RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE
Ministère de L'enseignement Supérieur et de la Recherche Scientifique
UNIVERSITÉ IBN KHALDOUN TIARET
FACULTÉ DE MATHÉMATIQUES ET DE L'INFORMATIQUES
Département de Mathématiques

MÉMOIRE de MASTER

Présenter en vue de l'obtention du diplôme de master

Spécialité :

Mathématiques

Option :

«Analyse fonctionnelle et équations différentielles »

Présenté Par :

RAHMOUNI Leila

Sous L'intitulé :

Monotone iterative method applied to systems of fractional differential equations

Soutenu publiquement le 29 / 06 / 2025
à Tiaret devant le jury composé de :

Mr ***BENHABI Mohamed***	***M.A.A***	Université *Tiaret*	Président
Mr ***MAHROUZ Tayeb***	***M.C.A***	Université *Tiaret*	Examinateur
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Méthode d'itérations monotones appliquée aux systèmes d'équations différentielles fractionnaires

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«اللهم لك الحمد»

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DEDICATIONS

I dedicate this modest work:

To my dearest parents, the light of my life, who have endured so much and made countless sacrifices to ensure my happiness.

I dedicate this modest work to my dear parents, who have always helped and supported me in completing my studies.

I am deeply grateful for their advice, affection, and unwavering support.

Thank you for all the efforts you have made for me. May God protect you, keep you safe, and bless you.

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To my dear brothers,

To my dear sisters,

To my little sweethearts: my nephews and nieces,

To my entire family,

To all my friends,

And to all my teachers, for their valuable advice, patience, and dedication.

This achievement is not mine alone, it belongs to all of you who believed in me.

RAHMOUNI Leila

Abstract

In this work, we present existence of extremal solutions for nonlinear Riemann-Liouville fractional differential equations with integral boundary conditions (nonlocal conditions) and for coupled systems of nonlinear Riemann-Liouville fractional differential equations with initial conditions. Also, we present the existence of extremal solutions for a coupled system of nonlinear fractional differential equations involving the ψ-Caputo derivative with initial conditions.

Our results will be obtained by using the monotone iterative technique combined with the method of upper and lower solutions.

Key words and phrases: fractional calculus, Riemann-Liouville fractional differential equations, ψ-Caputo fractional derivatives, systems of fractional differential equations, upper and lower solutions, monotone iterative technique.

Résumé

Nous présentons dans ce mémoire, l'existence de solutions extrêmes pour des équations différentielles à dérivées fractionnaires au sens de Riemann-Liouville avec condition intégrale aux limites, et pour un système couplè d'équations différentielles non linéaires à dérivées fractionnaires au sens de Riemann-Liouville avec des conditions initiales. Aussi, nous présentons l'existence de solutions extrêmes pour un système couplé d'equations différentielles non linéaires à dérivées fractionnaires au sens de ψ-Caputo avec conditions initiales.

Ces résultats sont obtenus grâce à la technique itérative monotone combinée à la méthode des sous et sur solutions.

Mots Clés: Calcul fractionnaire, dérivée fractionnaire de ψ-Caputo, équations différentielles à dérivées fractionnaires au sens de Riemann-Liouville, systèmes d'équations différentielles fractionnaires, sous et sur solutions, technique des itérations monotones.

Contents

Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. We can find numerous applications of differential equations of fractional order in various sciences such as physics, mechanics, electrochemistry, control, population dynamics, electrodynamics and electromagnetic, etc. For details, see [10, 29, 32, 34, 37, 40]. Several approaches to fractional derivatives exist: Caputo, Riemann-Liouville (RL), Hadamard, Grunwald-Letnikov (GL), etc., can be found in [11, 12, 16].

Recently, a new fractional derivative, called the ψ-Caputo fractional derivative, was introduced by Almeida in [4]. For recent results on ψ-Caputo fractional derivative we refer the reader to [2, 4, 5, 6, 7, 8, 20, 21, 39].

In this work, we present existence of extremal solutions for nonlinear Riemann-Liouville fractional differential equations with integral boundary conditions (nonlocal conditions) and for coupled systems of nonlinear Riemann-Liouville fractional differential equations with initial conditions. Also, we present the existence of extremal solutions for a coupled system of nonlinear fractional differential equations involving the ψ-Caputo derivative with initial conditions.

The monotone iterative technique combined with the method of upper and lower solutions has been applied by several authors, see [3, 9, 13, 14, 17, 18, 21, 22, 30, 31, 33, 42, 43, 46]. The purpose of this method is to constructing two monotone iterative sequences, by using x_0, y_0 the lower and upper solutions with $x_0 \leq y_0$, showing the convergence of the constructed sequences, and proving these two sequences approximate the extremal solutions of the given problem.

We have organized this work as follows:

In Chapter 1, we present some definitions and results which are used throughout this work.

In Chapter 2, we investigate the existence of extremal solutions of boundary value problem for the following nonlinear fractional differential equation with integral boundary conditions:

$$\begin{cases} {}^{RL}D^q x(t) = f(t, x(t)), & t \in [0,T],\ T > 0, \\ x(0) = \lambda \displaystyle\int_0^T x(s)ds + d, & d \in \mathbb{R}. \end{cases}$$

where $0 < q < 1$, $\lambda \geq 0$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^{RL}D^{\alpha}x$ denotes the Riemann-Liouville fractional derivative of x of order q.

