. *[6] **Al-Saphory, R. (2010)***
(Distributive Laws). Let A,B and C be fuzzy sets, then

1. $C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$
2. $C \cup (A \cap B) = (C \cup A) \cap (C \cup B)$

Proof. □

1. Without lose of the generality, we assume that

$$\mu_A(x) < \mu_B(x) < \mu_C(x) \text{ for all } x \in \chi$$

Therefore,

$$\begin{aligned} \mu_{C\cap(A\cup B)}(x) &= \min\{\mu_C(x), \max\{\mu_A(x), \mu_B(x)\}\} \\ &= \max\{\mu_C(x), \mu_B(x)\} \\ &= \mu_B(x) \end{aligned} \tag{2.4.5}$$

On the other hand,

$$\begin{aligned} \mu_{(C\cap A)\cup(C\cap B)}(x) &= \max\{\min\{\mu_C(x), \mu_A(x)\}, \min\{\mu_C(x), \mu_B(x)\}\} \\ &= \max\{\mu_A(x), \mu_B(x)\} \\ &= \mu_B(x) \end{aligned} \tag{2.4.6}$$

Hence from (2.4.5) and (2.4.6) we get,

$$\mu_{C\cap(A\cup B)}(x) = \mu_{(C\cap A)\cup(C\cap B)}(x) \text{ for all } x \in \chi$$

Therefore

$$C \cap (A \cup B) = (C \cap A) \cup (C \cap B).$$

2.5 Convex Fuzzy Subsets and The Cardinality

Definition 2.5.1. *[6] **Al-Saphory, R. (2010)***
*A fuzzy set A is **convex** if and only if its $\alpha-$ cuts are convex.*

Theorem 2.5.1. *[6] **Al-Saphory, R. (2010)***
*A is **convex** if and only if for all $x, y \in \chi$,*

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\} \textit{ for all } \lambda \in [0,1]. \tag{2.5.1}$$

Proof. □

$\Rightarrow$) Let $x, y \in \chi$, assume that $\alpha = \mu_A(x) \leq \mu_A(y)$.
the $A_\alpha = \{z \in \chi : \mu_A(z) \geq \alpha\} = \{z \in \chi : \mu_A(z) \geq \mu_A(x)\}$
the $x,\ y \in A_\alpha$
the A_α is convex set
the $\lambda x + (1-\lambda)y \in A_\alpha$
Hence,

$$\mu_A(\lambda x + (1-\lambda)y) \geq \mu_A(x) \tag{2.5.2}$$

Similarly, if $\alpha = \mu_A(y) \leq \mu_A(x)$, then

$$\mu_A(\lambda x + (1-\lambda)y) \geq \mu_A(y) \tag{2.5.3}$$

Therefore, from(2.5.2) and(2.5.3) we get

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x),\ \mu_A(y)\}.$$

$\Leftarrow$) Let $x \in \chi$ and $\alpha = \mu_A(x)$
Claim: A_α is convex set:
Let $u, v \in A_\alpha$ and $\lambda \in [0,1]$ the $\mu_A(u) \geq \mu_A(x)$ and$\mu_A(v) \geq \mu_A(x)$
By (2.5.1),

$$\begin{aligned}\mu_A(\lambda u + (1-\lambda)v) &\geq \min\{\mu_A(u),\ \mu_A(v)\}\\ &\geq \mu_A(x) = \alpha\end{aligned}$$

the $\lambda u + (1-\lambda)v \in A_\alpha$
the A_α is convex set.

Example 2.5.1.

$\mu_A(t) \geq \min(\mu_A(r),\ \mu_A(s))$
where $t = \lambda r + (1-\lambda)s,\ \ r,\ \ s \in \mathbb{R},\ \lambda \in [0,1]$

