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DEDICATIONS

After long years of study and work, I dedicate this modest work:
To my dear parents, may God bless and protect them for their unwavering moral and financial support, their encouragement, and the sacrifices they have made.
To my brothers:Abdelkader, Amro and sisters:Radia, Fatima, Nour.
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Introduction

The study of fractional calculus has gained increasing attention due to its wide range of applications in various fields such as physics, engineering, biology, and finance.

Unlike classical calculus,fractional calculus deals with derivatives and integrals of arbitrary(non-integer)order, offering more accurate models of memory and hereditary properties of different materials and processes. This report focuses on the theoretical and numerical aspects of fractional differential equations. It begins with a comprehensive preliminary section that introduces essential mathematical concepts such as the Riemann-Liouville fractional integral and derivative, the Gamma and Beta functions, and the notion of phase space.

Fundamental tools like fixed point theorems and different types of stability are also introduced as necessary foundations for the following chapters.

The final chapter explores various numerical methods for solving fractional differential equations. Techniques such as the finite difference method, Euler's discretization, and different approaches for approximating fractional derivatives are discussed in detail.

Numerical applications and results, especially concerning the Riemann-Liouville derivative, are provided to illustrate the effectiveness and accuracy of the proposed methods.

Through this work, we aim to contribute to the understanding and application of fractional calculus, providing both a solid theoretical background and practical computational tools for further research and development in the field.

Chapter 1

PRELIMINARY RATING

In this chapter we will clearly give some useful results concerning the integral and the derivative of fractional order in the sense of Riemann-Liouville and we will identify the relationship between the two .

1.1 Notations and definitions

The symbol $\Psi = C(D, \mathbb{R})$ represents the Banach space of continuous functions from D to $\mathbb{R}$ of norm

$$\|w\|_{\Psi} = \sup_{\varphi \in D} |w(\varphi)|$$

1.1.1 The Gamma function

One of the basic functions used in fractional calculus is the Euler Gamma function. It extends the factorial function to real numbers and even to complex numbers.

Definition 1.1.1. *[14] For $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$, Euler's Gamma function is defined by the following integral:*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt \tag{1.1}$$

Remark 1.1.1. *The Gamma function satisfies the following properties:*

1) **The recurrence relation** : for all $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$ on a :

$$\Gamma(z+1) = z\Gamma(z)$$

And for everything $n \in \mathbb{N}^*$ on a :

$$\Gamma(n) = (n-1)!$$

Especially $\Gamma(1) = 1$.

2) **The representation of the Gamma function by the limit:**

$$\Gamma(z) = \lim_{n\to+\infty} \frac{n!n^z}{(z+1)\cdots(z+n)}$$

with $\mathrm{Re}(z) > 0$.

3) **The derivative:** The Gamma function is indefinitely differentiable on $\mathbb{R}_+^*$ its derivative is:

$$\Gamma'(z) = \Gamma(z)\Psi(z)$$

Or

$$\Psi(z) = \frac{d}{dz}\ln[\Gamma(z)].$$

4) **Some special values of the Gamma function:**

For $z = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

For $z = n + \frac{1}{2}, n \in \mathbb{N}^*$,

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}.$$

1.1.2 The Beta function

Definition 1.1.2. *[14]The Beta function is a type of Euler integral defined for complex numbers z and w by:*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt. \tag{1.2}$$

With $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$.

Remark 1.1.2. *For $z, w \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$ the Beta function satisfies the following properties:*

1) The Beta function is linked to the Gamma function by the following relationship:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

2) The Beta function is symmetric, that is to say:

$$B(z,w) = B(w,z).$$

3) The Beta function can also take integral forms:

$$B(z,w) = 2\int_0^{\frac{\pi}{2}} \sin^{2z-1}(\theta)\cos^{2w-1}(\theta)d\theta$$
$$B(z,w) = \int_0^{+\infty} \frac{t^{z-1}}{(1+t)^{z+w}}dt.$$

4) The derivative of the Beta function is given by:

$$\frac{\partial}{\partial z}B(z,w) = B(z,w)\left(\frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+w)}{\Gamma(z+w)}\right).$$

Definition 1.1.3. *[14] The generalized binomial formula* $\binom{\alpha}{n}$ *for* $\alpha \in \mathbb{C}$ *where* $n \in \mathbb{N}$ *is given by :*

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, (n \in \mathbb{N}) \tag{1.3}$$

In particular, for $\alpha = m \in \mathbb{N}$ we a :

$$\binom{m}{n}\frac{m!}{n!(m-n)!}, (m, n \in \mathbb{N}; m \geq n) \tag{1.4}$$

And

$$\binom{m}{n} = 0, (m, n \in \mathbb{N}; 0 \leq m < n) \tag{1.5}$$

Definition 1.1.4. *[14] This formula can be expressed in terms of the Gamma function for* $\alpha \notin \mathbb{Z}_-^*$ *as follows:*

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)}, \left(\alpha \in \mathbb{C}, \alpha \notin \mathbb{Z}_-^*, n \in \mathbb{N}\right) \tag{1.6}$$

1.1.3 The L^p space

► $p = 1$

$$\left(\int_\Omega |f| dx\right) < \infty,$$

et

$$\|f\|_1 = \int_\Omega | f | \, dx.$$

► $1 < p < +\infty$

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R}; f \text{ is mesurable in } \Omega \text{ and } \int_\Omega |f(t)|^p dx < \infty \right\}.$$

et

$$\|f\|_p = \left(\int_\Omega | f |^p \, dx\right)^{\frac{1}{p}}.$$

► $p = +\infty$ There existe $c > 0$ such as : $|f(x)| < c$ and :

$$\|f\|_{+\infty} = \sup_{x\in\Omega} |f(x)|.$$

1.2 Fractional calculus of constant order for Riemann-Liouville

1.2.1 Riemann-Liouville fractional integral of constant order

Let $\Omega = [a, b] \subset \mathbb{R}$ be finite $g \in L^P(\Omega)$. The Riemann-Liouville fractional integrals $I^{\alpha}_{a+}g$ and $I^{\alpha}_{b-}g$ of real order $\alpha > 0$ are defined by:

$$I^{\alpha}_{a+}g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds, (t > a, \alpha > 0) \tag{1.7}$$

$$I^{\alpha}_{b-}g(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} g(s)ds, (t < b, \alpha > 0)$$

Where $\Gamma(\alpha)$ is the Gamma function, the formula $I^{\alpha}_{a+}g$ is called a fractionary integral of order α on the left and $I^{\alpha}_{b-}g$ is called a fractonary integral of order α on the right.

If $\alpha = n \in \mathbb{N}$, the Riemann-Liouville fractional integral (1.7), takes the following form:

$$I^{n}_{a+}g(t) = \frac{1}{(n-1)!} \int_\alpha^t (t-s)^{n-1} g(s)ds, \quad (n \in \mathbb{N})$$

$$I^{n}_{b-}g(t) = \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} g(s)ds, \quad (n \in \mathbb{N})$$

1.2.2 Riemann-Liouville fractional derivative of constant order

The Riemann-Liouville fractional derivative $D^{\alpha}_{a+}g$ of real order $\alpha \geq 0$ is defined by:

$$\begin{aligned} D^{\alpha}_{a+}g(t) &= \frac{d^n}{dt^n}\left(I^{n-\alpha}_{a+}g(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}g(s)ds, t > a \end{aligned} \tag{1.8}$$

where $n = [\alpha]+1$, $[.]$ is the integer part of a real number. In particular if $\alpha = n \in \mathbb{N}$, we obtain:

$$D^{0}_{a+}g(t) = g(t), \quad D^{n}_{a+}g(t) = g^{(n)}(t) \tag{1.9}$$

Where $g^{(n)}(t)$ denotes the usual derivative of order n of $g(t)$. If $0 < \alpha < 1$, then:

$$D^{\alpha}_{a+}g(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-s)^{-\alpha}g(s)ds, (t > a) \tag{1.10}$$

Proposition 1.2.1. *Let $f : [a, b] \to \mathbb{R}$ a continuous function. so, for all $\alpha > 0, \beta > 0$, we have :*

1. $I^{\alpha}_a\left[I^{\beta}_a[f(t)]\right] = I^{\alpha+\beta}_a[f(t)]$.
2. $I^{\alpha}_a\left[I^{\beta}_a[f(t)]\right] = I^{\beta}_a\left[I^{\alpha}_a[f(t)]\right]$.

Proof 1.2.1. *Indeed :*

$$\begin{aligned} I^{\alpha}_a\left[I^{\beta}f(t)\right] &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}I^{\beta}f(\tau)d\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left[(t-\tau)^{\alpha-1}\int_a^{\tau}(\tau-x)^{\beta-1}f(x)dx\right]d\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^{\tau}\left[(t-\tau)^{\alpha-1}(\tau-x)^{\beta-1}f(x)\right]dxd\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left[f(x)\int_t^{x}(t-\tau)^{\alpha-1}(\tau-x)^{\beta-1}d\tau\right]dx, \end{aligned}$$

si we take : $\tau = x + (t-x)y$, *so*

$$\int_t^x (t-\tau)^{\alpha-1}(\tau-x)^{\alpha-1}d\tau = \int_0^1 (t-x-(t-x)y)^{\alpha-1}(x+(t-x)y-x)^{\beta-1}(t-x)dy$$
$$=(t-x)^{\alpha+\beta-1}\int_0^1 (1-y)^{\alpha-1}y^{\beta-1}dy = B(\alpha,\beta)(t-x)^{\alpha+\beta-1}.$$

So

$$\left[I_a^\beta f(t)\right] = \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-x)^{\alpha+\beta-1}f(x)dx$$
$$= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t (t-x)^{\alpha+\beta-1}f(x)dx =_{R.L} I^{\alpha+\beta}f(t).$$

Now to demonstrate (2), using the previous property (1). So :

$$I_a^\alpha\left[I_a^\beta[f(t)]\right] = I_a^{\alpha+\beta}[f(t)] = I_a^{\beta+\alpha}[f(t)] = I_a^\beta\left[I_a^\alpha[f(t)]\right].$$

Example 1.2.1. *If we take :* $\alpha = \beta = \dfrac{1}{2}$ *and* $f(t) = t$, *we find :*

$$I_{0+}^\alpha\left[I_0^\beta[f(t)]\right] = I_{0+}^\alpha\left[\frac{\Gamma(2)}{\Gamma(\beta+2)}t^{\beta+1}\right]$$
$$= I_{0+}^\alpha\left[\frac{1}{\Gamma\left(\frac{3}{2}+1\right)}t^{\frac{3}{2}}\right]$$
$$= \frac{1}{\Gamma\left(\frac{3}{2}+1\right)}\left[\frac{\Gamma\left(\frac{3}{2}+1\right)}{\Gamma(\alpha+2)}t^{\alpha+\frac{3}{2}}\right] = \frac{1}{2}t^2.$$

On the other hand, we have:

$$I_{a+}^{\alpha+\beta}f(t) = {}_{R.L}I_0^1 f(t) = \int_0^t t\,dt = \frac{1}{2}t^2.$$

1.3 Fractional calculus of variable order for Riemann-Liouville

Definition 1.3.1. *([11], [34],[35]) Let* $-\infty < \nu_1 < \nu_2 < +\infty$, *and* $\wp(\varphi) : [\nu_1, \nu_2] \to (0, +\infty)$, *the left RLFIVO for function* $F(\varphi)$ *is defined by*

$$I_{\nu_1^+}^{\wp(\varphi)} F(\varphi) = \int_{\nu_1}^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} F(s) ds, \quad \varphi > \nu_1 \tag{1.11}$$

Definition 1.3.2. *([11], [34],[35]) Let* $-\infty < \nu_1 < \nu_2 < +\infty, n \in \mathbb{N}$ *and* $\wp(\varphi) : [\nu_1, \nu_2] \to (n-1, n)$, *the left(RLFDVO) for function* $F(\varphi)$ *is defined by*

$$D_{\nu_1^+}^{\wp(\varphi)} F(\varphi) = \left(\frac{d}{d\varphi}\right)^n I_{\nu_1^+}^{n-\wp(\varphi)} F(\varphi) = \left(\frac{d}{d\varphi}\right)^n \int_{\nu_1}^{\varphi} \frac{(\varphi - s)^{n-\wp(\varphi)-1}}{\Gamma(n - \wp(\varphi))} F(s) ds, \quad \varphi > \nu_1 \tag{1.12}$$

We notice that, if the order $\wp(\varphi)$ is a constant function $\wp$, then the (RLFDVO) (1.12) and (RLFIVO) (1.11) are the usual (RLFDI) respectively (see [1, 34, 35]).

Remark 1.3.1. *For arbitrary functions* $\wp(\varphi), \varrho(\varphi)$,*we notice that the semigroup property doesn't hold, i.e*

$$I_{\nu_1^+}^{\wp(\varphi)} I_{\nu_1^+}^{\varrho(\varphi)} F(\varphi) \neq I_{\nu_1^+}^{\wp(\varphi)+e(\varphi)} F(\varphi).$$

The above identity was very well proved and justified in the literature see ([2], [3], [4]).