In Chapter 3, we investigate the existence of solutions for the following system of nonlinear fractional differential equations:

$$\begin{cases} {}^{RL}D^{\alpha}x(t) = f(t, x(t), y(t)), & t \in I = (0, b], \\ {}^{RL}D^{\alpha}y(t) = g(t, y(t), x(t)), & t \in I = (0, b], \\ t^{1-\alpha}x(t)\big|_{t=0} = u_0, \quad t^{1-\alpha}y(t)\big|_{t=0} = v_0. \end{cases}$$

where, $0 < \alpha \leq 1$, ${}^{RL}D^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order α, $J = [0, b]$, $b > 0$, $u_0, v_0 \in \mathbb{R}$, $u_0 \leq v_0$ and $f, g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

In Chapter 4, we investigate the existence of extremal solutions for the following coupled systems of nonlinear fractional differential equations involving the ψ-Caputo derivative with initial conditions:

$$\begin{cases} {}^{C}D_{a+}^{\alpha;\psi}x(t) = h_1(t, x(t), y(t)), & t \in J = [a, b], \\ {}^{C}D_{a+}^{\alpha;\psi}y(t) = h_2(t, y(t), x(t)), & t \in J = [a, b], \\ x(a) = x_a, \ y(a) = y_a. \end{cases}$$

where $h_1, h_2 \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $x_a, y_a \in \mathbb{R}$, $x_a \leq y_a$ and ${}^{C}D_{a+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivative of order $0 < \alpha \leq 1$.

Existence results for these problems are obtained by using the monotone iterative technique combined with the method of upper and lower solutions, as presented respectively in the following articles [43, 42, 21].

Chapter 1

Preliminaries

We present in this Chapter some notations and definitions of Fractional Calculus Theory and some fixed point theorems.

Let $C(\mathcal{J},\mathbb{R})$ be the Banach space of continuous functions from $\mathcal{J}=[a,b]$, $a,b\in\mathbb{R}$ into $\mathbb{R}$ with the norm

$$\|y\| = \sup\{|y(t)| : t\in\mathcal{J}\}.$$

$C_{1-\alpha}(\mathcal{J},\mathbb{R}) = \{y\in C((a,b],\mathbb{R});\ t^{1-\alpha}y\in C(\mathcal{J},\mathbb{R}),\ \text{with } 0<\alpha<1\}$, is a Banach space with the norm

$$\|y\|_{C_{1-\alpha}} = \sup\{t^{1-\alpha}|y(t)| : t\in\mathcal{J}\}.$$

1.1 Elements of Functional Analysis

Definition 1.1.1 *[36]. Let E, F be Banach spaces and $T: E\to F$.*

(i) The operator T is said to be bounded if it maps any bounded subset of E into a bounded subset of F.

(ii) The operator T is called compact if $T(E)$ is relatively compact (i.e., $\overline{T(E)}$ is compact).

(iii) The operator T is said to be completely continuous if it is continuous and maps any bounded subset of E into a relatively compact subset of F.

Theorem 1.1.2 (*Arzela-Ascoli Theorem [35]*). *A subset $\mathcal{F}$ of $C([a,b],\mathbb{R})$ is relatively compact (i.e. $\overline{\mathcal{F}}$ is compact) if and only if the following conditions hold:*

1. $\mathcal{F}$ is uniformly bounded i.e, there exists $M>0$ such that

$$\|f(t)\| < M \textit{ for each } t\in[a,b] \textit{ and each } f\in\mathcal{F}.$$

2. $\mathcal{F}$ *is equicontinuous i.e, for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for each* $t_1, t_2 \in [a, b]$, $|t_2 - t_1| \leqslant \delta$ *implies* $\|f(t_2) - f(t_1)\| \leqslant \varepsilon$, *for every* $f \in \mathcal{F}$.

Theorem 1.1.3 (*Arzela-Ascoli Theorem [15](Sequential Version)*). *If* $\{f_n(t)\}$ *is a uniformly bounded and equicontinuous sequence of real functions on an interval* $[a, b]$, *then there is a subsequence which converges uniformly on* $[a, b]$ *to a continuous function.*

Theorem 1.1.4 (*Banach's fixed point theorem [26]*) *Let* C *be a non-empty closed subset of a Banach space* X, *then any contraction mapping* T *of* C *into itself has a unique fixed point.*

Theorem 1.1.5 (*Lebesgue dominated convergence theorem [25]*). *Suppose* $f_n : \mathbb{R} \to [-\infty, +\infty]$ *are (Lebesgue) measurable functions such that*

a. $\lim\limits_{n \to +\infty} f_n(x) = f(x)$.

b. *There is an integrable* $g : \mathbb{R} \to [0, +\infty]$ *with* —$f_n(x)| \leq g(x)$, *for each*$x \in \mathbb{R}$.

Then f *is integrable as is* f_n *for each* n, *and*

$$\lim_{n \to +\infty} \int_{\mathbb{R}} f_n d_\mu = \int_{\mathbb{R}} \lim_{n \to +\infty} f_n d_\mu = \int_{\mathbb{R}} f d_\mu.$$

1.2 Fractional Calculus.

In this section, we introduce some necessary definitions and properties of the fractional calculus which are used in this report and can be found in [1, 27, 28].