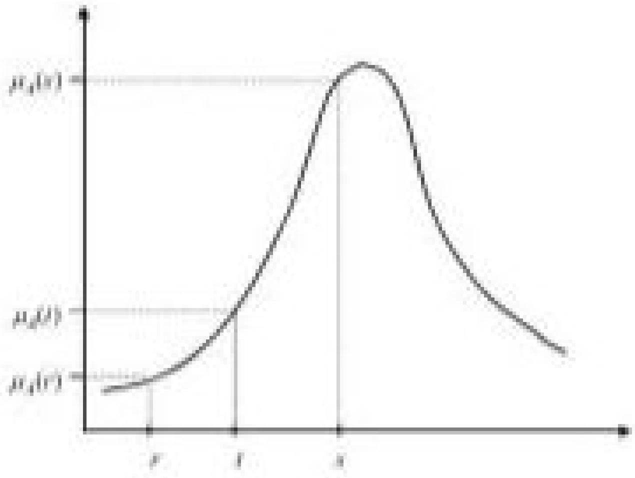

Figure 2.7: convex fuzzy set

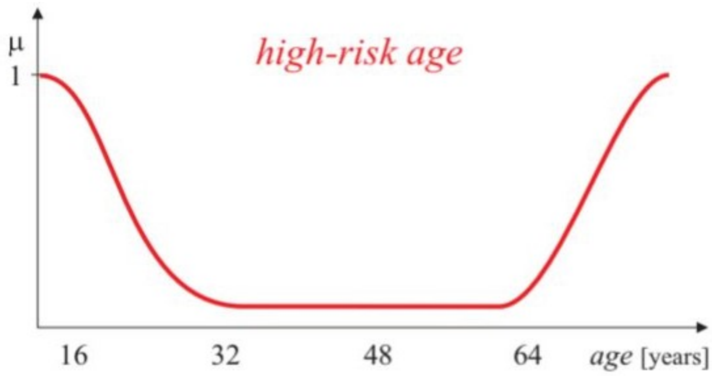

Figure 2.8: non-convex fuzzy set

Theorem 2.5.2. *[6]* ***Al-Saphory, R. (2010)***
If A and B are convex sets, so is their intersection.

Proof. □

Let $C = A \cap B$ where A and B are convex fuzzy sets.
Let $x, y \in \chi$ and $\lambda \in [0, 1]$, then

$$\mu_C(\lambda x + (1-\lambda)y) = \min\{\mu_A(\lambda x + (1-\lambda)y),\ \mu_B(\lambda x + (1-\lambda)y)\}$$

$\geq \min\{\min\{\mu_A(x), \mu_A(y)\}, \min\{\mu_B(x), \mu_B(y)\}\}$

$= \min\{\mu_A(x), \mu_A(y), \mu_B(x), \mu_B(y)\}$

$= \min\{\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(y), \mu_B(y)\}\}$

$= \min\{\mu_C(x), \mu_C(y)\}$

the $\mu_C(\lambda x + (1-\lambda)y) \geq \min\{\mu_C(x), \mu_C(y)\}$

Hence by theorem (2.5.2), $C = A \cap B$ is convex fuzzy set

Definition 2.5.2. *[6] **Al-Saphory, R. (2010)***
Let χ be the universe set, and A be a fuzzy set,

Scalar Cardinality ($| A |$)**:**

the scalar is defined as the sum of the grade of the membership of finite fuzzy set A. That is:

$$| A | := \sum_{x \in \chi} \mu_A(x).$$

Remarks 2.5.1. *[6] **Al-Saphory, R. (2010)***
For any fuzzy sets A and B,

1. *If for all $K, \mu_B(x_K) \leq \mu_A(x_K)$, then $| B | \leq | A |$.*

   Proof. □

   $| B | = \sum_K \mu_B(x_K) \leq \sum_K \mu_A(x_K) = | A |$

2. $| \tilde{A} | = | \chi | - | A |$.

   Proof. □

$$|\tilde{A}| = \sum_{K=1}^{n} \mu_{\tilde{A}}(x_K) = \sum_{K=1}^{n}(1 - \mu_A(x_K))$$

$$= n - \sum_{K=1}^{n} \mu_A(x_K) = |\chi| - |A|$$

3. $|A \cup B| + |A \cap B| = |A| + |B|$.

Relative Cardinality ($\| A \|$)**:**

$$\| A \| := \frac{|A|}{|\chi|}$$

Remarks 2.5.2. *[6] **Al-Saphory, R. (2010)***
For any fuzzy set A,

1. $0 \leq \| A \| \leq 1$.

2. $\| A \| = 0$ *if* $\mu_A(x_K) = 1$ *for all* K. *Since,*

$$\| A \| := \frac{|A|}{|\chi|} = \frac{0}{|\chi|} = 0$$

3. $\| A \| = 1$ *if* $\mu_A(x_K) = 1$ *for all* K. *Since,*

$$\| A \| := \frac{|A|}{|\chi|} = \frac{\sum_{K=1}^{n} \mu_A(x_K)}{|\chi|} = \frac{n}{n} = 1$$

Fuzzy Cardinality ($|A|_F$)**:**

$$|A|_F := \{(|A_\alpha|, \alpha) : \alpha \in [0,1]\}$$

Example 2.5.2.