Example 1.3.1. *Let*

$$u(\varphi) = \begin{cases} 2, & \varphi \in [0, 1], \\ 1, & \varphi \in]1, 3], \end{cases} \qquad v(\varphi) = \begin{cases} 1, & \varphi \in [0, 1], \\ 2, & \varphi \in]1, 3], \end{cases}$$

and $h(\varphi) = \varphi, \quad \varphi \in [0, 3]$.

$$\begin{aligned}
I_{0^+}^{u(\varphi)} I_{0^+}^{v(t)} h(\varphi) &= \int_0^1 \frac{(\varphi - s)^{u(s)-1}}{\Gamma(u(s))} \int_0^s \frac{(s-\tau)^{v(\tau)-1}}{\Gamma(v(\tau))} h(\tau) d\tau ds \\
&+ \int_1^\varphi \frac{(\varphi - s)^{u(s)-1}}{\Gamma(u(s))} \int_0^s \frac{(s-\tau)^{v(\tau)-1}}{\Gamma(v(\tau))} h(\tau) d\tau ds, \\
&= \int_0^1 \frac{(\varphi - s)^1}{\Gamma(2)} \int_0^s \frac{(s-\tau)^0}{\Gamma(1)} \tau d\tau ds \\
&+ \int_1^\varphi \frac{(\varphi - s)^0}{\Gamma(1)} \left[\int_0^1 \frac{(s-\tau)^0}{\Gamma(1)} \tau d\tau + \int_1^s \frac{(s-\tau)^1}{\Gamma(2)} \tau d\tau \right] ds \\
&+ \int_1^\varphi \frac{(\varphi - s)^{u(s)-1}}{\Gamma(u(s))} \int_0^s \frac{(s-\tau)^{v(\tau)-1}}{\Gamma(v(\tau))} h(\tau) d\tau ds \\
&= \int_0^1 \frac{(\varphi - s)^1}{\Gamma(2)} \int_0^s \frac{(s-\tau)^0}{\Gamma(1)} \tau d\tau ds \\
&+ \int_1^\varphi \frac{(\varphi - s)^0}{\Gamma(1)} \left[\int_0^1 \frac{(s-\tau)^0}{\Gamma(1)} \tau d\tau + \int_1^s \frac{(s-\tau)^1}{\Gamma(2)} \tau d\tau \right] ds, \\
&= \int_0^1 \frac{(\varphi - s) s^2}{2\Gamma(2)} ds + \int_1^\varphi \frac{s^3}{6} - \frac{s}{2} + \frac{5}{6} ds \\
I_{0^+}^{u(\varphi)+v(\varphi)} h(\varphi) &= \int_0^\varphi \frac{(\varphi - s)^{u(s)+v(s)-1}}{\Gamma(u(s)+v(s))} h(s) ds.
\end{aligned}$$

We see that

$$\begin{aligned}
I_{0^+}^{u(\varphi)} I_{a^+}^{v(\varphi)} h(\varphi) \Big|_{\varphi=2} &= \int_0^1 \frac{(2-s)s^2}{2\Gamma(2)} ds + \int_1^2 \frac{s^3}{6} - \frac{s}{2} + \frac{5}{6} ds, \\
&= \frac{5}{24} + \frac{17}{24} = \frac{22}{24}, \\
I_{0^+}^{u(\varphi)+v(\varphi)} h(\varphi) \Big|_{\varphi=2} &= \int_0^1 \frac{(2-s)^{2+1-1}}{\Gamma(2+1)} s ds + \int_1^2 \frac{(2-s)^{1+2-1}}{\Gamma(1+2)} s ds \\
&= \frac{11}{24} + \frac{5}{24} = \frac{16}{24}.
\end{aligned}$$

Therefore, we obtain

$$I_{0^+}^{u(\varphi)} I_{0^+}^{v(\varphi)} h(\varphi) \Big|_{\varphi=2} \neq I_{0^+}^{u(\varphi)+v(\varphi)} h(\varphi) \Big|_{\varphi=2}.$$

Lemma 1.3.1. *([38]) Let $\wp : D \to (0,1]$ be a continued fraction, then for $w \in C_\iota(D, \mathrm{I\!R}) = w(\varphi) \in \Psi, \varphi' w(\varphi) \in \Psi, (0 \le \iota \le 1)$ and for each points on D, the $I_{0+}^{p(\varphi)} w(\varphi)$ exists.*

Lemma 1.3.2. *([38]) Let $\wp : D \to (0,1]$ be a continued fraction, then*

$$I_{0+}^{\wp(\varphi)} w(\varphi) \in \Psi \text{ for } w \in \Psi.$$

Remark 1.3.2. *[38]. For $0 \le s \le \varphi \le \sigma$ we let $\wp_* = \min_{0 \le \varphi \le \sigma} |\wp(\varphi)|$, then we get if*

$$0 \le \sigma \le 1, \text{ then } \sigma^{\wp(s)-1} \le \sigma^{\wp_* - 1}$$

if

$$1 \le \sigma \le \infty, \text{ then } \sigma^{\wp(s)-1} \le 1$$

Thus for $-\infty \le \sigma \le +\infty$, we know

$$\sigma^{\wp(s)-1} \le \max\left\{1, \sigma^{\wp_* - 1}\right\} = \sigma^*$$

1.4 Phase Space

The notion of the phase space $\mathfrak{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [22] (see also Kappel and Schappacher [15] and Schumacher [27]). For a detailed discussion on this topic we refer the reader to the book by Hino et al [40]

Fractional differential equations have been of great interest recently. In cause, in part to both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc. For details, see the monographs of Miller and Ross [28], Podlubny [20] and Samko et al [33], and the papers of Delboso and Rodino [10], Diethelm et al ([24] , [25], [26]), Gaul et al [29], Glockle and Nonnenmacher

[39], Mainardi [16], Metzler et al [17], Momani and Hadid [36], Momani et al [37], Podlubny et al [21], Yu and Gao [9] and the references therein

Our approach is based on the Banach fixed point theorem and on the nonlinear alternative of Leray-Schauder type [6]. These results can be considered as a contribution to this emerging field.

In this thesis, we take on that the state space $(\mathfrak{B}, \|\cdot\|_\sigma)$ is a Semi-normed lineaire of function mapping $(-\infty, 0]$ into $\mathbb{R}$, and check the fundamental axioms of Hale and kate given in [22].

($\mathcal{E}$) If $w : (-\infty, \nu_2] \to \mathbb{R}$, and $w_0 \in \mathfrak{B}$, then $\forall \varphi \in D$ the following conditions are satisfied:

(i) w_φ is in $\mathfrak{B}$.

(ii) $\|w_\varphi\|_{\mathfrak{B}} \leq \kappa(\varphi)\sup\{|w(s)| : 0 \leq s \leq \varphi\} + L(\varphi)\,\|w_0\|_{\mathfrak{B}}$,

(iii) $|w(\varphi)| \leq T\,\|w_\varphi\|_{\mathfrak{B}}$,

where the constant $T \geq 0$, the Continued fraction $\kappa : D \to [0, \infty)$, the locally bounded $L\ [0, \infty) \to [0, \infty)$, the κ, T, L are independent of $w(.)$.
($\mathcal{E} - \mathbf{1}$) For the function $w(.)in(\mathcal{E}), w_\varphi$ is a $\mathfrak{B}$-valued continues function on D.
($\mathcal{E} - 2$) The space $\mathfrak{B}$ is complete.

1.4.1 Examples of Phase Spaces

In this section, we present some examples of phase spaces.

Example 1.4.1. *The space C_γ.*

For any real constant γ, we define the functional space C_γ by

$$C_\gamma = \left\{ \eta \in C((-\infty, 0], \Psi), \lim_{\jmath \to -\infty} \eta(\jmath) \text{ existe in } \Psi \right\}.$$

Endowed with the following norm

$$\|\eta\| = \sup\left\{e^{\gamma \jmath}|\eta(\jmath)|; \jmath < 0\right\}.$$

Then in the space C_γ(see [40]) the axioms $(\mathcal{E}) - (\mathcal{E} - 2)$ are satisfied.

Example 1.4.2. *The spaces banach contraction, bounded and uniformly continous, C^∞ and C^0.*

Let banach contraction the space of bounded continuous functions defined from $(-\infty, 0]$ to Ψ bounded and uniformly continous the space of bounded uniformly continous functions defined from $(-\infty, 0]$ to Ψ

$$C^\infty := \left\{\eta \in \text{ banach contraction} : \lim_{\jmath \to -\infty} \eta(\jmath) \text{ existe in } \Psi\right\}.$$

$C^0 := \{\eta \in$ banach contraction $: \lim_{\jmath \to -\infty} \eta(\jmath) = 0\}$, endowed with the uniform norm

$$\|\eta\| = \sup\{\eta(\jmath) : \jmath \leq 0\}.$$

We have ([40]) that the spaces Bounded and uniformly continous, C^∞ and C^0 satisfy conditions $(\mathcal{E})$- $(\mathcal{E} - 2)$. bounded and continous satisfies $(\mathcal{E}) - (\mathcal{E} - 2)$ but $(\mathcal{E})$ is not satisfied.

1.5 Some fixed point theorems

Theorem 1.5.1. *(banach contraction principle [6]). Let C be a non-empty closed subset of a banach's space Ψ, then any contraction mapping V of C into itself has a fixed point.*

Theorem 1.5.2. *(schauder fixed point theorem [23]). Let Ψ a Banach's space and Q be be a convex subset of Ψ and $V : Q \longrightarrow Q$ is completely continuity. Then V has* *at least one fixed point in Q.*

Theorem 1.5.3. *(Alternative non linear Leray schauder theorem) ([1]) Let Ψ a banach's space and Q be a convex, closed bounded non-empty of Ψ and $V \subset Q$ an open and such that $0 \in V$. Assume that $\Phi : V \to Q$ is completely continuity. If $\Phi(V)$ is Relatively compact then, either (i) : © has a Fixed point, or (ii) : there is a point $u \in \partial V$ and $\lambda \in (0;1)$ with $u = \lambda \Phi u$.*

Theorem 1.5.4. *(Arzéla-Ascoli). For $A \subset C[0,1], A$ is compact if, and only if, A is closed, bounded, and equicontinuous.*

Lemma 1.5.1. *Let $\phi : [0,\sigma] \to [0,\infty)$ be a Real function and $\psi(.)$ is à non negative, Locally integrable function on $[0,\sigma]$ and there are constants $\gamma > 0$ and
$0 < \wp(\varphi) \leq \wp^* \leq 1$ such that*

$$\phi(\varphi) \leq \psi(\varphi) + \gamma \int_0^{\varphi} \frac{\phi(s)}{(\varphi - s)^{\wp(s)-1}} ds,$$

then there exists a constant $\kappa = \kappa\left(\wp^*\right)$ such that

$$\phi(\varphi) \leq \psi(\varphi) + \kappa\gamma \int_0^{\varphi} \frac{\psi(s)}{(\varphi - s)^{\wp(s)-1}} ds.$$

For every $\varphi \in [0,\sigma]$.

1.6 Types of stability

Theorem 1.6.1. *The system is Ulam Hyers Stability if there exists $c_r > 0$, such that for each $\varepsilon > 0$ and for every solution $\chi \in \Psi$ of the following inequality*

$$\left| D_{0^+}^{\wp(\varphi)} \chi(\varphi) - \Upsilon(\varphi, \chi(\varphi)) \right| < \varepsilon, \quad \varphi \in D$$

There exists a solution $w \in \Psi$ *of (PNPS)*

$$|\chi(\varphi) - w(\varphi)| < c_r \varepsilon, \quad \varphi \in D$$

Study existence and uniqueness of solutions

2.1 Introduction and motivations

The piece-wise constant function will play a vital role in our study for converting the fractional problem of variable order to an equivalent standard fractional problem of the constant order.

Benchohra and al [30] studied the existence of solutions for the following nonlinear fractional derivative equations for constant order:

$$\begin{cases} D^{\varrho}_{0+}\xi(s) = \varphi\left(s, \xi_s\right); \quad s \in \mathcal{N} := [0, N] & (1) \\ \xi(s) = \chi(s); \quad s \in [-\gamma, 0], \gamma > 0 & (2) \end{cases} \tag{2.1}$$

Where D^{ϱ}_{0+} is standard Riemann-Liouville fractional derivative, $\varphi : \mathbb{N} \times C([-\gamma, 0], \mathbb{R}) \to \mathbb{R}$ is a given function $\chi \in C([-\gamma, 0], \mathbb{R})$ via $\chi(0) = 0$. For any function ξ defined on $[-\gamma, N]$ and any $s \in \mathcal{N}$, we denote by ξ_s the element of $C([-\gamma, 0], \mathbb{R})$ defined by

$$\xi_s(\tau) = \xi(s + \tau), \quad \tau \in [-\gamma, 0].$$

Since the authors in [30] consider an infinite delay, the obtained existence results can be examined as a generalization of several existence results for delayed fractional derivative equations with constant order. In fact, there have been some important existence results for

M. Benaouda, S. Sabit, H. Gunerhan, M. S. Souid, Modern Technique to Study Cauchy-Type Problem of Fractional Variable Order, Journal of Modern Physics Letters B, 2024(2024).

such equations where different techniques have been applied. [8, 18] How- ever, as stated above, the constant order corresponding results for delayed fractional variable order boundary-value problems are very few.

In this chapter we apply the new technique on the following fractional Cauchy-type problem (PNPS)

$$\begin{cases} D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(\varphi, w(\varphi)) \ \ \varphi \in D = [0, \sigma] \quad (1) \\ \\ w(\varphi) = \eta(\varphi), \quad \varphi \in]-\infty, 0] \qquad\qquad (2) \end{cases} \tag{PNPS}$$

Where $0 < \sigma < +\infty, 0 < \wp(\varphi) \leq 1, \Upsilon : D \times \mathfrak{B} \to \mathbb{R}$ is a Continuous Functions and $D_{0^+}^{\wp(\varphi)}$, is the R-Liouville Fractional Derivative of Variable-Order $\wp(\varphi), 0 < \wp(\varphi) \leq \wp^* \leq 1$ and $\eta(\varphi) \in \mathfrak{B}$ with $\eta(0) = 0$ and $\mathfrak{B}$ is Phase space

For each function w defined on $]-\infty, \sigma]$ and each $\varphi \in D$, we note by w_φ the element of $\mathfrak{B}$ defined by

$$w_\varphi(\jmath) = w(\varphi + \jmath), \quad \jmath \in]-\infty, 0]$$

2.2 Study existence and uniqueness of solutions

Let us start by defining what we mean by a solution of problem (PNPS). Let the space

$$\Omega = \Big\{ w : (\infty, \sigma] \to \mathbb{R} : w|_{(\infty,0]} \in \mathfrak{B} \text{ and } \ w|_{[0,\sigma]} \text{ is continuous } \Big\}.$$

We impose the following assumptions:

(C1) $\wp : \left[\frac{1}{2}, \sigma\right] \to \left(\frac{1}{2}, \wp^*\right]$ is continued fraction, such that $\frac{1}{2} < \wp(\varphi) \leq \wp^* \leq 1$.

(C2) Let $\varphi^\iota \Upsilon : D \times \mathfrak{B} \to \mathbb{R}$ is continued fraction$(0 \leq \iota \leq 1)$, there exist a constants ℓ, such that,

$$\varphi^\iota |\Upsilon(\varphi, u) - \Upsilon(\varphi, v)| \leq \ell \|u - v\|_B \text{ for each } u, v, \in \mathfrak{B} \text{ and } \varphi \in D.$$

(C3) There existe $\alpha, \beta \in C\left(D, \mathbb{R}^+\right)$such that

$$\varphi^\iota \|\Upsilon(\varphi, u)\| \leq \alpha(\varphi) + \beta(\varphi)\|u\|_B$$

for $\varphi \in D$ and each $u \in B$, and $\left\|I^{\wp^*} p\right\|_\Psi < \infty$.

Definition 2.2.1. *A function $w \in \Omega$ is said to be a solution for (PNPS(1)) and (PNPS(2)) if w satisfies the equation $D^{\phi(\varphi)} = \Upsilon(\varphi, w(\varphi))$ on D and the condition $w(\varphi) = \eta(\varphi)$ on $(-\infty, 0]$.*

For the existence of solutions for the (PNPS), an auxiliary lemma is needed as follows:

Lemma 2.2.1. *[38] Let $0 < \wp(\varphi) < 1$ and let $F : (0, \sigma] \to \mathbb{R}$ be continuous and $\lim_{\varphi \to 0^+} F(\varphi) = F\left(0^+\right) \in \mathbb{R}$. Then w is a solution of fractional derivative equations*

$$w(\varphi) = \int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} F(s) ds, \quad \varphi \in D \text{ and } \wp(s) > 0$$

if and only if w is a solution of the Initial value problem for the fractional derivative equations

$$\begin{cases} D^{\wp(\varphi)} = F(\varphi), \\ w(0) = 0. \end{cases} \quad \varphi \in (0, \sigma] \tag{2.2}$$

Lemma 2.2.2. *[38] Let (C1) hold. And let $w_n, w \in \mathfrak{B}$ assume that*

$$w_n(\varphi) \to w(\varphi), \varphi \in D \text{ as } n \to \infty$$

then

$$\int_0^{\varphi} \frac{(\varphi - s)^{\wp(\varphi)}}{\Gamma(1-\wp(\varphi))} w_n(s)ds \to \int_0^{\varphi} \frac{(\varphi - s)^{\wp(\varphi)}}{\Gamma(1-\wp(\varphi))} w(s)ds, \varphi \in [0,\sigma] \text{ as } n \to \infty$$

The first result obtained by using the Banach Contraction Principle.