Definition 1.2.1 *[29]* (*The Euler gamma function*)
The gamma function Γ *is defined by the following integral:*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathfrak{Re}(z) > 0.$$

This integral is convergent for all $z \in \mathbb{C}$ *with* $\mathfrak{Re}(z) > 0$.

Let us recall some properties of the gamma function:

1. $\Gamma(z + 1) = z\Gamma(z), \ \mathfrak{Re}(z) > 0.$
2. $\Gamma(z)\Gamma(1 - z) = \dfrac{\pi}{\sin(\pi z)}, \ z \in \mathbb{C} \backslash \mathbb{Z}$ and $\mathfrak{Re}(z) > 0$.
3. $\Gamma(1) = 1, \ \Gamma(2) = 1, \ \Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \sqrt{\pi}/2$.

Definition 1.2.2 *[29](Riemann-Liouville fractional integrals) The Riemann-Liouville fractional integral $I^{\alpha}_{a^+}f$ of order $\alpha > 0$ is defined for function $f : [a, b] \to \mathbb{R}$ by*

$$I^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds,$$

where Γ is the gamma function. For $a = 0$ we put $I^{\alpha}_0 f(t) = I^{\alpha}f(t)$.

Definition 1.2.3 *[29](Riemann-Liouville fractional Derivatives). The Riemann-Liouville fractional derivatives ${}^{RL}D^{\alpha}_{a^+}f$ of order $\alpha \in (n-1, n]$ is defined by*

$${}^{RL}D^{\alpha}_{a^+}f(t) = \left(\frac{d}{dt}\right)^n I^{n-\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}f(s)ds,$$

Here $n = [\alpha] + 1$. If $\alpha \in (0, 1]$, then

$${}^{RL}D^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-s)^{-\alpha}f(s)ds.$$

Definition 1.2.4 *[29](The Caputo fractional Derivatives) The Caputo fractional derivative of order $\alpha \in (n-1, n]$, of function f is given by*

$${}^{C}D^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\left(\frac{d}{dt}\right)^n f(s)ds.$$

Here $n = [\alpha] + 1$. If $\alpha \in (0, 1]$, then

$${}^{C}D^{\alpha}_{a^+}h(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}f'(s)ds.$$

For $a = 0$ we put ${}^{C}D^{\alpha}_{0^+}f(t) = {}^{C}D^{\alpha}f(t)$.

Caputo fractional order derivative of certain functions as follow:

1. ${}^{C}D^{\alpha}\lambda = 0, \quad \lambda \in \mathbb{R}$.
2. ${}^{C}D^{\alpha}(t^{\beta}) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, \quad \beta > 0$.
3. ${}^{C}D^{\alpha}(e^{\beta t}) = \beta^{\alpha}e^{\beta t}, \quad \beta > 0$.

Lemma 1.2.5 *[29] If $h \in C([a, b], \mathbb{R})$ and $0 < \alpha < 1$, then*

$$I^{\alpha}_{a^+}\,{}^{C}D^{\alpha}_{a^+}h(t) = h(t) - h(a).$$

Next, we present the defintions and some properties of Mittag-Leffler functions (see [19, 24, 38, 41]).

Definition 1.2.6 *[29] The classical Mittag-Leffler function is defined by*

$$E_{\alpha,1}(z) = E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad \alpha, z \in \mathbb{C} \text{ and } \mathfrak{Re}(\alpha) > 0.$$

The generalized Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha, \beta, z \in \mathbb{C}, \ \mathfrak{Re}(\alpha) > 0 \text{ and } \mathfrak{Re}(\beta) > 0.$$

In particular,

$$E_{\alpha,0}(z) = zE_{\alpha,\alpha}(z); \ E_{\alpha,1}(0) = E_\alpha(0) = 1; \ E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}.$$

Now, we present the defintions and some properties of ψ-Riemann-Liouville fractional integrals and ψ-Caputo fractional derivatives, (see [4, 6, 29]).

Definition 1.2.7 *[4](ψ-Riemann-Liouville fractional integrals)For $\alpha > 0$, the left-sided ψ-Riemann-Liouville fractional integral of order α of an integrable function $f : [a, b] \to \mathbb{R}$ with respect to the increasing differentiable function $\psi : [a, b] \to \mathbb{R}$ with $\psi^{'}(t) \neq 0$ for all $t \in J = [a, b]$ is defined as*

$$I_{a+}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds.$$

Definition 1.2.8 *[4](ψ-Caputo fractional derivatives) Let $n \in \mathbb{N}$ and let $\psi, f \in C^n(J, \mathbb{R})$ be two functions such that ψ is increasing with $\psi^{'}(t) \neq 0$ for all $t \in J = [a, b]$. The left-sided ψ-Caputo fractional derivative of a function f of order $\alpha > 0$ is defined by:*

$${}^C D_{a+}^{\alpha;\psi} f(t) = I_{a+}^{n-\alpha;\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n f(t).$$

where $n = [\alpha] + 1$ for $\alpha \in \mathbb{R} \backslash \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$.
From the above definition, we have:

$${}^C D_{a+}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^n f(s) ds, \quad if \ \alpha \in \mathbb{R} \backslash \mathbb{N},$$

$${}^C D_{a+}^{\alpha;\psi} f(t) = \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^n f(s), \quad if \ \alpha \in \mathbb{N}.$$