Let $\chi = \{1,2,3,4,5,6,7,8,9,10\}$
$A = 0.2/1 + 0.5/2 + 0.8/3 + 1/4 + 0.7/5 + 0.3/6$,
$B = 0.3/1 + 0.9/3 + 0.8/6 + 0.1/7 + 0.4/8 + 0.6/9 + 1/10$,
then

Scalar Cardinality ($\mid A \mid$)**:**

$\mid A \mid = \sum_{x \in \chi} \mu_A(x) = 0.2 + 0.5 + 0.8 + 1 + 0.7 + 0.3 = 3.5$
$\mid B \mid = \sum_{x \in \chi} \mu_B(x) = 0.3 + 0.9 + 0.8 + 0.1 + 0.4 + 0.6 + 1 = 4.1$

While,

$\mid A \cup B \mid = 0.3 + 0.5 + 0.9 + 1 + 0.7 + 0.8 + 0.1 + 0.4 + 0.6 + 1 = 6.3$
$\mid A \cap B \mid = 0.2 + 0.8 + 0.3 = 1.3$

Thus,

$$\mid A \cup B \mid + \mid A \cap B \mid = 7.6 = \mid A \mid + \mid B \mid$$

Relative Cardinality ($\| A \|$)**:**

$$\| A \| := \frac{\mid A \mid}{\mid \chi \mid} = \frac{3.5}{10} = 0.35$$

Fuzzy Cardinality ($\mid A \mid_F$)**:**

$$A_{0.2} = \{1, 2, 3, 4, 5, 6\}$$

$$A_{0.3} = \{2, 3, 4, 5, 6\}$$

$$A_{0.5} = \{2, 3, 4, 5\}$$

$$A_{0.7} = \{3, 4, 5\}$$

$$A_{0.8} = \{3, 4\}$$

$$A_1 = \{4\}$$

Thus,

$$\mid A_{0.2} \mid = 6$$

$$| A_{0.3} |= 5$$

$$| A_{0.5} |= 4$$

$$| A_{0.7} |= 3$$

$$| A_{0.8} |= 2$$

$$| A_1 |= 1$$

$| A |_F = \{(| A_\alpha |, \alpha) : \alpha \in [0, 1]\}$
$= \{(6, 0.2), (5, 0.3), (4, 0.5), (3, 0.7), (2, 0.8), (1, 1)\}$

Example 2.5.3.

1. Let $\chi = [1, 5]$ and $A := \int_x \frac{1}{x}/x$. Is A convex?

2. According to example (2.3.7). Calculate the scalar cardinality of young fuzzy set, the relative cardinality to adult fuzzy set, and the fuzzy cardinality of senior fuzzy set.

3. Compute the relative cardinality of $A \cup B$ and the scalar cardinality of $\widetilde{A \cap B}$ where,

$$A = \{(x, 0.4), (y, 0.5), (z, 0.9), (w, 1)\}$$

$$B = 0.5/u + 0.8/v + 0.9/w + 0.1/x$$

2.6 Expansion of Fuzzy Set

Definition 2.6.1. *[6]* ***Al-Saphory, R. (2010)***
If the value of membership function is given by a fuzzy set, it is type-2 fuzzy set. This concept can be extended up to Typen fuzzy set.

Example 2.6.1.

Consider set $A =$ adult. The membership function of this set maps whole age to youth, manhood and senior. For instance, for any person $x, y,$ and z,

$$\mu_A(x) = \text{youth}$$

$$\mu_A(y) = \text{manhood}$$

$$\mu_A(z) = \emptyset$$

The values of membership for youth and manhood are also fuzzy sets, and thus the set adult is a type-2 fuzzy set. The sets youth and manhood are type-1 fuzzy sets.

Remark 2.6.1. *[6] **Al-Saphory, R. (2010)***
In the same manner, if the values of membership function of youth and manhood are type-2, the set adult is type-3.

Definition 2.6.2. *[6] **Al-Saphory, R. (2010)***
consider a fuzzy set satisfying $A \neq \emptyset$ and $A \neq \chi$.
*The pair $(A, \tilde{A})$ is defined as **fuzzy partition**. More generally, if there are m subsets defined in χ, $(A_1, A_2, ..., A_n)$ holding the following conditions is called a **fuzzy partition**.*

1. $\forall i,\ \ A_i \neq \emptyset$.