Theorem 2.2.1. *Supposing that conditions $(C1)$, $(C2)$ and $(C3)$ are hold, if*

$$\frac{\sigma^* \sigma^{-\iota+1} \left(\|\alpha\|_\Psi + \|\beta\|_\Psi L_\sigma \|\eta\|_{\mathfrak{B}}\right)}{(-\iota+1)\Gamma\left(\wp^*\right) - \sigma^*\sigma^{-\iota+1}\|\beta\|_\Psi \kappa_\sigma} < 1$$

Then the (PNPS) has at least one solution on D.

Proof 2.2.1. *We give the operator $\Im : M_0 \to M_0$ defined by:*

$$(\Im v)(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon\left(s, v^*(s) + \varsigma(s)\right) ds, \quad \varphi \in D. \tag{2.3}$$

We shall use the Schauder Fixed Point Theorem to prove that $\Im$ is Fixed Point. We consider the set

$$B_{R_\iota} = \{v \in M_0, \|v\|_\sigma \le R_\iota\}$$

where

$$R_\iota = \frac{\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\left(\|\alpha\|_\Psi + \|\beta\|_\Psi L_\sigma \|\eta\|_{\mathfrak{B}}\right)\sigma^{\wp_*-\iota}}{\Gamma\left(1-\iota+\wp^*\right) - \sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\|\beta\|_\Psi \kappa_\sigma \sigma^{\wp_*-\iota}}.$$

Clearly B_{R_ι} is Convex,closed bounded non-empty .
We proved this in three steps. Step 1: $\Im$ is Continues operator.
Let v_n be a sequence such that $v_n \to v$ in M_0 then

$$\|(\Im v_n) - (\Im v)\|_\sigma \to 0$$

For $\varphi \in [0, \sigma]$, we find

$$\begin{aligned}
\left|\Im\left(v_{n}\right)(\varphi)-\Im(v)(\varphi)\right| &= \left| \int_{0}^{\varphi} \frac{(\varphi-s)^{\varphi(s)-1}}{\Gamma(\wp(s))} \Upsilon\left(s, v_{n}^{*}(s)+\varsigma(s)\right) ds \right. \\
&\quad \left. -\int_{0}^{\varphi} \frac{(\varphi-s)^{\varphi(s)-1}}{\Gamma(\wp(s))} \Upsilon\left(s, v^{*}(s)+\varsigma(s)\right) ds \right| \\
&\leq \int_{0}^{\varphi} \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\left|\Upsilon\left(s, v_{n}^{*}(s)+\varsigma(s)\right)-\Upsilon\left(s, v^{*}(s)+\varsigma(s)\right)\right| ds \\
&\leq \frac{\ell}{\Gamma\left(\wp^{*}\right)} \int_{0}^{\varphi}(\varphi-s)^{\wp(s)-1} s^{-\iota}\left\|v_{n}^{*}(s)-v^{*}(s)\right\|_{\mathfrak{B}} ds \\
&\leq \frac{\ell}{\Gamma\left(\wp^{*}\right)} \int_{0}^{\varphi} \sigma^{\wp(s)-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp(s)-1} s^{-\iota}\left\|v_{n}^{*}(s)-v^{*}(s)\right\|_{\mathfrak{B}} ds \\
&\leq \frac{\ell}{\Gamma\left(\gamma^{*}\right)} \int_{0}^{\varphi} \sigma^{\wp_{*}-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp_{*}-1} s^{-\iota} \kappa(s) sup_{s[0,\varphi]\}}\left\|v_{n}(s)-v(s)\right\| ds \\
&\leq \frac{\ell \sigma^{*} \kappa_{\sigma}}{\Gamma\left(\wp^{*}\right)} \sigma^{1-\wp_{*}} \int_{0}^{\varphi}(\varphi-s)^{\wp_{*}-1} s^{-\iota} ds\left\|v_{n}-v\right\|_{\sigma} \\
&\leq \frac{\ell \sigma^{*} \sigma^{1-\wp_{*}} \kappa_{\sigma} \Gamma\left(\wp^{*}\right) \Gamma(1-\iota)}{\Gamma\left(\wp^{*}\right) \Gamma\left(1-\iota+\wp^{*}\right)} \sigma^{\wp_{*}-\iota}\left\|v_{n}-v\right\|_{\sigma}
\end{aligned}$$

Then

$$\begin{aligned}
\left\|\Im\left(v_{n}\right)-\Im(v)\right\|_{\sigma} &\leq \frac{\ell \sigma^{*} \sigma^{1-\wp_{*}} \kappa_{\sigma} \Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^{*}\right)} \sigma^{\wp_{*}-\iota}\left\|v_{n}-v\right\|_{\sigma}, \\
&\quad \left\|\left(\Im v_{n}\right)-(\Im v)\right\|_{\sigma} \rightarrow 0 \text{ as } n \rightarrow \infty
\end{aligned}$$

Consequently, $\Im$ is Continous operator on M_0.

Step 2: $\Im\left(B_{R_{\iota}}\right) \subseteq \left(B_{R_{\iota}}\right)$.

For $v \in B_{R_\iota}$, and by $(C3)$, we get:

$$
\begin{aligned}
|\Im(v)(\varphi)| &= \left|\int_0^\varphi \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon\left(s, v_s^*+\varsigma_s\right) ds\right| \\
&\leq \int_0^\varphi \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\left|\Upsilon\left(s, v_s^*+\varsigma_s\right)\right| ds \\
&\leq \frac{1}{\Gamma\left(\wp^*\right)} \int_0^\varphi \sigma^{\wp(s)-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp(s)-1} s^{-\iota}\left(|\alpha(s)|+\mid \beta(s)\mid \left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}\right) ds \\
&\leq \frac{1}{\Gamma\left(\wp^*\right)} \int_0^\varphi \sigma^{\wp_*-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp_*-1}\left(s^{-\iota}|\alpha(s)|+s^{-\iota} \mid \beta(s)\mid \left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}\right) ds \\
&\leq \frac{\sigma^*}{\Gamma\left(\wp^*\right)} \sigma^{1-\wp_*} \int_0^\varphi(\varphi-s)^{\wp_*-1}\left(s^{-\iota}|\alpha(s)|+s^{-\iota} \mid \beta(s)\mid \left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}\right) ds \\
&\leq \frac{\sigma^* \sigma^{1-\wp_*}}{\Gamma\left(\wp^*\right)} \int_0^\varphi(\varphi-s)^{\wp_*-1} s^{-\iota}|\alpha(s)| ds \\
&+\frac{\sigma^* \sigma^{1-\wp_*}}{\Gamma\left(\wp^*\right)} \int_0^\varphi(\varphi-s)^{\wp_*-1} s^{-\iota}|\beta(s)|\left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}) ds \\
&\leq \frac{\sigma^* \sigma^{1-\wp_*}\|\alpha\|_\Psi}{\Gamma\left(\wp^*\right)} \int_0^\varphi(\varphi-s)^{\wp_*-1} s^{-\iota} ds \\
&+\frac{\sigma^* \sigma^{1-\wp_*}\|\beta\|_\Psi}{\Gamma\left(\wp^*\right)} \int_0^\varphi(\varphi-s)^{\wp_*-1} s^{-\iota}\left(\left\|v_s^*\right\|_{\mathfrak{B}}+\left\|\varsigma_s\right\|_{\mathfrak{B}}\right) ds \\
&\leq \frac{\sigma^* \sigma^{1-\wp_*}\|\alpha\|_\Psi}{\Gamma\left(\wp^*\right)} \int_0^\varphi(\varphi-s)^{\wp_*-1} s^{-\iota} ds \\
&+\frac{\sigma^* \sigma^{1-\wp_*}\|\beta\|_\Psi}{\Gamma\left(\wp^*\right)}\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right) \int_0^\varphi(\varphi-s)^{\wp_*-1} s^{-\iota} ds \\
&\leq \frac{\sigma^* \sigma^{1-\wp_*}\|\alpha\|_\Psi \Gamma\left(\wp^*\right) \Gamma(1-\iota)}{\Gamma\left(\wp^*\right) \Gamma\left(1-\iota+\wp^*\right)} \sigma^{\wp_*-\iota} \\
&+\frac{\sigma^* \sigma^{1-\wp_*}\|\beta\|_\Psi \Gamma\left(\wp^*\right) \Gamma(1-\iota)}{\Gamma\left(\wp^*\right) \Gamma\left(1-\iota+\wp^*\right)} \sigma^{\wp_*-\iota}\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right) \\
&\leq \frac{\sigma^* \sigma^{1-\wp_*}\|\alpha\|_\Psi \Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^*\right)} \sigma^{\wp_*-\iota}+\frac{\sigma^* \sigma^{1-\wp_*}\|\beta\|_\Psi \Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^*\right)} \sigma^{\wp_*-\iota}\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\sigma^{*}\sigma^{1-\wp_{*}}\Gamma(1-\iota)\left(\|\alpha\|_{\Psi}+\|\beta\|_{\Psi}L_{\sigma}\|\eta\|_{\mathfrak{B}}\right)}{\Gamma\left(1-\iota+\wp^{*}\right)}\sigma^{\wp_{*}-\iota}+\frac{\sigma^{*}\sigma^{1-\wp_{*}}\Gamma(1-\iota)\|\beta\|_{\Psi}\kappa_{\sigma}}{\Gamma\left(1-\iota+\wp^{*}\right)}\sigma^{\wp_{*}-\iota}\|v\|_{\sigma}\\
&\leq \frac{\sigma^{*}\sigma^{1-\wp_{*}}\Gamma(1-\iota)\left(\|\alpha\|_{\Psi}+\|\beta\|_{\Psi}L_{\sigma}\|\eta\|_{\mathfrak{B}}\right)}{\Gamma\left(1-\iota+\wp^{*}\right)}\sigma^{\wp_{*}-\iota}\\
&\times \frac{\Gamma\left(1-\iota+\wp^{*}\right)}{\Gamma\left(1-\iota+\wp^{*}\right)-\sigma^{*}\sigma^{1-\wp_{*}}\Gamma(1-\iota)\|\beta\|_{\Psi}\kappa_{\sigma}\sigma^{\wp_{*}-\iota}}\\
&\leq \frac{\sigma^{*}\sigma^{1-\wp_{*}}\Gamma(1-\iota)\left(\|\alpha\|_{\Psi}+\|\beta\|_{\Psi}L_{\sigma}\|\eta\|_{\mathfrak{B}}\sigma^{\wp_{*}-\iota}\right.}{\Gamma\left(1-\iota+\wp^{*}\right)-\sigma^{*}\sigma^{1-\wp_{*}}\Gamma(1-\iota)\|\beta\|_{\Psi}\kappa_{\sigma}\sigma^{\wp_{*}-\iota}}\\
&\leq R_{\iota},
\end{aligned}
$$

Where:

$$
\|v_{s}^{*}+\varsigma_{s}\|_{\mathfrak{B}}\leq\|v_{s}^{*}\|_{\mathfrak{B}}+\|\varsigma_{s}\|_{\mathfrak{B}}\leq\kappa_{\sigma}\|v\|_{\sigma}+L_{\sigma}\|\eta\|_{\mathfrak{B}},
$$

and

$$
L_{\sigma}=\sup\{|L(\varphi)|:\varphi\in D\},
$$

which means that $\Im\left(\mathfrak{B}_{R_{\iota}}\right)\subseteq\left(B_{R_{\iota}}\right)$.

`Step 3` :$\Im$ is Continuous operator

Now, we will show that $\Im\left(B_{R_{\iota}}\right)$ is relatively compact, meaning that $\Im$ is Coop. Clearly $\Im\left(B_{R_{\iota}}\right)$ is uniformly bounded because by Step 2, we obtain $\Im\left(B_{R_{\iota}}\right)=\{\Im(v):v\in B_{R_{\iota}}\}\subset B_{R_{\iota}}$ thus for each $v\in B_{R_{\iota}}$ we get $\|\Im(v)\|_{M_0}\leq R_{\iota}$ which means that $\Im\left(B_{R_{\iota}}\right)$ is uniformly bounded. It remains to show that $\Im\left(B_{R_{\iota}}\right)$ is Equicontinuity.

For $\varphi_1,\varphi_2\in D,\varphi_1<\varphi_2$ and $\varsigma\in B_{R_{\iota}}$, we have:

$$
\begin{aligned}
&\left|\Im(v)\left(\varphi_2\right)-\Im(v)\left(\varphi_1\right)\right|\\
&=\left|\int_0^{\varphi_2}\frac{\left(\varphi_2-s\right)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon\left(s,v_s^{*}+\varsigma_s\right)ds-\int_0^{\varphi_1}\frac{\left(\varphi_1-s\right)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon\left(s,v_s^{*}+\varsigma_s\right)ds\right|\\
&=\left|\int_0^{\varphi_1}\frac{\left(\varphi_2-s\right)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon\left(s,v_s^{*}+\varsigma_s\right)ds+\int_{\varphi_1}^{\varphi_2}\frac{\left(\varphi_2-s\right)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon\left(s,v_s^{*}+\varsigma_s\right)ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^{\varphi_1}\frac{(\varphi_1-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon\left(s,v_s^*+\varsigma_s\right)ds\\
&\le\int_0^{\varphi_1}\left[(\varphi_2-s)^{\wp(s)-1}-(\varphi_1-s)^{\wp(s)-1}\right]\left|\frac{\Upsilon\left(s,v_s^*+\varsigma_s\right)}{\Gamma(\wp(s))}\right|ds\\
&+\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp(s)-1}\left|\frac{\Upsilon\left(s,v_s^*+\varsigma_s\right)}{\Gamma(\wp(s))}\right|ds\\
&\le\frac{1}{\Gamma\left(\wp^*\right)}\int_0^{\varphi_1}\left[\sigma^{\wp(s)-1}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp(s)-1}-\sigma^{\wp(s)-1}\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp(s)-1}\right]s^{-\iota}\left(\alpha(s)+\beta(s)\left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}\right)ds\\
&+\frac{1}{\Gamma\left(\wp^*\right)}\int_{\varphi_1}^{\varphi_2}\sigma^{\wp(s)-1}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp(s)-1}s^{-\iota}\left(\alpha(s)+\beta(s)\left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}\right)ds\\
&\le\frac{1}{\Gamma\left(\wp^*\right)}\int_0^{\varphi_1}\left[\sigma^{\varphi^{\beta*-1}}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\sigma^{\wp_*-1}\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]s^{-\iota}\left(\alpha(s)+\beta(s)\left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}\right)ds\\
&+\frac{1}{\Gamma\left(\wp^*\right)}\int_{\varphi_1}^{\varphi_2}\sigma^{\wp^{\beta+1}}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}\left(\alpha(s)+\beta(s)\left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}\right)ds\\
&\le\frac{\sigma^*}{\Gamma\left(\wp^*\right)}\int_0^{\varphi_1}\left[\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]s^{-\iota}\left(\alpha(s)+\beta(s)\left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}\right)ds\\
&+\frac{\sigma^*}{\Gamma\left(\wp^*\right)}\int_{\varphi_1}^{\varphi_2}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}\left(\alpha(s)+\beta(s)\left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}\right)ds\\
&\le\frac{\sigma^*}{\Gamma\left(\wp^*\right)}\int_0^{\varphi_1}\left[\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]s^{-\iota}\alpha(s)ds\\
&+\frac{\sigma^*}{\Gamma\left(\wp^*\right)}\int_0^{\varphi_1}\left[\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]s^{-\iota}\beta(s)\left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}ds\\
&+\frac{\sigma^*}{\Gamma\left(\wp^*\right)}\int_{\varphi_1}^{\varphi_2}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}\alpha(s)ds+\frac{\sigma^*}{\Gamma\left(\wp^*\right)}\int_{\varphi_1}^{\varphi_2}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}\beta(s)\left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}ds\\
&\le\frac{\sigma^*\|\alpha\|_\Psi}{\Gamma\left(\wp^*\right)}\sigma^{1-\wp_*}\int_0^{\varphi_1}\left[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\right]s^{-\iota}ds\\
&+\frac{\sigma^*\|\beta\|_\Psi\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right)}{\Gamma\left(\wp^*\right)}\sigma^{1-\wp_*}\int_0^{\varphi_1}\left[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\right]s^{-\iota}ds\\
&+\frac{\sigma^*\|\alpha\|_\Psi}{\Gamma\left(\wp^*\right)}\sigma^{1-\wp_*}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds\\
&+\frac{\sigma^*\|\beta\|_\Psi\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right)}{\Gamma\left(\wp^*\right)}\sigma^{1-\wp_*}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds\\
&\le\frac{\sigma^*\sigma^{1-\delta_*}\left(\|\alpha\|_\Psi+\|\beta\|_\Psi\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right)\right)}{\Gamma\left(\wp^*\right)}\int_0^{\varphi_1}\left[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\right]s^{-\iota}ds\\
&+\frac{\sigma^*\sigma^{1-\wp_*}\left(\|\alpha\|_\Psi+\|\beta\|_\Psi\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right)\right)}{\Gamma\left(\wp^*\right)}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds
\end{aligned}
$$