Remark 1.2.9 *Note that, if we take $\psi(t) = t$, then we have*

$$I_{a+}^{\alpha;\psi} f(t) = I_{a+}^{\alpha} f(t), \quad and \quad {}^C D_{a+}^{\alpha;\psi} f(t) = {}^C D_{a+}^{\alpha} f(t).$$

Proposition 1.2.10 *[6, 29] Let $\alpha, \beta > 0$, $f : J = [a, b] \to \mathbb{R}$ be continuous function. Then for all $t \in J$ we have,*

1). $I_{a+}^{\alpha,\psi} I_{a+}^{\beta,\psi} f(t) = I_{a+}^{\alpha+\beta,\psi} f(t)$.

2). ${}^{C}D_{a+}^{\alpha,\psi} I_{a+}^{\alpha,\psi} f(t) = f(t)$.

3). ${}^{C}D_{a+}^{\alpha,\psi} (\psi(t) - \psi(a)) = 0,$.

Lemma 1.2.11 *[6] If $f \in C^1([a, b], \mathbb{R})$ and $0 < \alpha < 1$, then*

$$I_{a+}^{\alpha,\psi C} D_{a+}^{\alpha,\psi} f(t) = f(t) - f(a).$$

Chapter 2

Existence of solutions for nonlinear fractional differential equations with integral boundary conditions

In this chapter, we mainly investigate the existence of extremal solutions of boundary value problem for the following nonlinear fractional differential equation with integral boundary conditions:

$$\begin{cases} {}^{RL}D^q x(t) = f(t, x(t)), & t \in [0, T], \ T > 0, \\ x(0) = \lambda \displaystyle\int_0^T x(s)ds + d, & d \in \mathbb{R}. \end{cases} \tag{2.1}$$

where $0 < q < 1$, $\lambda \geq 0$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^{RL}D^\alpha x$ denotes the Riemann-Liouville fractional derivative of x of order q.
The existence of solutions for (2.1) is proved by using the monotone iterative technique and the method of coupled upper and lower solution. The original results of this chapter are found in [43].

In [45], Wang and Xie, developed monotone iterative method for the following fractional differential equations with integral boundary conditions with Hölder continuity and obtained existence and uniqueness of solution of the problem

$$\begin{cases} {}^{RL}D^q x(t) = f(t, x(t)), & t \in [0, T], \ T > 0, \\ x(0) = \lambda \displaystyle\int_0^T x(s)ds + d, & d \in \mathbb{R}. \end{cases}$$

where $0 < q < 1$, λ, is 1 or -1, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^{RL}D^\alpha x$ denotes the Riemann-Liouville fractional derivative of x of order q.

D. Dhaigude et al. in [23], developed monotone iterative technique by introducing upper and lower solutions to Riemann-Liouville fractional differential equations with deviating arguments and integral boundary conditions:

$$\begin{cases} {}^{RL}D^q x(t) = f(t, x(t), x(\theta(t))), & t \in I = [0,T],\ T > 0, \\ x(0) = \lambda \displaystyle\int_0^T x(s)ds + d, \quad d \in \mathbb{R}. \end{cases}$$

where $0 < q < 1$, $\lambda \geq 0$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\theta : I \to I$ are a continuous functions and $\theta(t) \leq t, t \in I$.

G. Wang in [44], studied the existence of solutions to the following boundary value problems for fractional differential equations with nonlinear boundary conditions and deviating arguments:

$$\begin{cases} {}^{c}D^q y(t) = f(t, y(t), x(\theta(t))), & t \in I = [0,T], \\ g(\widehat{y}(0), \widehat{y}(T)) = 0, \end{cases}$$

where $0 < \alpha \leq 1$, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\theta : I \to I$ are a a continuous functions, $\widehat{y}(0) = t^{1-q}y(t)\Big|_{t=0}$, $\widehat{y}(T) = t^{1-q}y(t)\Big|_{t=T}$ and ${}^{RL}D^q x$ denotes the Riemann-Liouville fractional derivative of x of order q.

2.1 Linear fractional differential equations

In this section, we study the expression of the solutions of a linear fractional differential equation involving integral boundary problem:

$$\begin{cases} {}^{RL}D^q x(t) = g(t), & t \in I = [0,T],\ T > 0, \\ x(0) = \lambda \displaystyle\int_0^T x(s)ds + d, \end{cases} \tag{2.2}$$

where $g \in C(I, \mathbb{R})$, $0 < q < 1$ and $\lambda \geq 0$.