2. $A_i \cap A_j = \emptyset$ *for all* $i \neq j$.

3. $\forall x \in \chi,\ \sum_{i=1}^{m} \mu_{A_i}(x) = 1$.

CHAPTER 3

MAXIMUM SOLUTION OF FUZZY RELATION EQUATIONS

In this chapter we will discuss the methods of finding the maximal solution of FRE (FUZZY RELATION EQUATIONS) with respect to unknowns R and Q. We will discuss the methods for finding the maximal "Q" and maximal "R" respectively for FRE of the from $R \circ Q = T$. Here "∇" denotes the maximal solution.

3.1 Determination of Maximal Q

[7] **Ahmed, U. saqib, M. (2010)**
For determination of maximal Q, first, we will discuss some results.

Lemma 3.1.1. *[7] **Ahmed, U. saqib, M. (2010)***
If we have two fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$ then the following inclusion will hold:

$$Q \subseteq R^{-1}L(R \circ Q) \tag{3.1.1}$$

where "$\circ$" denotes the max-min composition and "L" is the composition made by $\alpha -$ operator.

$$\mu_{R^{-1}}(y, x) = R(x, y) \tag{3.1.2}$$

$$a\alpha(b\Delta c) \geq aLc \tag{3.1.3}$$

Proof. □

Let $A = R^{-1}L(R \circ Q) \subseteq Y \times Z$.
Then by using (3.1.2) and (3.1.3), we have

$$\mu_A(y,z) = \bigwedge_{x \in X} \{\mu_{R^{-1}}(y,x)\alpha\mu_{R\circ Q}(x,z)\}$$

$$= \bigwedge_{x \in X} \{\mu_R(x,y)\alpha\mu_{R\circ Q}(x,z)\}$$

$$= \bigwedge_{x \in X} (x,y)\alpha(\bigvee\{\mu_R(x,t) \bigwedge \mu_Q(t,z)\}))$$

$$\bigwedge_{x \in X} (\mu_R(x,y)\alpha(\mu_R(x,y) \wedge \mu_Q(y,z) \wedge \bigvee_{t \in Y t \neq y} (\mu_R(x,t) \wedge \mu_Q(t,z))))$$

so it becomes

$$\mu_A(y,z) \geq \bigwedge_{x \in X} \{\mu_R(x,y)\alpha(\mu_R(x,y) \wedge \mu_Q(y,z))\}$$

as we know that

$$a\,\alpha(a \wedge b) \geq b \tag{3.1.4}$$

then by using (3.1.4), we have

$$\mu_A(y,z) \geq \mu_Q(y,z)\,\forall y \in Y \text{ and } Z \in Z$$

3.1.1 Example of lemma 3.1.1

Consider $X = \{x_1, x_2, x_3, x_4\}, Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2, z_3\}$.

$$R = \begin{bmatrix} & y_1 & y_2 & y_3 \\ x_1 & 0.1 & 0 & 0.9 \\ x_2 & 0.5 & 0.8 & 1 \\ x_3 & 0.6 & 0.3 & 0.2 \\ x_4 & 1 & 0.1 & 0.4 \end{bmatrix}$$

$$Q = \begin{bmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.1 & 0.6 \\ y_2 & 0 & 0.3 & 0.2 \\ y_3 & 0.6 & 1 & 0.8 \end{bmatrix}$$

$$R \circ Q = \begin{bmatrix} 0.6 & 0.9 & 0.8 \\ 0.6 & 1 & 0.8 \\ 0.4 & 0.3 & 0.6 \\ 0.4 & 0.4 & 0.6 \end{bmatrix} \textbf{ and } R^{-1} = \begin{bmatrix} 0.1 & 0.5 & 0.6 & 1 \\ 0 & 0.8 & 0.3 & 0.1 \\ 0.9 & 1 & 0.2 & 0.4 \end{bmatrix}$$

$$R^{-1}L(R \circ Q) = \begin{bmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.4 & 0.3 & 0.6 \\ y_2 & 0.6 & 1 & 1 \\ y_3 & 0.6 & 1 & 0.8 \end{bmatrix}$$

So this example shows that $Q \subset R^{-1}L(R \circ Q)$. Hence it clearly satisfies the lemma (3.1.1)