$$\leq \frac{\sigma^*\sigma^{1-\wp_*}\left(\|\alpha\|_\Psi+\|\beta\|_\Psi\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right)\right)}{\Gamma\left(\wp^*\right)}\left[\int_0^{\varphi_1}\left(\varphi_2-s\right)^{\wp_*-1}s^{-\iota}ds-\int_0^{\varphi_1}\left(\varphi_1-s\right)^{\wp_*-1}s^{-\iota}ds\right]$$
$$+\frac{\sigma^*\sigma^{1-\wp_*}\left(\|\alpha\|_\Psi+\|\beta\|_\Psi\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right)\right)}{\Gamma\left(\wp^*\right)}\varphi_1^{-\iota}\left(\varphi_2-\varphi_1\right)^{\rho_*}$$
$$\leq \frac{\sigma^*\sigma^{1-\wp_*}\left(\|\alpha\|_\Psi+\|\beta\|_\Psi\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right)\right)}{\Gamma\left(\wp^*\right)}\left[\frac{\Gamma\left(\wp^*\right)\Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^*\right)}\varphi_1^{\wp_*-\iota}-\frac{\Gamma\left(\wp^*\right)\Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^*\right)}\varphi_2^{\wp_*+\iota}\right]$$
$$+\frac{\sigma^*\sigma^{1-\wp_*}\left(\|\alpha\|_\Psi+\|\beta\|_\Psi\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right)\right)}{\Gamma\left(\wp^*\right)}\varphi_1^{-\iota}\left(\varphi_2-\varphi_1\right)^{\wp_*}$$

Consequently, we obtain

$$\begin{aligned}\left|\Im(v)\left(\varphi_2\right)-\Im(v)\left(\varphi_1\right)\right| \leq & \frac{\sigma^*\sigma^{1-\wp_*}\left(\|\alpha\|_\Psi+\|\beta\|_\Psi\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right)\right)}{\Gamma\left(1-\iota+\wp^*\right)}\left[\Gamma(1-\iota)\varphi_1^{\wp_*-\iota}-\Gamma(1-\iota)\varphi_2^{\wp_*-\iota}\right] \\ & +\frac{\sigma^*\sigma^{1-\wp_*}\left(\|\alpha\|_\Psi+\|\beta\|_\Psi\left(\kappa_\sigma\|v\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\right)\right)}{\Gamma\left(\wp^*\right)}\varphi_1^{-\iota}\left(\varphi_2-\varphi_1\right)^{\wp_*}\end{aligned}$$

Hence $\left|\left(\Im v\right)\left(\varphi_2\right)-\left(\Im v\right)\left(\varphi_1\right)\right| \rightarrow 0$ as $\left|\varphi_2-\varphi_1\right| \rightarrow 0$. It implies that $\Im\left(B_{R_\iota}\right)$ is Equicontinuity As a sequence of Steps 1 to 3 together with Arzella Ascolli theorem, we conclude that $\Im$ is Completely continuity.

Step 4 (A priori bounds) : Now we can expose there exists an open set $V \subseteq M_0$ with $v \neq \lambda\Im(v)$ for some $0<\lambda<1$. Then for each $\varphi \in [0,\sigma]$ we have

$$v(\varphi)=\lambda\left[\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon\left(s, v_s^*+\varsigma_s\right)ds\right]$$

This implies by $(C3)$

$$\begin{aligned}|v(\varphi)| & \leq \int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\left|\Upsilon\left(s, v_s^*+\varsigma_s\right)\right|ds \\ & \leq \frac{1}{\Gamma\left(\wp^*\right)}\int_0^{\varphi}(\varphi-s)^{\wp(s)-1}s^{-\iota}\left(\alpha(s)+\beta(s)\left\|v_s^*+\varsigma_s\right\|_{\mathfrak{B}}\right)ds\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\wp^*)} \int_0^{\varphi} \sigma^{\wp(s)-1} \left(\frac{\varphi - s}{\sigma} \right)^{\wp(s)-1} \left(s^{-\iota}\alpha(s) + s^{-\iota}\beta(s) \left\| v_s^* + \varsigma_s \right\|_{\mathfrak{B}} \right) ds \\
&\leq \frac{1}{\Gamma(\wp^*)} \int_0^{\varphi} \sigma^{\wp_*-1} \left(\frac{\varphi - s}{\sigma} \right)^{\wp_*-1} \left(s^{-\iota}\alpha(s) + s^{-\iota}\beta(s) \left\| v_s^* + \varsigma_s \right\|_{\mathfrak{B}} \right) ds \\
&\leq \frac{\sigma^*}{\Gamma(\wp^*)} \sigma^{1-\wp_*} \int_0^{\varphi} (\varphi - s)^{\wp_*-1} s^{-\iota}\beta(s) \| v_s^* + \varsigma_s \|_{\mathfrak{B}} ds \\
&+ \frac{\sigma^*\sigma^{1-\wp_*} \|\alpha\|_{\Psi} \Gamma(\wp^*) \Gamma(1-\iota)}{\Gamma(\wp^*) \Gamma(1-\iota+\wp^*)} \sigma^{\wp_*-\iota} \\
&\leq \frac{\sigma^*}{\Gamma(\wp^*)} \sigma^{1-\wp_*} \int_0^{\varphi} (\varphi - s)^{\wp_*-1} s^{-\iota}\beta(s) \left\| v_s^* + \varsigma_s \right\|_{\mathfrak{B}} ds + \frac{\sigma^*\sigma^{1-\wp_*} \|\alpha\|_{\Psi} \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)} \sigma^{\wp_*-\iota}
\end{aligned}
$$

but

$$
\begin{aligned}
\|v_s^* + \varsigma_s\|_{\mathfrak{B}} &\leq \|v_s^*\|_{\mathfrak{B}} + \|\varsigma_s\|_{\mathfrak{B}} \\
&\leq \kappa(\varphi) \sup\{|v(s)| : 0 \leq s \leq 1\} + L(\varphi) \left\| v_0 \right\|_{\mathfrak{B}} \\
&+ \kappa(\varphi) \sup\{|\varsigma(s)| : 0 \leq s \leq 1\} + L(\varphi) \left\| \varsigma_0 \right\|_{\mathfrak{B}} \\
&\leq \kappa_\sigma \sup\{|v(s)| : 0 \leq s \leq 1\} + L_\sigma \|\eta\|_{\mathfrak{B}}. \qquad (2.4)
\end{aligned}
$$

If we name $\psi(\varphi)$ the right-hand side of (2.4), then we get

$$
\|v_s^* + \varsigma_s\|_{\mathfrak{B}} \leq \psi(t)
$$

and therefor

$$
|v(\varphi)| \leq \frac{\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)} \int_0^{\varphi} (\varphi - s)^{\wp_*-1} s^{-\iota}\beta(s)\psi(s) ds + \frac{\sigma^*\sigma^{1-\wp_*} \|\alpha\|_{\Psi} \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)} \sigma^{\wp_*-\iota}, \quad \varphi \in [0, \sigma].
$$

Using the above inequality and the definition of ψ we have that

$$
\psi(\varphi) \leq L_\sigma \|\eta\|_{\mathfrak{B}} + \frac{\kappa_\sigma \sigma^*\sigma^{1-\wp_*} \|\alpha\|_{\Psi} \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)} \sigma^{\wp_*-\iota} + \frac{\kappa_\sigma \sigma^*\sigma^{1-\wp_*} \|\beta\|_{\Psi}}{\Gamma(\wp^*)} \int_0^{\varphi} (\varphi - s)^{\wp_*-1} s^{-\iota}\psi(s) ds, \varphi \in [0, \sigma].
$$

Then from Lemma (1.5.1), there exists $\kappa=\kappa\left(\wp^{*}\right)$ in this manner we have
$|\psi(\varphi)| \leq L_{\sigma}\|\eta\|_{\mathfrak{B}}+\frac{\kappa_{\sigma} \sigma^{*} \sigma^{1-\wp_{*}}\|\alpha\|_{\Psi} \Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^{*}\right)} \sigma^{\wp_{*}-\iota}+\kappa\left(\wp^{*}\right) \frac{\kappa_{\sigma} \sigma^{*} \sigma^{1-\wp_{*}}\|\beta\|_{\Psi}}{\Gamma\left(\wp^{*}\right)} \int_{0}^{\varphi}(\varphi-s)^{\wp_{*}-1} s^{-\iota} \delta d s,$
$\varphi \in[0, \sigma]$.
Where

$$\delta=L_{\sigma}\|\eta\|_{\mathfrak{B}}+\frac{\kappa_{\sigma} \sigma^{*} \sigma^{1-\wp_{*}}\|\alpha\|_{\Psi} \Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^{*}\right)} \sigma^{\wp_{*}-\iota}$$

hence

$$\|\psi\|_{\Psi} \leq \delta+\frac{\delta \kappa\left(\wp^{*}\right) \kappa_{\sigma} \sigma^{*} \sigma^{1-\wp_{*}} \Gamma(1-\iota) \sigma^{\wp_{*}-\iota}}{\Gamma\left(1-\iota+\wp^{*}\right)}=\widehat{L}$$

then

$$\|v\|_{\Psi} \leq \widehat{L}\left\|I^{\wp^{*}} \beta\right\|_{\Psi}+\frac{\sigma^{*} \sigma^{1-\wp_{*}}\|\alpha\|_{\Psi} \Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^{*}\right)}=L^{*}$$

Set

$$V=\left\{v \in M_{0}:\|v\|_{\sigma}<L^{*}+1\right\} .$$

$\Im: \bar{V} \rightarrow M_{0}$ is continuous and Completely continuity. From the choice of V, there is no $v \in \partial V$ such that $v=\lambda \Im(v)$ for $\lambda \in(0,1)$. As a consequence of Alternative non linear schauder theorem ([7]), we deduce that $\Im$ has a fixed point v in V.

The **1** result obtained by using the Banach Contraction Principle.

Theorem 2.2.2. *Suppossing that conditions (C1), (C2) are hold, and if*

$$\frac{\sigma^{*} \sigma^{1-\wp_{*}} \ell \kappa_{\sigma} \Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^{*}\right)} \sigma^{\wp_{*}-\iota}<1 \tag{2.5}$$

Where

$$\kappa_{\sigma}=\sup \{|\kappa(\varphi)|: \varphi \in D\}$$

Then the (PNPS) has a unique solution in Ψ.

Proof 2.2.2. *We give the operator*
$\Phi : \Omega \to \Omega$
defined by :

$$\Phi(w)(\varphi) = \begin{cases} \eta(\varphi), & \text{if } \varphi \in (-\infty, 0] \\ \displaystyle\int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon\left(s, w_s\right) ds, & \text{if } \varphi \in D \quad \text{and } \wp(s) > 0 \end{cases} \tag{2.6}$$

Let $\varsigma() : .(-\infty, \sigma] \to \mathbb{R}$ *be the function defined by*

$$\varsigma(\varphi) = \begin{cases} 0, & \text{if } \varphi \in [0, \sigma] \\ \eta(\varphi), & \text{if } \varphi \in (-\infty, 0] \end{cases}$$

Then $\varsigma_0 = \eta$. For each $v \in C([0, \sigma], \mathbb{R})$ with $v(0) = 0$, we denote v^* the function defined by

$$v^*(\varphi) = \begin{cases} v(\varphi), & \text{if } \varphi \in [0, \sigma] \\ 0, & \text{if } \varphi \in (-\infty, 0] \end{cases}$$

If $w(.)$ satisfies the integral equation

$$w(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon\left(s, w_s\right) ds$$

we can decompose $w(.)asw(\varphi) = v^*(\varphi) + \varsigma(\varphi), 0 \leq \varphi \leq \sigma$, and the function $v(.)satisfies$

$$v(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon\left(s, v^*(s) + \varsigma(s)\right) ds$$

Set M_0 is bound set with norm $\| \cdot \|_\sigma$ such that
$M_0 = \{v \in C([0, \sigma], \mathbb{R}) : v_0 = 0\}$,and let $\| \cdot \|_\sigma$ in M_0 defined by:

$$\|v\|_\sigma = \|v_0\|_{\mathfrak{B}} + \sup\{|v(\varphi)| : 0 \leq \varphi \leq \sigma\} = \sup\{|v(\varphi)| : 0 \leq \varphi \leq \sigma\}, \quad v \in M_0$$

We give the operator $\Im : M_0 \to M_0$ defined by:

$$(\Im v)(\varphi) = \int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon\left(s, v^*(s) + \varsigma(s)\right) ds, \quad \varphi \in D. \tag{2.7}$$

That the operator Φ admits a unique Fixed point is equivalent to $\Im$ has a Fixed point, and so we turn to proving that $\Im$ has a Fixed point. We shall that $\Im : M_0 \to M_0$ is a contraction map.