We introduce the following spaces:

$$C_{1-\alpha}(I, \mathbb{R}) = \{f \in C((0,T], \mathbb{R});\ t^{1-\alpha} f \in C(I, \mathbb{R}),\ \text{with } 0 < \alpha < 1\}$$
$$C^1(I, \mathbb{R}) = \{f : I \to \mathbb{R},\ \text{ is differentiable on } I \text{ and } f' \in C(I, \mathbb{R})\}.$$

Lemma 2.1.1 *Let $0 < q < 1$, $\lambda \geq 0$, $d \in \mathbb{R}$ and $g \in C(I, \mathbb{R})$.*
A function $x \in C^1(I, \mathbb{R})$ is a solution of the problem (2.2) if and only if x is a solution of the following integral equation:

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s)ds + \lambda \int_0^T x(s)ds + d. \tag{2.3}$$

Proof. Assume that $x \in C^1(I, \mathbb{R})$ satisfies the integral equation (2.3). Applying the Riemann-Liouville operator ${}^{RL}D^q$ to both sides of the integral equation (2.3), we have

$$\begin{aligned}
{}^{RL}D^q x(t) &= {}^{RL}D^q \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds + \lambda \int_0^T x(s) ds + d \right), \\
&= {}^{RL}D^q \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds + x(0) \right) \\
&= {}^{RL}D^q \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds \right) + {}^{RL}D^q x(0), \\
&= {}^{RL}D^q I^q g(t) + {}^{RL}D^q x(0), \\
&= g(t) + {}^{RL}D^q x(0).
\end{aligned}$$

On the other hand by, we have

$${}^{RL}D^q x(t) = \frac{d}{dt} \left(I^{1-q} x(t) \right) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} x(s) ds.$$

So,

$${}^{RL}D^q x(0) = \frac{1}{\Gamma(1-q)} .0 = 0.$$

Finally we have

$${}^{RL}D^q x(t) = g(t).$$

In addition, we have $x(0) = \lambda \int_0^T x(s) ds + d$ from the integral equation (2.3). Consequently x is solution of problem (2.2).

Conversely, Assume that x satisfies the problem (2.2). if ${}^{RL}D^q x(t) = g(t)$ then $I^q({}^{RL}D^q x(t)) = I^q g(t)$. So we obtain

$$\begin{aligned}
I^q({}^{RL}D^q x(t)) &= x(t) - \left[{}^{RL}D^{q-1} x(t) \right]_{t=0} \frac{(t-0)^{q-1}}{\Gamma(q)} \\
&= x(t) - x(0) \frac{(t-0)^{1-q}}{\Gamma(2-q)} \frac{(t-0)^{q-1}}{\Gamma(q)} \\
&= x(t) - x(0).
\end{aligned}$$

Then,

$$\begin{aligned}
x(t)) &= I^q({}^{RL}D^q x(t)) + x(0) \\
&= I^q g(t) + \lambda \int_0^T x(s) ds + d
\end{aligned}$$

$$= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s)ds + \lambda\int_0^T x(s)ds + d.$$

□

Lemma 2.1.2 *Let $0 < q < 1$, $\lambda \geq 0$, $d \in \mathbb{R}$ and $g \in C(I,\mathbb{R})$. If $\lambda < \frac{1}{T}$, then (2.2) has a unique solution $x \in C(I,\mathbb{R})$.*

Proof. Define an operator $\mathcal{A} : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ par

$$\mathcal{A}(x)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s)ds + \lambda\int_0^T x(s)ds + d, \tag{2.4}$$

For any $x, y \in C(I,\mathbb{R})$, we have

$$\begin{aligned} |\mathcal{A}(x(t)) - \mathcal{A}(y(t))| &= \lambda\int_0^T (x(s)-y(s))ds, \\ &\leq \lambda\int_0^T |x(s)-y(s)|ds \\ &\leq \lambda T|x(s)-y(s)| < |x(s)-y(s)|. \end{aligned}$$

Therefore, $\|\mathcal{A}x - \mathcal{A}y\| < \|x-y\|$, we know that $\mathcal{A}$ is a contraction operator on $C(I,R)$. Consequently, by the Banach fixed point theorem, the operator $\mathcal{A}$ has a unique fixed point x, i.e. $x(t)$ is a unique solution of (2.2). □

Lemma 2.1.3 *Let $0 < q < 1$ and $m \in C_{1-\alpha}(I,\mathbb{R})$. Suppose that for any $t \in (0,T]$, we have $m(t_1) = 0$ and $m(t) \leq 0$ for $0 \leq t \leq t_1$. Then it follows that*

$${}^{RL}D^q m(t_1) \geq 0.$$

Proof. Consider $m \in C_{1-\alpha}(I,\mathbb{R})$, such that $m(t_1) = 0$ and $m(t) \leq 0$ for $0 \leq t \leq t_1$. Then, $m(t)$ is continuous on $(0,T]$ and $t^{1-q}m(t)$ is continuous on $I = [0,T]$.
Since $m(t)$ is continuous on $(0,T]$, given any t_1 such that $0 < t_1 < T$, there exists a $k(t_1) > 0$ and $h > 0$ such that

$$-k(t_1)(t_1-s) \leq m(t) - m(s) \leq k(t_1)(t_1-s), \ \ for \ 0 < t_1 - h \leq s \leq t_1 + h < T.$$

We have ${}^{RL}D^q m(t) = \dfrac{1}{\Gamma(1-q)}\dfrac{d}{dt}\int_0^t (t-s)^{-q}m(s)ds$, set $H(t) = \int_0^t (t-s)^{-q}m(s)ds$ and consider

$$H(t_1) - H(t_1-h) = \int_0^{t_1-h} [(t_1-s)^{-q} - (t_1-h-s)^{-q}]m(s)ds + \int_{t_1-h}^{t_1} (t_1-s)^{-q}m(s)ds.$$