Lemma 3.1.2. *[7] **Ahmed, U. saqib, M. (2010)***
Assume that we have two fuzzy relations $R \subseteq X \times Y$ and $T \subseteq X \times Z$ then the following inclusion holds:

$$R \circ (R^{-1}LT) \subset T \tag{3.1.5}$$

where " $\circ$ " denotes the max-min composition and "L" is the composition of $\alpha -$ operator. The proof of this lemma is analogous to the proof of lemma (3.1.1)

Lemma 3.1.3. *[7] **Ahmed, U. saqib, M. (2010)***
Consider we have two fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$ then the following inclusion holds:

$$R \subseteq (QL(R \circ Q)^{-1})^{-1} \tag{3.1.6}$$

Lemma 3.1.4. *[7] **Ahmed, U. saqib, M. (2010)***
Consider we have two fuzzy relations $Q \subseteq Y \times Z$ and $T \subseteq X \times Z$ then the following inclusion holds:

$$(QLT^{-1})^{-1} \circ Q \subset T \tag{3.1.7}$$

Theorem 3.1.1. *[7] **Ahmed, U. saqib, M. (2010)***
Let $R \subseteq X \times Y$ and $T \subseteq X \times Z$ be the two fuzzy relations, $S(Q)$ be the set of fuzzy relations $Q \in Y \times Z$ such that $R \circ Q = T$.
$S(Q) = \{Q \in Y \times Z | R \circ Q = T\} \neq \phi$, if and only if
$R^{-1}LT \in S(Q)$ then" $R^{-1}LT$" is the the greatest element in $S(Q)$

Theorem 3.1.2. *[7]* ***Ahmed, U. saqib, M. (2010)***
Let $R \subseteq X \times Y$ and $T \subseteq X \times Z$ be the two fuzzy relations, the set of fuzzy relations $Q \in Y \times Z$ such that $R \circ Q \subseteq T$ contains a greatest element $R^{-1}LT$.

Proof. □

Let $S(Q)^* = \{\text{Fuzzy } Q \in (Y \times Z) | R \circ Q \subseteq T\}$ and$(R)^* \neq \phi$.
because of the null relation

$$0(y,z) = 0 \ \ \forall (y,z) \in Y \times Z, \in S(Q)^*$$

let $Q \subseteq S(R)^* : R \circ Q = T$
then we have

$$R^{-1}L(R \circ Q) \subseteq R^{-1}LT,$$

but from lemma (3.1.1), we have

$$Q \subset R^{-1}L(R \circ Q)$$

then it shows that

$$Q \subset R^{-1}LT$$

now from Theorem we have

$$R^{-1}LT \in S(Q).$$

Then it shows that $R^{-1}LT \in S(Q)^*$, then $R^{-1}LT$ will be the greatest element in $S(Q)^*$. Hence $R^{-1}LT$ be the greatest element in $S(Q)^*$. Then

$$Q^{\nabla} = R^{-1}LT \tag{3.1.8}$$

which is the maximum relation "Q" satisfying the equation $R \circ Q = T$.

3.1.2 Necessary Condition For Existence Of Q^{∇}

[7] **Ahmed, U. saqib, M. (2010)**
The necessary condition for the existence of Q^{∇} satisfying the FRE (2.1) is

$$\mu_T(x,z) \leq \bigvee_{y \in Y} \mu_R(x,y) \qquad \forall x \in X \text{ and } z \in Z \tag{3.1.9}$$

3.1.3 Example of Determining the Maximal Q

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and $Z = \{z_1, z_2, z_3\}$.
Suppose $R \subseteq X \times Y$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively

$$\begin{matrix} & y_1 & y_2 & y_3 & y_4 \end{matrix}$$

$$R = \begin{bmatrix} x_1 & 1 & 0.3 & 0.2 & 0.7 \\ x_2 & 0.1 & 0 & 0.9 & 0.4 \\ x_3 & 0.5 & 0.6 & 0 & 0.7 \end{bmatrix}$$

and

$$T = \begin{bmatrix} & z_1 & z_2 & z_3 \\ x_1 & 0.3 & 0.6 & 0.9 \\ x_2 & 0.7 & 0.4 & 0.4 \\ x_3 & 0.6 & 0.6 & 0.5 \end{bmatrix}$$

We are to compute "Q^{∇}"
First we check the necessary condition for the existence of "Q^{∇}" using (3.1.9).