For $v_1(\varphi), v_2(\varphi) \in M_0$, we obtain that:

$$\begin{aligned}
\left|\Im\left(v_1\right)(\varphi) - \Im\left(v_2\right)(\varphi)\right| &\leq \int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \left|\Upsilon\left(s, v_1^*(s) + \varsigma(s)\right) - \Upsilon\left(s, v_2^*(s) + \varsigma(s)\right)\right| ds \\
&\leq \frac{1}{\Gamma\left(\wp^*\right)} \int_0^\varphi \sigma^{\wp(s)-1} \left(\frac{\varphi - s}{\sigma}\right)^{\wp(s)-1} \ell s^{-\iota} \left\|v_1^*(s) - v_2^*(s)\right\|_{\mathfrak{B}} ds \\
&\leq \frac{\ell}{\Gamma\left(\wp^*\right)} \int_0^\varphi \sigma^{\wp_*-1} \left(\frac{\varphi - s}{\sigma}\right)^{\wp_*-1} s^{-\iota} \kappa(s) sup_{s\in[0,\varphi]} \left\|v_1(s) - v_2(s)\right\| ds \\
&\leq \frac{\sigma^* \ell \kappa_\sigma}{\Gamma\left(\wp^*\right)} \sigma^{1-\wp_*} \int_0^\varphi (\varphi - s)^{\wp_*-1} s^{-\iota} \left\|v_1 - v_2\right\|_\sigma ds \\
&\leq \frac{\sigma^* \ell \kappa_\sigma}{\Gamma\left(\wp^*\right)} \sigma^{1-\wp_*} \left\|v_1 - v_2\right\|_\sigma \int_0^\varphi (\varphi - s)^{\wp_*-1} s^{-\iota} ds \\
&\leq \frac{\sigma^* \sigma^{1-\wp_*} \ell \kappa_\sigma \Gamma\left(\wp^*\right) \Gamma(1-\iota)}{\Gamma\left(\wp^*\right) \Gamma\left(1 - \iota + \wp^*\right)} \sigma^{\wp_*-\iota} \left\|v_1 - v_2\right\|_\sigma
\end{aligned}$$

by bound φ on D we find

$$\left\|\Im\left(v_1\right) - \Im\left(v_2\right)\right\|_\sigma \leq \frac{\sigma^* \sigma^{1-\wp_*} \ell \kappa_\sigma \Gamma(1-\iota)}{\Gamma\left(1 - \iota + \wp^*\right)} \sigma^{\wp_*-\iota} \left\|v_1 - v_2\right\|_\sigma,$$

Consequently by (2.5), the operator $\Im$ is a contraction. Hence, by Banach Contraction Principle, $\Im$ has a unique Fixed Point $v \in M_0$, which is a unique solution of the (PNPS).

2.3 Ulam Hyers stability

Theorem 2.3.1. *Let the conditions (C1) and (C2),(C3) hold, then the (PNPS) is Ulam Hyers Stability.*

Proof 2.3.1. *Let* $\varepsilon > 0$ *an arbitrary number and the function* $w(\varphi)$ *from* $v \in \mathfrak{B}$ *satisfy the following inequality*

$$\left|D^{\wp(\varphi)}\chi(\varphi) - \Upsilon(\varphi, \chi(\varphi))\right| < \varepsilon, \quad \varphi \in D$$

we have

$$D^{\wp(\varphi)}\chi(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{1-\wp(s)}}{\Gamma(1-\wp(s))}\chi(s)ds$$

we obtain

$$\begin{aligned}
\left|\chi(\varphi) - \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} r(s, \chi(s))ds\right| &\leq \varepsilon \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))}ds \\
&\leq \frac{\varepsilon}{\Gamma(\wp^*)} \int_0^{\varphi} \sigma^{\wp(s)-1} \left(\frac{\varphi - s}{\sigma}\right)^{\wp(s)-1} ds \\
&\leq \frac{\varepsilon}{\Gamma(\wp^*)} \int_0^{\varphi} \sigma^{\wp_*-1} \left(\frac{\varphi - s}{\sigma}\right)^{\wp_*-1} ds \\
&\leq \frac{\varepsilon\sigma^*}{\Gamma(\wp^*)}\sigma^{1-\wp_*} \int_0^{\varphi} (\varphi - s)^{\wp_*-1} s^{-\iota} s^{\iota} ds \\
&\leq \frac{\varepsilon\sigma^*\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{1-\wp_*}\sigma^{\wp_*-\iota} \int_0^{\varphi} s^{\iota} ds \\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\left[\frac{s^{\iota+1}}{\iota+1}\right]_0^{\varphi} \\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1}.
\end{aligned}$$

Let $\varphi \in D$, we get

$$\begin{aligned}
|\chi(\varphi) - w(\varphi)| &= \Bigg| \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,\chi(s))ds \\
&\quad - \int_0^{\varphi}\frac{(t-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,w(s))ds\Bigg| \\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}|\Upsilon(s,\chi(s)) - \Upsilon(s,w(s))|ds \\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell}{\Gamma\left(\wp^*\right)}\int_0^{\varphi}\sigma^{\wp(s)-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp(s)-1}s^{-\iota}|\chi(s)-w(s)|ds \\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell}{\Gamma\left(\wp^*\right)}\int_0^{\varphi}\sigma^{\wp_*-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}|\chi(s)-w(s)|ds \\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell\sigma^*\sigma^{1-\wp_*}}{\Gamma\left(\wp^*\right)}\|\chi-w\|_{\Psi}\int_0^{\varphi}(\varphi-s)^{\wp_*-1}s^{-\iota}ds \\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma\left(\wp^*\right)\Gamma(1-\iota)}{\Gamma\left(\wp^*\right)\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\|\chi-w\|_{\Psi} \\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\|\chi-w\|_{\Psi}.
\end{aligned}$$

Then

$$\|\chi-w\|_{\Psi}\left(1 - \frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\right) \leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\varphi^{\iota+1}.$$

We obtain, for each $\varphi \in D$

$$\begin{aligned}
|\chi(\varphi) - w(\varphi)| \leq \|\chi-w\|_{\Psi} &\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma\left(1-\iota+\wp^*\right)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} \\
&\quad\times\frac{\Gamma\left(1-\iota+\wp^*\right)}{\Gamma\left(1-\iota+\wp^*\right) - \ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}} \\
&\leq \varepsilon\frac{\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}\varphi^{\iota+1}}{(\iota+1)\left(\Gamma\left(1-\iota+\wp^*\right) - \ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}\right)} = c_{\Upsilon}\varepsilon.
\end{aligned}$$

Then the (PNPS) is Ulam Hyers Stability.

Chapter 3

The numerical methods

3.1 The finite difference method

3.1.1 Taylor's formula

Let f be an n + 1 function differentiable on [a, b], for all to $x_0 \in [a, b]$, there exists ξ between x and x_0 such as

$$\begin{aligned} f(x) &= f\left(x_0\right) + \frac{df}{dx}\left(x_0\right)\left(x - x_0\right) + \frac{d^2 f}{dx^2}\left(x_0\right)\frac{\left(x - x_0\right)^2}{2!} + \frac{d^3 f}{dx^3}\left(x_0\right)\frac{\left(x - x_0\right)^3}{3!} \\ &+ \cdots + \frac{d^n f}{dx^n}\left(x_0\right)\frac{\left(x - x_0\right)^n}{n!} + \frac{d^{n+1} f}{dx^{n+1}}(\xi)\frac{\left(x - x_0\right)^{n+1}}{(n+1)!} \end{aligned}$$

We pose:

$$R(\xi) = \frac{d^{n+1} f}{dx^{n+1}}(\xi)\frac{\left(x - x_0\right)^{n+1}}{(n+1)!}$$

So :

$$\begin{aligned} f(x) &= f\left(x_0\right) + \frac{df}{dx}\left(x_0\right)\left(x - x_0\right) + \frac{d^2 f}{dx^2}\left(x_0\right)\frac{\left(x - x_0\right)^2}{2!} + \frac{d^3 f}{dx^3}\left(x_0\right)\frac{\left(x - x_0\right)^3}{3!} \\ &+ \cdots + \frac{d^n f}{dx^n}\left(x_0\right)\frac{\left(x - x_0\right)^n}{n!} + R(\xi) \end{aligned} \tag{3.1}$$

3.1.2 Approximation of first derivative of f by forward finite differences with three points

We pose $x_j = x_0 + jh$ in (3.1). if $x = x_{j+1}$ and $x_0 = x_j$, so :

$$\begin{aligned} f(x_{j+1}) &= f(x_j) + f'(x_j)(x_{j+1} - x_j) + f''(x_j)\frac{(x_{j+1}-x_j)^2}{2!} + f'''(\xi_1)\frac{(x_{j+1}-x_j)^3}{3!} \\ &= f(x_j) + hf'(x_j) + \frac{h^2}{2!}f''(x_j) + \frac{h^3}{3!}f'''(\xi_1). \end{aligned} \tag{3.2}$$

if $x = x_{j+2}$ and $x_0 = x_j$, so

$$\begin{aligned} f(x_{j+2}) &= f(x_j) + f'(x_j)(x_{j+2} - x_j) + f''(x_j)\frac{(x_{j+2}-x_j)^2}{2!} + f'''(\xi_2)\frac{(x_{j+2}-x_j)^3}{3!} \\ &= f(x_j) + hf'(x_j) + \frac{h^2}{2!}f''(x_j) + \frac{h^3}{3!}f'''(\xi_1) \end{aligned} \tag{3.3}$$

We multiply (3.2) by (-4) then we add the result to (3.3) we find :

$$f'(x_j) = \frac{-3f(x_j) + 4f(x_{j+1}) - f(x_{j+2})}{2h} - \frac{2h^2}{3!}f'''(\xi_1) + \frac{4h^2}{3!}f'''(\xi_2). \tag{3.4}$$

So the error is

$$\text{error} = -\frac{2h^2}{3!}f'''(\xi_1) + \frac{4h^2}{3!}f'''(\xi_2) = O\left(h^2\right).$$

3.1.3 Approximation of the second derivative of f by forward finite differences with three points

if $x = x_{j+1}$ and $x_0 = x_j$, so :

$$\begin{aligned} f(x_{j+1}) &= f(x_j) + f'(x_j)(x_{j+1} - x_j) + f''(x_j)\frac{(x_{j+1}-x_j)^2}{2!} + f'''(\xi_1)\frac{(x_{j+1}-x_j)^3}{3!} \\ &= f(x_j) + hf'(x_j) + \frac{h^2}{2!}f''(x_j) + \frac{h^3}{2!}f'''(\xi_1). \end{aligned} \tag{3.5}$$

if $x = x_{j+2}$ and $x_0 = x_j$, so :

$$\begin{aligned} f(x_{j+2}) &= f(x_j) + f'(x_j)(x_{j+2} - x_j) + f''(x_j)\frac{(x_{j+2} - x_j)^2}{2!} + f'''(\xi_2)\frac{(x_{j+2} - x_j)^3}{3!} \\ &= f(x_j) + 2hf'(x_j) + \frac{(2h)^2}{2!}f''(x_j) + \frac{(2h)^2}{3!}f'''(\xi_2). \end{aligned} \tag{3.6}$$

Multiplying (3.5) by (-2) and summing with (3.6) we get:

$$f''(x_j) = \frac{f(x_j) - 2f(x_{j+1}) + f(x_{j+2})}{h^2} + \frac{2h}{3!}f'''(\xi_1) - \frac{8h}{3!}f'''(\xi_2).$$

So the error is

$$\text{error } = \frac{2h}{3!}f'''(\xi_1) + \frac{8h}{3!}f'''(\xi_2) = O(h).$$

3.1.4 Approximation of the second derivative of f by backward finite differences with three points

if $x = x_{j-1}$ and $x_0 = x_j$, so :

$$\begin{aligned} f(x_{j-1}) &= f(x_j) + f'(x_j)(x_{j-1} - x_j) + f''(x_j)\frac{(x_{j-1} - x_j)^2}{2!} + f'''(\xi_1)\frac{(x_{j-1} - x_j)^3}{3!} \\ &= f(x_j) + hf'(x_j) + \frac{h^2}{2!}f''(x_j) + \frac{h^3}{3!}f'''(\xi_1). \end{aligned} \tag{3.7}$$

if $x = x_{j-2}$ and $x_0 = x_j$, we have acconstant orderding to(3.1):

$$\begin{aligned} f(x_{j-2}) &= f(x_j) + f'(x_j)(x_{j-2} - x_j) + f''(x_j)\frac{(x_{j-2} - x_j)^2}{2!} + f'''(\xi_2)\frac{(x_{j-2} - x_j)^3}{3!} \\ &= f(x_j) + 2hf'(x_j) + \frac{(2h)^2}{2!}f''(x_j) + \frac{(2h)^3}{3!}f'''(\xi_2). \end{aligned} \tag{3.8}$$

By multiplying (3.7) by (-2) then adding with (3.8) we find:

$$f''(x_j) = \frac{f(x_{j-2}) - 2f(x_{j-1}) + f(x_j)}{h^2} - \frac{2h}{3!}f'''(\xi_1) + \frac{8h}{3!}f'''(\xi_2), \tag{3.9}$$

And

$$\text{error } = -\frac{2h}{3!} f'''(\xi_1) + \frac{8h}{3!} f'''(\xi_2) = O(h).$$

3.1.5 Approximation of the second derivative of f by finite differences in the center with three points

if $x = x_{j+1}$ and $x_0 = x_j$, so :

$$\begin{aligned} f(x_{j+1}) &= f(x_j) + f'(x_j)(x_{j+1} - x_j) + f''(x_j)\frac{(x_{j+1} - x_j)^2}{2!} + f'''(\xi_1)\frac{(x_{j+1} - x_j)^3}{3!} \\ &+ f''''(\xi_2)\frac{(x_{j+1} - x_j)^4}{4!} \\ &= f(x_j) + hf'(x_j) + \frac{h^2}{2!} f''(x_j) + \frac{h^3}{3!} f'''(x_j) + \frac{h^4}{4!} f''''(\xi_1). \end{aligned} \tag{3.10}$$

if $x = x_{j-1}$ and $x_0 = x_j$, so :

$$\begin{aligned} f(x_{j-1}) &= f(x_j) + f'(x_j)(x_{j-1} - x_j) + f''(x_j)\frac{(x_{j-1} - x_j)^2}{2!} + f'''(\xi_1)\frac{(x_{j-1} - x_j)^3}{3!} \\ &+ f''''(\xi_2)\frac{(x_{j-1} - x_j)^4}{4!} \\ &= f(x_j) - hf'(x_j) + \frac{h^2}{2!} f''(x_j) - \frac{h^3}{3!} f'''(x_j) + \frac{h^4}{4!} f''''(\xi_2). \end{aligned} \tag{3.11}$$

We add (3.10) to (3.11) we find:

$$f''(x_j) = \frac{f(x_{j-1}) - 2f(x_j) + f(x_{j+1})}{h^2} - \frac{h^2}{4!} f''''(\xi_1) - \frac{h^2}{4!} f''''(\xi_2), \tag{3.12}$$

And

$$\text{error } = -\frac{h^2}{4!} f''''(\xi_1) - \frac{h^2}{4!} f''''(\xi_2) = O\left(h^2\right).$$

3.2 The Euler's discretization method

The discretization process is introduced to discretize fractional-order differential equations/systems. It has been observed that as the fractional-order parameter approaches one, Euler's discretization method is recovered. This discretization method has been applied to fractional-order versions of the Riccati differential equation and Chua's system [12, 13, 28, 32]. In this context, we are particularly interested in applying the discretization method to the Cauchy-type problem for fractional variable order. Let $0 < \wp(\varphi) \leq 1$, and consider the fractional-order differential equation given by the system (1.5). The constant orderresponding equation with a piecewise constant argument is

$$D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon\left(r\frac{\varphi}{r}, w\left(r\frac{\varphi}{r}\right)\right), \quad \varphi \in D = [0, \sigma]$$

Let $\varphi \in [0, r]$, then $\frac{\varphi}{r} \in [0, 1]$. We get $D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(0, w(0)), \varphi \in [0, r]$.
Thus $w_1 = w_0 + \frac{\varphi^{\wp(0)}}{\Gamma(1+\wp(0))}\Upsilon(0, w_0)$.
Let $\varphi \in [r, 2r]$, then $\frac{\varphi}{r} \in [1, 2]$. We get $D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(r, w(r)), \varphi \in [r, 2r]$.
Thus $w_2 = w_1 + \frac{(\varphi - r)^{\wp(r)}}{\Gamma(1+\wp(r))}\Upsilon(r, w_1)$.
Let $\varphi \in [2r, 3r]$, then $\frac{\varphi}{r} \in [2, 3]$. We get $D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(2r, w(2r)), \varphi \in [2r, 3r]$.
Thus $w_3 = w_2 + \frac{(\varphi - 2r)^{\wp(2r)}}{\Gamma(1+\wp(2r))}\Upsilon(2r, w_2)$.
Repeating the process, we get when $\varphi \in [nr, (n+1)r]$, then $\frac{\varphi}{r} \in [n, n+1]$. So we get $D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(nr, w(nr)), \varphi \in [nr, (n+1)r]$. thus

$$w_{n+1}(\varphi) = w_n(nr) + \frac{(\varphi - nr)^{\wp(nr)}}{\Gamma(1+\wp(nr))}\Upsilon(nr, w_n(nr))$$

We can compute this sequence w_n for larger n to obtain a more accurate approximation.