Let $I_1 = \int_0^{t_1-h}[(t_1-s)^{-q} - (t_1-h-s)^{-q}]m(s)ds$ and $I_2 = \int_{t_1-h}^{t_1}(t_1-s)^{-q}m(s)ds$.
Since $t_1 - s > t_1 - h - s$ and $-q < 0$, we have $(t_1-s)^{-q} < (t_1-h-s)^{-q}$.
This, coupled with the fact that $m(t) \leq 0,\ 0 < t < t_1$, implies that $I_1 \geq 0$. Now, consider $I_2 = \int_{t_1-h}^{t_1}(t_1-s)^{-q}m(s)ds$. Using (2.5) and the fact that $m(t_1) = 0$, for $s \in]t_1-h, t_1+h[$, we obtain,

$$m(s) \geq -k(t_1)(t_1-s), \quad and \quad I_2 \geq k(t_1)\int_{t_1-h}^{t_1}(t_1-s)^{1-q}ds = -k(t_1)\frac{h^{2-q}}{2-q}.$$

Thus, we have

$$H(t_1) - H(t_1 - h \geq -\frac{k(t_1)h^{2-q}}{2-q}.$$

Then dividing through by h and taking limits as $h \to 0$, we have

$$\lim_{h\to 0}\left[\frac{H(t_1)-H(t_1-h)}{h} + \frac{k(t_1)h^{2-q}}{h(2-q)}\right] \geq 0.$$

Since $0 < 1-q < 1$ and $1 < 2-q < 2$, we conclude that $\frac{d}{dt}H(t_1) \geq 0$, which implies that ${}^{RL}D^q m(t_1) \geq 0$. □

Lemma 2.1.4 *(Comparison result) Let* $0 < q < 1$, $M \in C(I, \mathbb{R}^+)$ *and* $\mathcal{M} = \sup_{t\in I} M(t)$. *Suppose that* $p \in C_{1-\alpha}(I, \mathbb{R})$ *satisfies*

$$\begin{cases} {}^{RL}D^q p(t) \geq -M(t)p(t), \\ t^{1-q}p(t)\big|_{t=0} \geq 0. \end{cases} \tag{2.5}$$

If $\mathcal{M}T^q\Gamma(1-q) < 1$, *then,* $p(t) \geq 0$, $\forall\ t \in I$.

Proof. We put $p_\theta(t) = p(t) + \theta$ with $\theta > 0,\ t \in I$. Then

$$\begin{aligned}
{}^{RL}D^q p_\theta(t) &= {}^{RL}D^q p(t) + {}^{RL}D^q\theta, \\
&\geq -M(t)p(t) + \frac{\theta t^{-q}}{\Gamma(1-q)} \\
&\geq -M(t)p_\theta(t) - M(t)\theta + \frac{\theta}{t^q\Gamma(1-q)} \\
&\geq -M(t)p_\theta(t) - \mathcal{M}\theta + \frac{\theta}{T^q\Gamma(1-q)} \\
&= -M(t)p_\theta(t) + \theta\frac{1-\mathcal{M}T^q\Gamma(1-q)}{T^q\Gamma(1-q)} \\
&> -M(t)p_\theta(t),
\end{aligned}$$

and

$$t^{1-q}p_\theta(t)\big|_{t=0} = t^{1-q}p(t)\big|_{t=0} + t^{1-q}\theta\big|_{t=0} > 0.$$

Next, we prove that $p_\theta(t) > 0$ on I. Assume that $p_\theta(t) > 0$ is not true. Then, by $t^{1-q}p_\theta\big|_{t=0} > 0$, it follows that there exists a $t_1 \in (0, T]$ such that $p_\theta(t_1) = 0$ and $p_\theta(t) > 0$, $0 < t < t_1$. Let $m(t) = -p_\theta(t)$, by Lemma 2.1.3, we have ${}^{RL}D^q m(t_1) \geq 0$. So, we have ${}^{RL}D^q p_\theta(t_1) \leq 0.$, i.e. $-M(t_1)p_\theta(t_1) < 0$, which implies $p_\theta(t_1) > 0$. It is a contradiction. So we have $p_\theta(t) > 0$, $t \in I = [0, T]$ is true. That is $p(t) + \theta > 0$, $t \in I$. By the arbitrariness of θ, we can get $p(t) \geq 0$, $t \in I$. □

2.2 Main Result

In this section, we prove the existence of extremal solutions for problem (2.1). Let us defining what we mean by a solution of this problem.

Definition 2.2.1 *A solution of problem* (2.1) *will be a function* $x \in C^1(I = [0, T], \mathbb{R})$ *for which* (2.1) *is satisfied.*

Next, we introduce the concept of coupled lower and upper solutions of this problem as follows.