$$\mu_T(x, z) \leq \bigvee_{y \in Y} \mu_R(x, y).$$

$$\begin{bmatrix} 0.3 & 0.6 & 0.9 \\ 0.7 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.5 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 0.9 \\ 0.7 \end{bmatrix}$$

Hence the necessary condition (3.1.9) is satisfied. So

$$R^{-1} = \begin{bmatrix} & x_1 & x_2 & x_3 \\ y_1 & 1 & 0.1 & 0.5 \\ y_2 & 0.3 & 0 & 0.6 \\ y_3 & 0.2 & 0.9 & 0 \\ y_4 & 0.7 & 0.4 & 0.7 \end{bmatrix}$$

now we compute $R^{-1}LT$

$$R^{-1}LT = \begin{bmatrix} 1 & 0.1 & 0.5 \\ 0.3 & 0 & 0.6 \\ 0.2 & 0.4 & 0.7 \\ 0.7 & 0.4 & 0.7 \end{bmatrix} @ \begin{bmatrix} 0.3 & 0.6 & 0.9 \\ 0.7 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} \bigwedge(0.3,1,1) & \bigwedge(0.6,1,1) & \bigwedge(0.9,1,1) \\ \bigwedge(1,1,1) & \bigwedge(1,1,1) & \bigwedge(1,1,0.5) \\ \bigwedge(1,0.7,1) & \bigwedge(1,0.4,1) & \bigwedge(1,0.4,1) \\ \bigwedge(0.3,1,0.6) & \bigwedge(0.6,1,0.6) & \bigwedge(1,1,0.5) \end{bmatrix}$$

$$Q^{\nabla} = R^{-1}LT = \begin{bmatrix} & z_1 & z_2 & z_3 \\ y_1 & 0.3 & 0.6 & 0.9 \\ y_2 & 1 & 1 & 0.5 \\ y_3 & 0.7 & 0.4 & 0.4 \\ y_4 & 0.3 & 0.6 & 0.5 \end{bmatrix}$$

Check:

Here we check whether "Q^{∇}" satisfies the FRE (2.1) i.e $R \circ Q^{\nabla} = T$ or nor?

$$R \circ Q^{\nabla} = \begin{bmatrix} 1 & 0.3 & 0.2 & 0.7 \\ 0.1 & 0 & 0.9 & 0.4 \\ 0.5 & 0.6 & 0 & 0.7 \end{bmatrix} \circ \begin{bmatrix} 0.3 & 0.6 & 0.9 \\ 1 & 1 & 0.5 \\ 0.2 & 0.4 & 0.7 \\ 0.7 & 0.4 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.6 & 0.9 \\ 0.7 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.5 \end{bmatrix} = T$$

3.1.4 Example of Determining the Maximal Q

Consider $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2\}$.
Suppose $R \subseteq X \times Y$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively.
We are to compute "Q^{∇}" when

$$R = \begin{bmatrix} 0.1 & 0.2 & 1 \\ 0 & 0.9 & 0.3 \\ 0.8 & 0.5 & 0.1 \\ 1 & 0.2 & 0 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.6 & 1 \\ 0.7 & 1 \\ 0.2 & 0 \end{bmatrix}$$

Since the condition (3.1.9) is satisfied for the above given R and T then, by using (3.1.8), our result will be

$$Q^{\nabla} = R^{-1}LT = \begin{bmatrix} 0.6 & 1 \\ 0.7 & 1 \\ 0.2 & 0.1 \end{bmatrix}$$

Q^{∇} also satisfies the FRE i.e $R \circ Q = T$.

3.1.5 Example of Determining the Maximal Q

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2, z_3\}$.
Suppose $R \subseteq X \times Y$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively. We are to compute "Q" for

$$R = \begin{bmatrix} 0.1 & 1 & 3 \\ 0.2 & 0 & 0.9 \\ 1 & 0.5 & 0.4 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.3 & 0.6 & 0.5 \\ 0.3 & 0.9 & 0.2 \\ 0.3 & 0.5 & 0.5 \end{bmatrix}$$

Since the condition (3.1.9) is fulfilled for the above given R and T.
So by using (3.1.8), our result will be

$$Q^{\nabla} = R^{-1}LT = \begin{bmatrix} 0.3 & 0.5 & 0.5 \\ 0.3 & 0.6 & 0.5 \\ 0.3 & 1 & 0.2 \end{bmatrix}$$

Q^{∇} also satisfies the FRE i.e $R \circ Q = T$.