3.3 The principle of the finite difference method

The principle of this methode is explained in the following lines: It is assumed that the interval $[a, b]$ is subdivided into n subintervals $[x_k, x_{k+1}]$ length $h = \dfrac{b-a}{n}$ using equally spaced knots $x_k = a + kh$ for $k = 0, 1, \ldots, n$ [31, 41]. The composite trapezoidal rule for n subintervals allows us to write:

$$T(f, h) = \frac{h}{2} \sum_{k=1}^{n} \left[f(x_{k-1}) + f(x_k) \right], \tag{3.13}$$

and

$$\int_a^b f(x)dx \approx T(f, h). \tag{3.14}$$

3.4 Approximation of fractional derivative in the Rieman-Liouville sense by the finite difference method

From the relationship between the fractional derivative in the sense of Caputo and the fractional derivative in the sense of Riemann-Liouville,we have:

$${}^{RL}D_a^\alpha yt = {}^{C} D_a^\alpha y(t) + \sum_{k=0}^{m-1} \frac{y^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}.$$

So :

$${}^{RL}D_a^\alpha y(t) \approx \frac{1}{\Gamma(2-\alpha)} \left[A + \frac{h}{2}(\beta + 2C + D) \right] + \sum_{k=0}^{m-1} \frac{y^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}.$$

3.5 Approximation of the fractional derivative in the sense of Caputo by the finite difference method

We have :

$$^{c}D^{\alpha}y(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{y'(t)}{(t-x)^{\alpha}}dx,$$

suchas : $0<\alpha<1$ and $t\geq 0$.

Integrating by parts we find :

$$^{c}D^{\alpha}y(t)=\frac{1}{(1-\alpha)\Gamma(1-\alpha)}\left(y'(0)t^{1-\alpha}+\int_0^t(t-x)^{1-\alpha}y''(x)dx\right).$$

We have the approximation of y in point t_i

$$^{C}D^{\alpha}y\left(t_i\right)=\frac{1}{(1-\alpha)\Gamma(1-\alpha)}\left(y'(0)t_i^{1-\alpha}+\int_0^{t_i}\left(t_i-x\right)^{1-\alpha}y''(x)dx\right).$$

We pose : $f(x)=\left(t_i-x\right)^{1-\alpha}y''(x)$ and we use (3.13) and (3.14), we obtain :

$$\begin{aligned}\int_0^{t_i}\left(t_i-x\right)^{1-\alpha}y''(x)dx &\approx \frac{h}{2}\sum_{k=1}^{i}\left[\left(t_i-x_{k-1}\right)^{1-\alpha}y''\left(x_{k-1}\right)+\left(t_i-x_k\right)^{1-\alpha}y''\left(x_k\right)\right]\\ &\approx \frac{h}{2}\sum_{k=1}^{i}\left(t_i-x_{k-1}\right)^{1-\alpha}y''\left(x_{k-1}\right)+\frac{h}{2}\sum_{k=1}^{i}\left(t_i-x_k\right)^{1-\alpha}y''\left(x_k\right)\\ &\approx \frac{h}{2}\left[\left(t_i-x_0\right)^{1-\alpha}y''(0)+2\sum_{k=1}^{i-1}\left(t_i-x_k\right)^{1-\alpha}y''\left(x_k\right)\right].\end{aligned}$$

So :

$$^{c}D^{\alpha}y\left(t_i\right)=\frac{h\left[\left(t_i-x_0\right)^{1-\alpha}y''(0)+2\sum_{k=1}^{i-1}\left(t_i-x_k\right)^{1-\alpha}y''\left(x_k\right)\right]}{2(1-\alpha)\Gamma(1-\alpha)}+\frac{\left(t_i-x_0\right)^{1-\alpha}y'(0)}{(1-\alpha)\Gamma(1-\alpha)},$$

and since $a\Gamma(a) = \Gamma(a+1)$, so we have :

$$c_{D^\alpha} y(t) = \frac{h\left[(t_i - x_0)^{1-\alpha} y''(0) + 2\sum_{k=1}^{i-1} (t_i - x_k)^{1-\alpha} y''(x_k)\right]}{2\Gamma(2-\alpha)} + \frac{(t_i - x_0)^{1-\alpha} y'(0)}{\Gamma(2-\alpha)}. \tag{3.15}$$

According to (3.4), (3.9) and (3.12), we find :

$$y'(x_k) \approx \frac{-3y(x_k) + 4y(x_{k+1}) - y(x_{k+2})}{2h}.$$
$$y''(x_k) \approx \frac{y(x_{k-2}) - 2y(x_{k-1}) + y(x_k)}{h^2}.$$

So :

$$\begin{aligned} y'(x_0) &\approx \frac{-3y(x_0) + 4y(x_1) - y(x_2)}{2h}. \\ y''(x_0) &\approx \frac{y(x_0) - 2y(x_1) + y(x_2)}{h^2}. \end{aligned} \tag{3.16}$$

We remplace with (3.16) in (3.15), on obtient :

$$\begin{aligned} {}^C D^\alpha y(t_i) \approx& \frac{1}{\Gamma(2-\alpha)}\left[\frac{y(x_0) - 2y(x_1) + y(x_2)}{2h}(t_i - x_0)^{1-\alpha}\right] \\ &+ \frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^{i-1}\left[\frac{y(x_{k-1}) - 2y(x_k) + y(x_{k+1})}{h}(t_i - x_k)^{1-\alpha}\right] \\ &+ \frac{-3y(x_0) + 4y(x_1) - y(x_2)}{2h\Gamma(2-\alpha)}(t_i - x_0)^{1-\alpha}. \forall i = 1..n \end{aligned}$$

3.6 Approximation of fractional derivative in the Riemann-Liouville seense by the finite difference method:

According to the relationship between the fractional derivative in the sense of Caputo and the fractional derivative in the sense of Riemann-Liouville, we have :

$$ {}^{RL}D_a^{\alpha} y(t) = {}^{C}D_a^{\alpha} y(t) + \sum_{k=0}^{m-1} \frac{y^{(k)}(a)(t-a)^{k-a}}{\Gamma(k-\alpha+1)}. $$

So :

$$ \begin{aligned} {}^{RL}D_a^{\alpha} y\left(t_i\right) \approx & \frac{1}{\Gamma(2-\alpha)}\left[\frac{y\left(x_0\right)-2 y\left(x_1\right)+y\left(x_2\right)}{2 h}\left(t_i-x_0\right)^{1-\alpha}\right] \\ & +\frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{i-1}\left[\frac{y\left(x_{k-1}\right)-2 y\left(x_k\right)+y\left(x_{k+1}\right)}{h}\left(t_i-x_k\right)^{1-\alpha}\right] \\ & +\frac{-3 y\left(x_0\right)+4 y\left(x_1\right)-y\left(x_2\right)}{2 h \Gamma(2-\alpha)}\left(t_i-x_0\right)^{1-\alpha} \\ & +\sum_{k=0}^{m-1} \frac{y^{(k)}(a)\left(t_i-a\right)^{k-a}}{\Gamma(k-\alpha+1)}, \forall i=1..n. \end{aligned} $$

3.6.1 Numerical applications for the Riemann-Liouville

We have:

$$ \begin{aligned} {}^{RL}D^{\alpha} y(t) \approx & \frac{-3 y\left(x_0\right)+4 y\left(x_1\right)-y\left(x_2\right)}{2 h \Gamma(2-\alpha)}\left(t-x_0\right)^{1-\alpha}+\frac{y\left(x_0\right)-2 y\left(x_1\right)+y\left(x_2\right)}{2 h \Gamma(2-\alpha)}\left(t-x_0\right)^{1-\alpha} \\ & +\sum_{k=1}^{n-1}\left[\frac{y\left(x_{k-1}\right)-2 y\left(x_k\right)+y\left(x_{k+1}\right)}{h \Gamma(2-\alpha)}\left(t-x_k\right)^{1-\alpha}\right] \\ & +\frac{y\left(x_{n-2}\right)-2 y\left(x_{n-1}\right)+y\left(x_n\right)}{2 h \Gamma(2-\alpha)}\left(t-x_n\right)^{1-\alpha}+\sum_{k=0}^{m-1} \frac{y^{(k)}\left(x_0\right)\left(t-x_0\right)^{k-a}}{\Gamma(k-\alpha+1)}. \end{aligned} $$

Remark 3.6.1. *We note $y(x_i)$ by y_i.*

We can write 4.4 in the following form :

$$^{RL}D^{\alpha}y(t) = \sum_{k=0}^{n} a_{k,i}y_k + \sum_{k=0}^{m-1} \frac{y^{(k)}(x_0)(t-x_0)^{k-a}}{\Gamma(k-\alpha+1)} = f(t_i)$$

With

$$a_{0,i} = \frac{(t_i - x_1)^{1-\alpha} - (t_i - x_0)^{1-\alpha}}{h\Gamma(2-\alpha)}$$

$$a_{1,i} = \frac{(t_i - x_0)^{1-\alpha} - 2(t_i - x_1)^{1-\alpha} + (t_i - x_2)^{1-\alpha}}{h\Gamma(2-\alpha)},$$

$$a_{2,i} = \frac{(t_i - x_1)^{1-\alpha} - 2(t_i - x_2)^{1-\alpha} + (t_i - x_3)^{1-\alpha}}{h\Gamma(2-\alpha)},$$

$$a_{k,i} = \frac{(t_i - x_{k-1})^{1-\alpha} - 2(t_i - x_k)^{1-\alpha} + (t_i - x_{k+1})^{1-\alpha}}{h\Gamma(2-\alpha)},$$

$$a_{n-2,i} = \frac{(t_i - x_n)^{1-\alpha}}{2h\Gamma(2-\alpha)} + \frac{(t_i - x_{n-3})^{1-\alpha} - 2(t_i - x_{n-2})^{1-\alpha} + (t_i - x_{n-1})^{1-\alpha}}{h\Gamma(2-\alpha)},$$

$$a_{n-1,i} = \frac{-(t_i - x_n)^{1-\alpha} - 2(t_i - x_{n-1})^{1-\alpha} + (t_i - x_{n-2})^{1-\alpha}}{h\Gamma(2-\alpha)}$$

$$a_{n,i} = \frac{(t_i - x_n)^{1-\alpha}}{2h\Gamma(2-\alpha)} + \frac{(t_i - x_{n-1})^{1-\alpha}}{h\Gamma(2-\alpha)}.$$

Example 3.6.1. *We define this fractional problem*

$$\begin{cases} D^{\wp(\varphi)}w(\varphi) = \dfrac{re^{-\theta\varphi+\varphi}\,|w_\varphi|}{(\varphi+1)^{\frac{1}{6}}\,(e^{\varphi}+e^{-\varphi})\,(1+|w_\varphi|)} & ,\varphi \in D = [0,1], \text{ and } \theta > 0. \\ w(\varphi) = \eta(\varphi), \varphi \in (-\infty, 0] \end{cases}$$

Where $\wp(\varphi) = \frac{1}{2}\varphi + \frac{1}{2}$ is a Continued fraction on $[0,1]$.

$$B_\theta = \left\{ w \in C(]-\infty, 0[, \mathbb{R}), \lim_{j \to -\infty} e^{j\theta} w(j) \text{ exist } \text{ in } \mathbb{R} \right\}$$

$$\|w_\theta\|_{B_\theta} = \sup_{-\infty < j \leqslant 0} e^{\theta j} |w(j)|$$

We take $\omega_\varphi(j) = \omega(\varphi + j) \in B_\theta$

$$\begin{aligned} |w_\varphi(j)| &= \sup_{-\infty < j \leqslant 0} e^{\theta j} \left| w_\varphi(j) \right| . \\ &= \sup_{-\infty < j \leqslant 0} e^{\theta j} |\omega(\varphi + j)| \end{aligned}$$

If $\varphi + j \leqslant 0 \Rightarrow |w_\varphi(j)| \leq \sup\{|w(s)|, s \leqslant 0\}$.
If $\varphi + j \geqslant 0 \Rightarrow |w_\varphi(j)| \leqslant \sup\{|w(s)|, \varphi \geqslant s \geqslant 0\}$.
Than

$$|w_\varphi(j)| \leqslant \sup\{|\omega(s)|, s \leq 0\} + \sup\{|\omega(s)|, \varphi \geqslant s \geqslant 0\}.$$

Than

$$k(\varphi) = 1, L(\varphi) = 1$$

$$\Rightarrow K_\sigma = \sup\{|k(\varphi)|\} = 1.$$

We have

$$\Upsilon(\varphi, w) = \frac{r e^{-\theta\varphi + \varphi} w}{\left((\varphi + 1)^{\frac{1}{6}} (e^\varphi + e^{-\varphi} \right) (1 + w)} \quad \varphi \in [0, 1], w \in B_\theta.$$