Definition 2.2.2 *We say that* $(x_0, y_0) \in (C^1(I, \mathbb{R}))^2$ *is a pair of coupled lower and upper solutions of the problem* (2.1), *respectively, if* $x_0(t) \leq y_0(t)$ *for all* $t \in I$ *and the following inequalities hold:*

$$\begin{cases} {}^{RL}D^q x_0(t) \leq f(t, x_0(t)), \quad t \in I, \quad x_0(0) \leq \lambda \displaystyle\int_0^T x_0(s)ds + d, \\ {}^{RL}D^q y_0(t) \geq f(t, y_0(t)), \quad t \in I, \quad y_0(0) \geq \lambda \displaystyle\int_0^T y_0(s)ds + d. \end{cases} \tag{2.6}$$

We define the sector:

$$\mathfrak{D} = [x_0, y_0] = \{x \in C^1(I, \mathbb{R}) : \ x_0(t) \leq x(t) \leq y_0(t), \ t \in I = [0, T]\}.$$

We assume the following hypothesis:

(G_1) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous function.

(G_2) There exists $(x_0, y_0) \in (C^1(I, \mathbb{R}))^2$ a pair of coupled lower and upper solutions of (2.1), with $x_0(t) \leq y_0(t)$ for $t \in I$.

(G_3) There exist $M > 0$ with $\mathcal{M}T^q\Gamma(1 - q) < 1$ such that

$$f(t, x) - f(t, y) \leq M(x - y), \tag{2.7}$$

where $x_0(t) \leq x \leq y \leq y_0(t)$, for all $t \in I$.

Now, We have the following results.

Theorem 2.2.3 *Assume that* (G_1)*,* (G_2) *and* (G_3) *hold. If*

$$\begin{cases} {}^{RL}D^q x(t) = f(t, x_0(t)) - M\left(x(t) - x_0(t)\right), \ t \in I, \quad x(0) = \lambda \displaystyle\int_0^T x_0(s)ds + d, \\ {}^{RL}D^q y(t) = f(t, y_0(t)) - M\left(y(t) - y_0(t)\right), \ t \in I, \quad y(0) = \lambda \displaystyle\int_0^T y_0(s)ds + d. \end{cases} \tag{2.8}$$

Then,

$$x_0(t) \leq x(t) \leq y(t) \leq y_0(t), \quad t \in I,$$

with x, y *are lower and upper solutions of problem* (2.1)*, respectively.*

Proof. Note that there exist unique solutions (x, y) to the boundary value problem (2.1).
Let $p = x - x_0$ and $\varphi = y_0 - y$, we have

$$\begin{cases} {}^{RL}D^q p(t) & ={}^{RL} D^q x(t) -{}^{RL} D^q x_0(t), \\ & \geq f(t, x_0(t)) - M\left(x(t) - x_0(t)\right) - f(t, x_0(t)) = -Mp(t), \ t \in I, \\ p(0) & \geq \lambda \displaystyle\int_0^T x_0(s)ds - \lambda \int_0^T x_0(s) = 0, \end{cases}$$

i.e.,

$$\begin{cases} {}^{RL}D^q p(t) \geq -Mp(t), \ t \in I, \\ p(0) \geq 0. \end{cases}$$

And

$$\begin{cases} {}^{RL}D^q \varphi(t) & ={}^{RL} D^q y_0(t) -{}^{RL} D^q y(t), \\ & \geq f(t, y_0(t)) - M\left(y(t) - y_0(t)\right) - f(t, y_0(t)) = -M\varphi(t), \ t \in I, \\ \varphi(0) & \geq \lambda \displaystyle\int_0^T y_0(s)ds - \lambda \int_0^T y_0(s) = 0, \end{cases}$$

i.e.,

$$\begin{cases} {}^{RL}D^q \varphi(t) \geq -M\varphi(t), \ t \in I, \\ \varphi(0) \geq 0. \end{cases}$$

Therefore, by Lemma 2.1.4, we have $p(t) \geq 0, \varphi(t) \geq 0$, $t \in I$, then $x(t) \geq x_0(t)$, $y_0(t) \geq y(t)$, $t \in I$.

Now let $m = y - x$. Assumption (G_2) yields

$$\begin{cases} {}^{RL}D^q m(t) &= {}^{RL}D^q y(t) - {}^{RL}D^q x(t) \\ &= f(t, y_0(t)) - M(y(t) - y_0(t)) - f(t, x_0(t)) - M(x(t) - x_0(t)) \\ &= f(t, y_0(t)) - f(t, x_0(t)) - M(y(t) - y_0(t) - x(t) + x_0(t)) \\ &\geq -M(y_0(t) - x_0(t)) + M(y_0(t) - x_0(t)) - M(y(t) - x(t)) = -Mm(t), \quad t \in I. \\ m(0) &= \lambda \int_0^T (y_0(s) - x_0(s)) \geq 0, \end{cases}$$

i.e.,

$$\begin{cases} {}^{RL}D^q m(t) \geq -Mm(t), \ t \in I, \\ m(0) \geq 0. \end{cases}$$

Hence $m(t) \geq 0,$, $t \in I$, then $y(t) \geq x(t), t \in I$. So $x_0(t) \leq x(t) \leq y(t) \leq y_0(t), \quad t \in I$.