3.2 Determination of Maximal R

[7] **Ahmed, U. saqib, M. (2010)**
First we will discuss some results in order to determine the maximal R.

Theorem 3.2.1. *[7] **Ahmed, U. saqib, M. (2010)***
Let $Q \subseteq Y \times Z$ and $T \subseteq X \times Z$ be the two fuzzy relations, $S(R)$ be the set of fuzzy relations $R \in X \times Y$ such that $R \circ Q = T$.
$S(R) = \{FuzzyR \in X \times Y | R \circ Q = T\} \neq \phi$, if and only if $(QLT^{-1})^{-1} \in S(R)$ and it is the greastest element in $S(R)$.

Theorem 3.2.2. *[7] **Ahmed, U. saqib, M. (2010)***
Let $Q \subseteq Y \times Z$ and $T \subseteq X \times Z$ be the two fuzzy relations, then the set of fuzzy relations $R \in X \times Y$ such that $R \circ Q \subseteq T$ contains a greatest element $(QLT^{-1})^{-1}$

Proof. □

Let $S(R)^* = \{\text{Fuzzy } R \in (X \times Y) | R \circ Q \subseteq T\}$ and $S(R)^* \neq \phi$ because of the null relation

$$0(x, y) = 0 \ \ \forall (x, y) \in Y \times Z, \in S(R)^*$$

Let $R \subseteq S(R)^* : R \circ Q = T$, then we have

$$(QL(R \circ Q)^{-1})^{-1} \subseteq (QLT^{-1})^{-1}$$

but from lemma (3.1.3), we have

$$R \subset (QL(R \circ Q)^{-1})^{-1}$$

then it shows that

$$R \subset (QLT^{-1})^{-1}$$

now from Theorem (3.2.2), we have

$$(QLT^{-1})^{-1} \in S(R).$$

Then it shows that $(QLT^{-1})^{-1} \in S(R)^*$, then $(QLT^{-1})^{-1}$ will be the greatest element in $S(R)^*$. Hence $(QLT^{-1})^{-1}$ be the greatest element in $S(R)^*$. So

$$R^{\nabla} = (QLT^{-1})^{-1} \tag{3.2.1}$$

which is the maximum relation for "R" satisfying the equation $R \circ Q = T$.

3.2.1 Necessary Condition for Existence of R^{∇}

[7] **Ahmed, U. saqib, M. (2010)**
The necessary condition for the existence of R^{∇} satisfying the FRE (2.1) is

$$\mu_T(x,z) \leq \bigvee_{y \in Y} \mu_Q(y,z) \;\; \forall x \in X \text{ and } z \in Z \tag{3.2.2}$$

3.2.2 Example of Determining the Maximal R

Consider $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3, y_4\}$ and $Z = \{z_1, z_2\}$. Suppose $Q \subseteq Y \times Z$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively

$$Q = \begin{bmatrix} & z_1 & z_2 \\ y_1 & 0.6 & 0 \\ y_2 & 0.3 & 0.5 \\ y_3 & 0.7 & 0.1 \\ y_4 & 0.5 & 1 \end{bmatrix}$$

and

$$T = \begin{bmatrix} & z_1 & z_2 \\ x_1 & 0.7 & 0.4 \\ x_2 & 0.5 & 0.7 \\ x_3 & 0.6 & 0.5 \end{bmatrix}$$

For the compute "R^{∇}".
First we check the necessary condition for the existence of "R^{∇}" using (3.2.2)
Since it is clear from the above given $Q$and Tthat $\mu_T(x,z) \leq \bigvee_{y\in Y} \mu_Q(y,z)$.
So now we compute $(QLT^{-1})^{-1}$.

$$T^{-1} = \begin{bmatrix} & x_1 & x_2 & x_3 \\ z_1 & 0.7 & 0.5 & 0.6 \\ z_2 & 0.4 & 0.7 & 0.5 \end{bmatrix}$$