Lets $w_1, w_2 \in B_\theta$, we have:

$$
\begin{aligned}
\varphi^l \left|\Upsilon(\varphi, w_1) - \Upsilon(\varphi, w_2)\right| &= \varphi^l \left| \frac{e^{-\theta\varphi+\varphi} w_1}{2\left((\varphi+1)^{\frac{1}{6}}(e^{\varphi}+e^{-\varphi}\right)(1+w_1)} - \frac{e^{-\theta\varphi+\varphi} w_2}{2(\varphi+1)(e^{\varphi}+e^{-\varphi})(1+w_2)} \right| \\
&= \varphi^l \frac{e^{-\theta\varphi+\varphi}}{2\left((\varphi+1)^{\frac{1}{6}}(e^{\varphi}+e^{-\varphi}\right)} \left| \frac{w_1}{1+w_1} - \frac{w_2}{1+w_2} \right| \\
&\leq \varphi^l \frac{1}{2(\varphi)^{\frac{1}{6}}} \left| \frac{w_1}{1+w_1} - \frac{w_2}{1+w_2} \right| \\
&\leq \frac{1}{2} \left| \frac{w_1}{1+w_1} - \frac{w_2}{1+w_2} \right| \\
&\leq \frac{1}{2} \left| \frac{w_1(1+w_2) - w_2(1+w_1)}{(1+w_1)(1+w_2)} \right| \\
&\leq \frac{1}{2} \left| \frac{w_1 - w_2}{(1+w_1)(1+w_2)} \right| \\
&\leq \frac{1}{2} \left| w_1 - w_2 \right| \\
&\leq \frac{1}{2} \left| w_1 - w_2 \right|_{B_\theta}
\end{aligned}
$$

Than $l = \frac{1}{2}, i = \frac{1}{6}, \sigma = 1$
$\sigma^* = \max\left\{\sigma, \sigma^{\varphi-1}\right\} = 1,\ k_\sigma = 1,$
We have:

$$
\begin{aligned}
& l = \frac{1}{2}, \quad i = \frac{1}{6}, \sigma = 1, P = 1 \\
& \sigma^* = \max\left\{\sigma; \sigma^{\varphi^*-1}\right\}. \\
& \sigma^* = \max\{1,1\} \quad \sigma^* = 1, k_\sigma = 1
\end{aligned}
$$

By substituting into the equation (2.5): We find:

$$
\frac{\sigma^* \sigma^{1-\wp_*} \ell \kappa_\sigma \Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^*\right)} \sigma^{\wp_*-\iota} = \frac{\frac{1}{2}\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{11}{6}\right)}
$$

And from it according to the property of Gamma, We find:

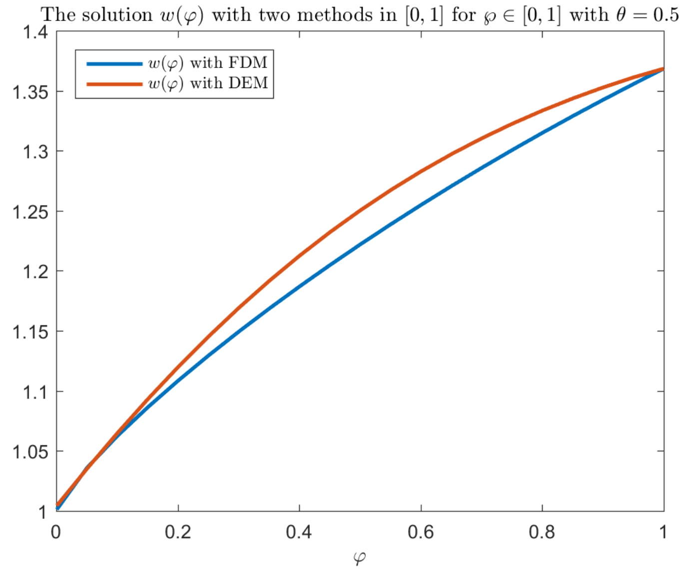
$$\frac{\sigma^{*}\sigma^{1-\wp_{*}}\ell\kappa_{\sigma}\Gamma(1-\iota)}{\Gamma\left(1-\iota+\wp^{*}\right)}\sigma^{\wp_{*}-\iota} = \frac{\frac{1}{2}\Gamma\left(\frac{5}{6}\right)}{\frac{5}{6}T\left(\frac{5}{6}\right)}$$
$$= \frac{3}{5} = 0.6 < 1$$

According to theorem (2.2.2), there is only one solution.

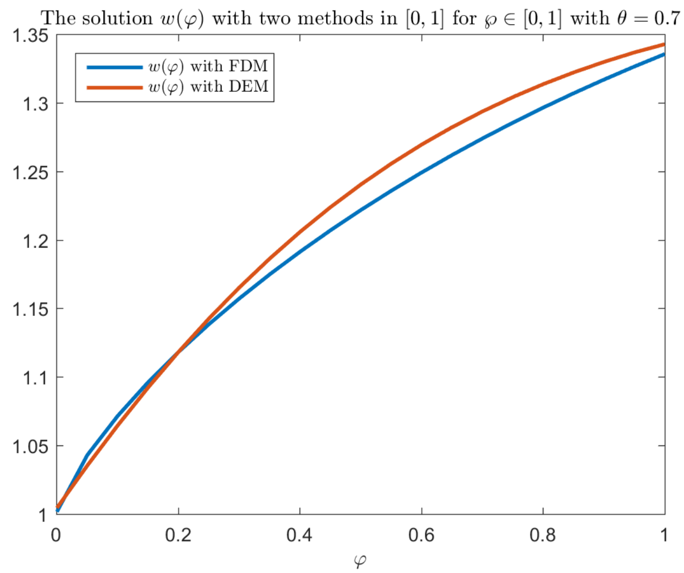
3.6.2 Numerical results

In this section, we use two numerical methods. The first is the finite difference method (**FDM**)[19], the second method is the Euler's discretization method (**DEM**)[5]. Both methods are based on the subdivision of the interval, we took 1000 points. We calculated the solution $w_{FDM}(\varphi)$ with the (**FDM**) method and the solution $w_{DEM}(\varphi)$ with the (**DEM**) method for $\wp(\varphi) = \frac{1}{2}\varphi + \frac{1}{2}$ with $\varphi \in [0, 1]$.

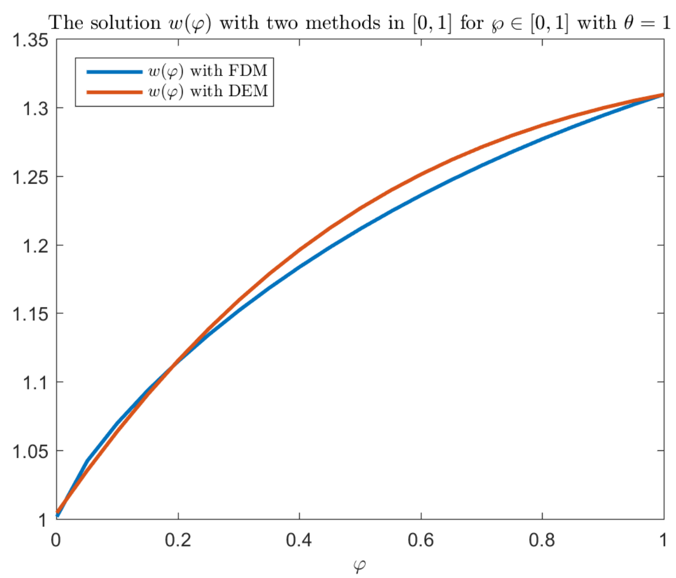
In figure (3.1), we plot the solutions $w_{FDM}(\varphi)$ and $w_{DEM}(\varphi)$ depending on φ and different θ.



The solution $w(\varphi)$ with two methods in $[0,1]$ for $\wp \in [0,1]$ with $\theta = 0.5$
$w(\varphi)$ with FDM
$w(\varphi)$ with DEM
1.4
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ



The solution $w(\varphi)$ with two methods in $[0,1]$ for $\wp \in [0,1]$ with $\theta = 0.7$
$w(\varphi)$ with FDM
$w(\varphi)$ with DEM
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ



The solution $w(\varphi)$ with two methods in $[0,1]$ for $\wp \in [0,1]$ with $\theta = 1$
$w(\varphi)$ with FDM
$w(\varphi)$ with DEM
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ

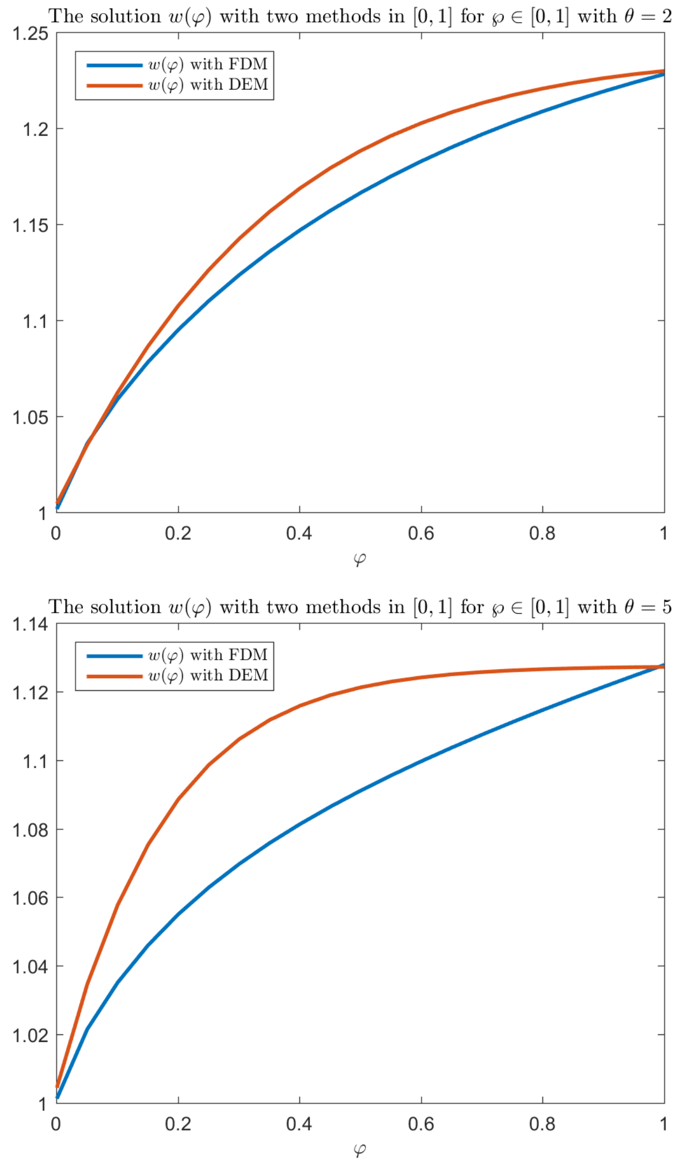


Figure 3.1: The solutions $w_{FDM}(\varphi)$ and $w_{DEM}(\varphi)$ in $[0,1]$ with $\wp(\varphi) = \frac{1}{2}\varphi + \frac{1}{2}$ and different θ.

In this table, we present the $Norm = \max_{\varphi\in[0,1]} \Big| w_{FDM}(\varphi) - w_{DEM}(\varphi) \Big|$ for $\wp(\varphi) \in [0,1]$.

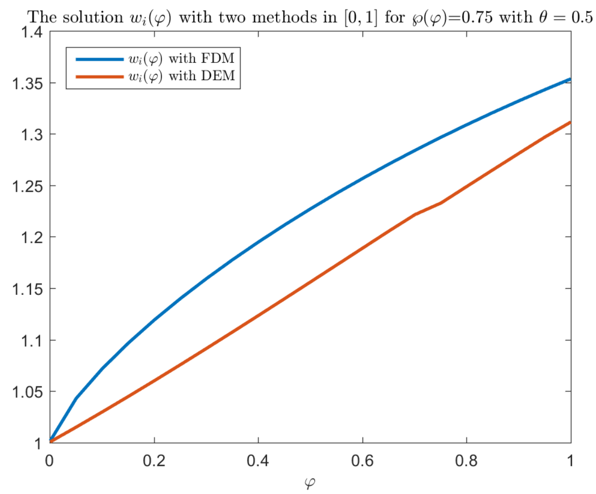
θ	$\theta = 0.5$	$\theta = 0.7$	$\theta = 1$	$\theta = 2$	$\theta = 5$
$Norm$	0.02665	0.02454	0.01541	0.02008	0.02940

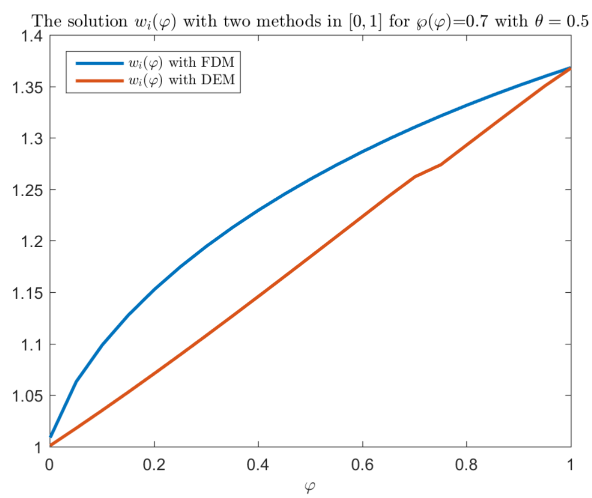
We notice that the error between the two methods is of order 2.
We observe that for $\theta \leq 1$, the approximate solution w_{FDM} is close to the approximate solution w_{DEM} when θ is proche to 1. For $\theta > 1$, the two approximate solution is the same when θ is proche to 1(see figure(3.1)).
Now,we calculated the solution $w_{i,FDM}(\varphi)$ with the (**FDM**) method and the solution $w_{i,DEM}(\varphi)$ with the (**DEM**) method for $\wp(\varphi_i) = \frac{1}{2}\varphi_i + \frac{1}{2}$ where φ_i is fixed and for different θ.

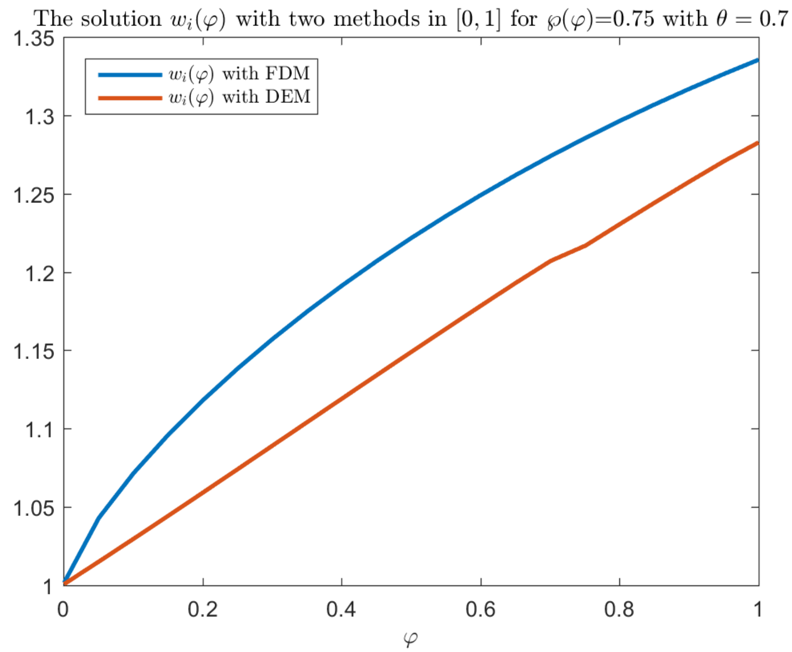
In this table, we present the $Norm_i = \max_{\varphi\in[0,1]} \Big| w_{i,FDM}(\varphi) - w_{i,DEM}(\varphi) \Big|$ for $\wp(\varphi)$ fixed in $[0,1]$.

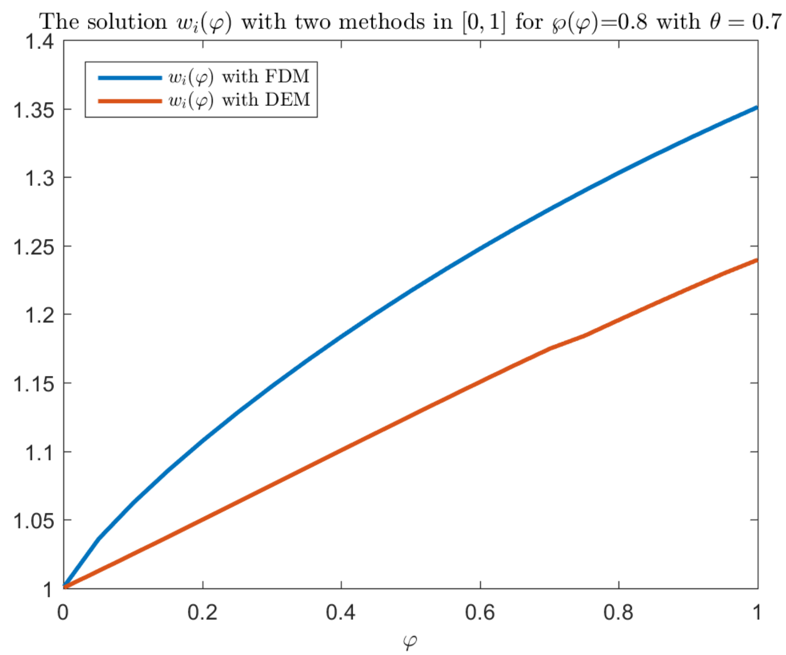
φ	0.2	0.4	0.5	0.6	0.8	1.0
$\wp(\varphi)$	0.6	0.7	0.75	0.8	0.9	1
$Norm_i, \theta = 0.5$	0.13623	0.03688	0.02726	0.02962	0.05435	0.07210
$Norm_i, \theta = 0.7$	0.13608	0.03703	0.02102	0.02450	0.04922	0.06663
$Norm_i, \theta = 1$	0.12351	0.02798	0.01608	0.03119	0.05407	0.07028
$Norm_i, \theta = 2$	0.02132	0.02152	0.02279	0.03419	0.05312	0.06625
$Norm_i, \theta = 5$	0.02124	0.02177	0.02989	0.03681	0.04750	0.05489

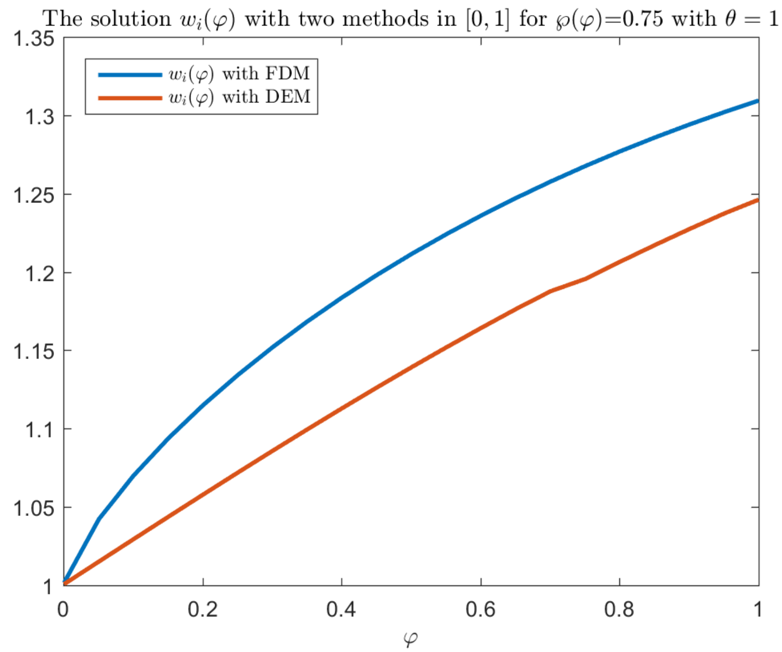
We observe that $\theta \leq 1$ the $Norm_i$ is small at $\varphi = 0.5, \wp = 0.75$, it decreasing in $\varphi \in [0, 0.5]$ and crescent in $\varphi \in [0.5, 1]$ but when $\theta > 1$ is creasing in $\varphi \in [0, 1]$. Then for $\theta \leq 1$, the approximate solution $w_{i,FDM}$ is close to the approximate solution $w_{i,DEM}$ when φ is proche to $0.5(\wp = 0.75)$. For $\theta > 1$, the two approximate solution is the same when φ is proche to $0(\wp = 0.5)$(see figure(3.2)).
In the figure (3.2), we present a comparison between the solution $w_{i,FDM}(\varphi)$ and the solutions $w_{i,DEM}(\varphi)$ with a $\wp$ different and different θ.

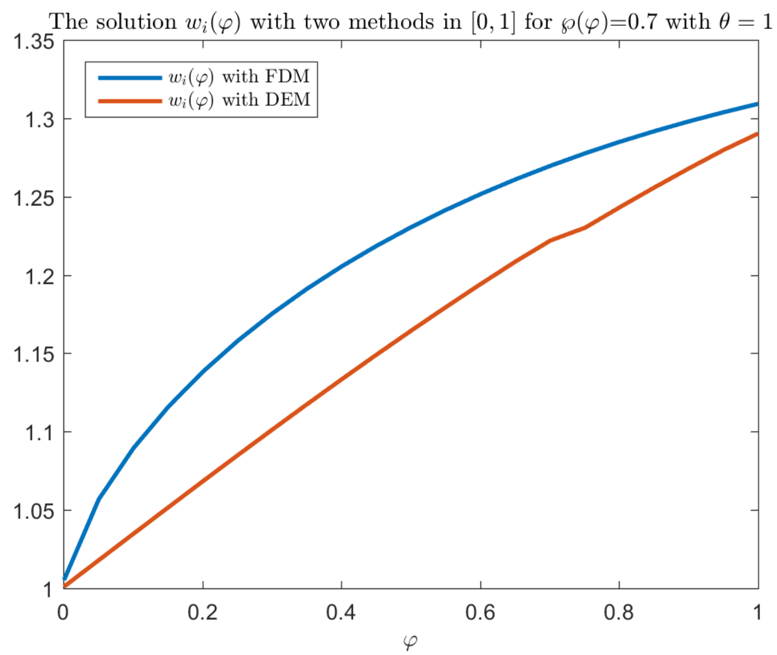


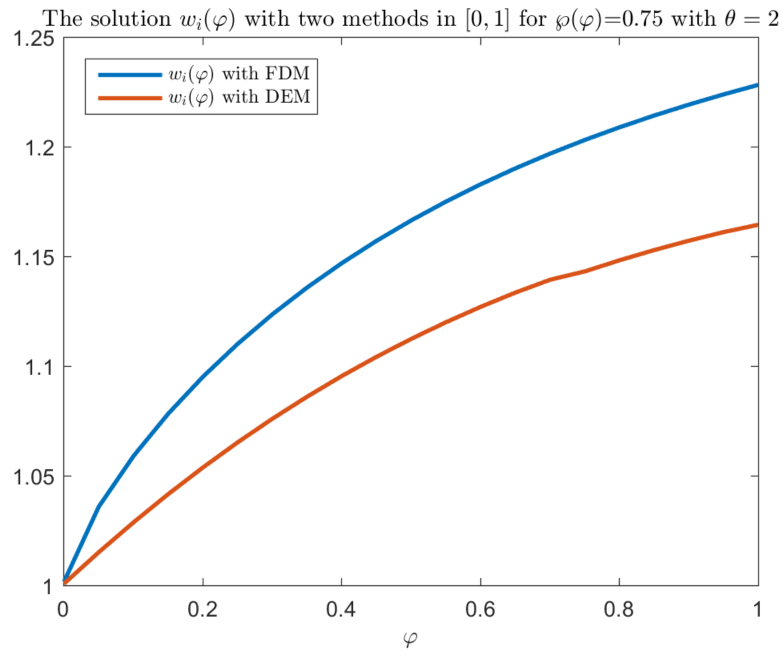


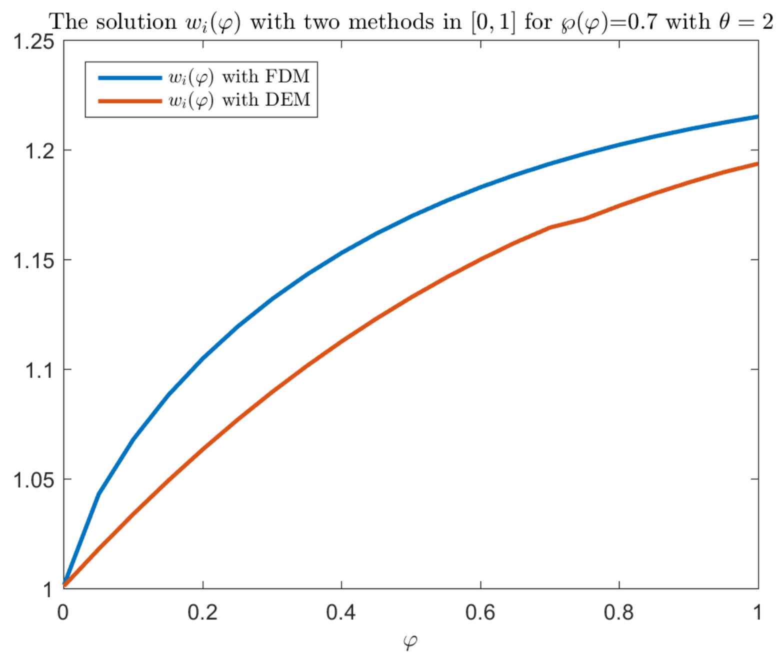

The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.75 with $\theta = 0.7$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ


The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.8 with $\theta = 0.7$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.4
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ


The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.75 with $\theta = 1$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ


The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.7 with $\theta = 1$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ


The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.75 with $\theta = 2$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ


The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.7 with $\theta = 2$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ

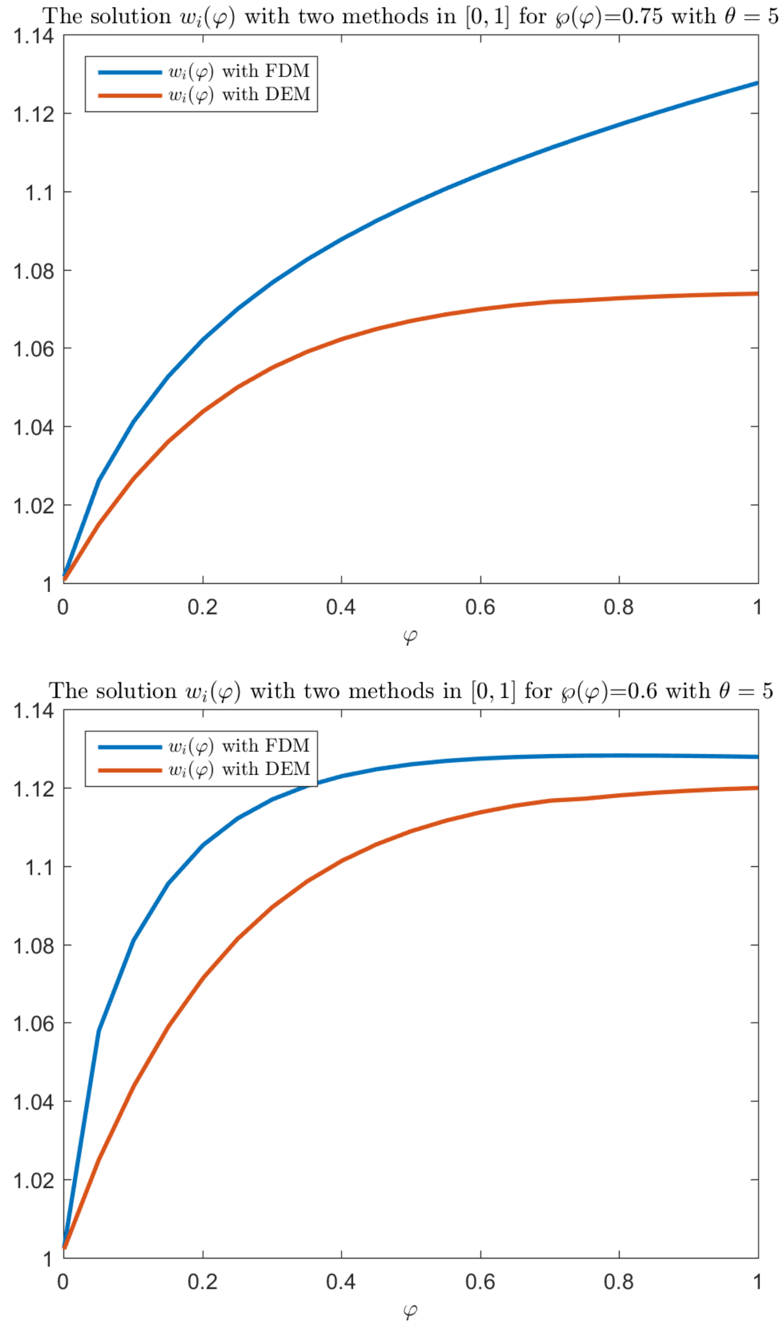


Figure 3.2: A plot of $w_i(\varphi)$ with two methods for different $\wp(\varphi)$ and different θ.

Conclusion

In this work, we have explored the theoretical and numerical aspects of fractional calculus, with a particular focus on fractional derivatives and their approximation using finite difference methods. The study began with foundational concepts, such as the Riemann-Liouville fractional integral and derivative, the Gamma and Beta functions,and the notion of phase space. These elements provided the essential mathematical tools required to model and analyze systems governed by fractional differential equations.

The second part of the study dealt with the existence and uniqueness of solutions. By examining fixed point theorems and different types of stability, especially Ulam's stability, we were able to build a rigorous theoretical framework ensuring the validity and reliability of the fractional models being used.

The core contribution of this work lies in the third part, which focused on numerical methods. We reviewed and implemented various finite difference techniques to approximation both integer-order and fractional-order derivatives.Special attentions was given to the Riemann-Liouville and Caputo fractional derivatives. We derived and analyzed different schemes, including forward, backward, and central difference approximations, and applied Euler's discretization method to demonstrate their practical utility. Moreover, we examined the effectiveness of these methods through numerical applications and results, highlighting their advantages and limitations.

In conclusion, this research provides a comprehensive approach to understanding and solving fractional differential aquations using finite difference methods. The results affirm that numerical methods are powerful tools for approximating solutions where analytical methods fall short, particularly in complex systems described by fractional dynamics. Future work may extend this approach to more advanced fractional operators and higher-dimensional problems,with possible applications in physics, engineering,and biology.

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