So

$$QLT^{-1} = \begin{bmatrix} 0.6 & 0 \\ 0.3 & 0.5 \\ 0.7 & 0.1 \\ 0.5 & 1 \end{bmatrix} L \begin{bmatrix} 0.7 & 0.5 & 0.6 \\ 0.4 & 0.7 & 0.5 \end{bmatrix} = \begin{bmatrix} \bigwedge(1,1) & \bigwedge(0.5,1) & \bigwedge(1,1) \\ \bigwedge(1,0.4) & \bigwedge(1,1) & \bigwedge(1,1) \\ \bigwedge(1,1) & \bigwedge(0.5,1) & \bigwedge(0.6,1) \\ \bigwedge(1,0.4) & \bigwedge(1,0.7) & \bigwedge(1,0.5) \end{bmatrix}$$

$$QLT^{-1} = \begin{bmatrix} 1 & 0.5 & 1 \\ 0.4 & 1 & 1 \\ 1 & 0.5 & 0.6 \\ 0.4 & 0.7 & 0.5 \end{bmatrix}$$

$$R^{\nabla} = (QLT^{-1})^{-1} = \begin{bmatrix} & y_1 & y_2 & y_3 & y_4 \\ x_1 & 1 & 0.4 & 1 & 0.4 \\ x_2 & 0.5 & 1 & 0.5 & 0.7 \\ x_3 & 1 & 1 & 0.6 & 0.5 \end{bmatrix}$$

Check:
Here we check whether "R^{∇}" satisfies the FRE i.e $R \circ Q = T$ or nor?

$$R^{\nabla} \circ Q = \begin{bmatrix} 1 & 0.4 & 1 & 0.4 \\ 0.5 & 1 & 0.5 & 0.7 \\ 1 & 1 & 0.6 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.6 & 0 \\ 0.3 & 0.5 \\ 0.7 & 0.1 \\ 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.4 \\ 0.5 & 0.7 \\ 0.6 & 0.5 \end{bmatrix} = T$$

Hence, Q^{∇} satisfies the FRE i.e $R \circ Q = T$.

3.2.3 Example of Determining the Maximal R

Consider $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2, z_3, z_4\}$.
Suppose $Q \subseteq Y \times Z$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively

$$Q = \begin{bmatrix} 0.2 & 0 & 0.9 & 1 \\ 1 & 0.5 & 0.3 & 0.6 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0.3 & 0.3 & 0.9 & 1 \\ 0.7 & 0.5 & 0.3 & 0.6 \\ 0.2 & 0.2 & 0.9 & 0.9 \end{bmatrix}$$

We compute "R^{∇}":
Since the necessary condition (3.2.2) is satisfied for the above given Q and T. Then by using (3.2.1), the result will be

$$R^{\nabla} = (QLT^{-1})^{-1} \begin{bmatrix} 1 & 0.3 & 0.9 \\ 0.3 & 0.7 & 0.2 \end{bmatrix}$$

R^{∇} satisfies the FRE i.e $R \circ Q = T$.

Conclusion

In conclusion, thus memory has shown that fuzzy set theory represents a significant shift in handling imprecise concepts and uncertain data by offering a flexible mathematical framework that allows for gradual membership instead of traditional binary classification.

Throughout thus study, we explored the fundamental principles of fuzzy set theory, its basic operations, and some of its practical applications.

The importance of thus theory lies in its ability to simulate human reasoning and deal with vague information, making it a powerful tool in various fields such as artificial intelligence, decision-making systems, and control engineering.

Despite the simplicity of its foundational concepts, the applications of fuzzy set theory continue to expand and evolve across multiple domains.

We hope that thus memory serves as a concise and useful introduction for those interested in the field and opens the door to further research and deeper exploration into fuzzy sets and their advance...

BIBLIOGRAPHY

[1] **D. Dubois, H. Prade**, Operations on Fuzzy Numbers, Int J. Syst Sci. 9(1978), 613-626.

[2] **R. Goetschel Jr, W. Voxamn**, Elementary Fuzzy Calculus, Fuzzy Sets Syst. 18(1986), 31-43.

[3] **Novak, Vilem (1989)**. Fuzzy Sets and Their Applications. Bristol: Adam Hilger.

[4] **Hajek, Petr (August 1995)**. "Fuzzy logic and arithmetical hierarchy". Fuzzy Sets and System. 73(3):

[5] **M. Ma, M. Friedman, A. Kandel**, A New Fuzzy Arithmetic, Fuzzy Sets Syst. 108(1999), 83-90.

[6] **Al-Saphory, R. (2010)**. Strategic sensors and regional observation in Neumann boundary conditions. Zanco Journal of Pure and Applied Science, 22, 227-237

[7] **Ahmed, U. saqib, M. (2010)**. Optimal Solutions Of Fuzzy Relation Equations