Now, we need to show that y, x are upper and lower solutions of problem (2.1), respectively.
Using Assumption (G_2), we have

$$\begin{cases} {}^{RL}D^q y(t) &= f(t, y_0(t)) - M(y(t) - y_0(t)), \\ &= f(t, y_0(t)) - M(y(t) - y_0(t)) - f(t, y(t)) + f(t, y(t)) \\ &\geq f(t, y_0(t)) - M(y(t) - y_0(t)) + M(y(t) - y_0(t)) = f(t, y(t)). \\ y(0) &= \lambda \int_0^T y_0(s)ds + d \geq \lambda \int_0^T y(s)ds + d, \end{cases}$$

i.e.,

$$\begin{cases} {}^{RL}D^q y(t) \geq f(t, y(t)), \ t \in I, \\ y(0) \geq \lambda \int_0^T y(s)ds + d. \end{cases}$$

Similarly, we have

$$\begin{cases} {}^{RL}D^q x(t) &= f(t, x_0(t)) - M\left(x(t) - x_0(t)\right), \\ &= f(t, x_0(t)) - M\left(x(t) - x_0(t)\right) - f(t, x(t)) + f(t, x(t)) \\ &\leq f(t, x(t)) - M(x(t) - x_0(t)) + M(x(t) - x_0(t)) = f(t, x(t)). \\ x(0) &= \lambda \int_0^T x_0(s)ds + d \leq \lambda \int_0^T x(s)ds + d, \end{cases}$$

i.e.,

$$\begin{cases} {}^{RL}D^q x(t) \leq f(t,x(t)),\ t \in I, \\ x(0) \leq \lambda \displaystyle\int_0^T x(s)ds + d. \end{cases}$$

So, y, x are upper and lower solutions of (2.1), respectively. □

Now we give the main result on the existence of extremal solutions for the nonlinear problem (2.1).

Theorem 2.2.4 *Assume that* (G_1), (G_2) *and* (G_3) *hold. Then there exist monotone iterative sequences* $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subset C(I,\mathbb{R})$ *converging uniformly to* x^*, y^*, *respectively, (i.e.,* $\lim_{n\to\infty} x_n = x^*$, $\lim_{n\to\infty} y_n = y^*$*), and* x^*, y^* *are the extremal solutions of problem* (2.1) *in the sector* $\mathfrak{D} = [x_0, y_0]$, *such that*

$$x_0 \leq ... \leq x_n \leq ... \leq y_n \leq ... \leq y_0, \quad on\ I\ for\ all\ n \in \mathbb{N}.$$

Proof. For all $x_n, y_n \in C(I,\mathbb{R})$, let

$$\begin{cases} {}^{RL}D^q x_{n+1}(t) = f(t,x_n(t)) - M\left(x_{n+1}(t) - x_n(t)\right),\ t \in I, \quad x_{n+1}(0) = \lambda \int_0^T x_n(s)ds + d. \\ {}^{RL}D^q y_{n+1}(t) = f(t,y_n(t)) - M\left(y_{n+1}(t) - y_n(t)\right),\ t \in I, \quad y_{n+1}(0) = \lambda \int_0^T y_n(s)ds + d, \end{cases} \tag{2.9}$$

obviously, by Theorem 2.2.3, we have that $x_0 \leq x_1 \leq y_1 \leq y_0$, on I, for all $n \in \mathbb{N}$., and y_1, x_1 are upper and lower solutions of (2.1), respectively.

Assume that

$$x_0 \leq x_1 \leq ... \leq x_k \leq x_{k+1} \leq y_{k+1} \leq y_k \leq ... \leq y_1 \leq y_0,$$

for some $k \geq 1$ and let x_k, y_k be lower and upper solutions of (2.1), respectively. Then, using again Theorem 2.2.3, we get $x_k(t) \leq x_{k+1}(t) \leq y_{k+1}(t) \leq y_k(t)$, $t \in I$. By induction, we have that

$$x_0(t) \leq x_1(t) \leq ... \leq x_n(t) \leq y_n(t) \leq ... \leq y_1(t) \leq y_0(t),\ t \in I.$$

Obviously, the sequences x_n, y_n are uniformly bounded and equicontinuous, applying the standard arguments, we have

$$\lim_{n\to+\infty} x_n = x^*(t); \quad \lim_{n\to+\infty} y_n = y^*(t);$$

uniformly on I. indeed, x^* and y^* are extremal generalized solutions of (2.1). To prove that x^* and y^* are extremal generalized solutions of (2.1), Assume that for some k, $x_k(t) \leq w(t) \leq y_k(t); t \in I$. Put $p = w - x_{k+1}, \varphi = y_{k+1} - w$. Then

$$\begin{cases} {}^{RL}D^q p(t) & = f(t,w(t)) - f(t,x_k(t)) + M\left(x_{k+1}(t) - x_k(t)\right), \\ & \geq -M(w(t)-x_k(t)) - M(x_k(t)) - x_{k+1}(t) = -Mp(t), \\ p(0) & \geq \lambda \int_0^T (w(s)-x_k(s))ds \geq 0, \end{cases}$$

and

$$\begin{cases} {}^{RL}D^q \varphi(t) & = f(t,y_k(t)) - f(t,w(t)) - M\left(y_{k+1}(t) - y_k(t)\right), \\ & \geq -M(y_k(t)-w(t)) - M(y_{k+1}(t) - y_k(t)) = -M\varphi(t), \\ \varphi(0) & \geq \lambda \int_0^T (y_k(t)-w(t))ds \geq 0. \end{cases}$$

By, Lemma 2.1.4, we have $x_{k+1}(t) \leq w(t) \leq y_{k+1}(t); t \in I$. It proves, by induction, that

$$x_n(t) \leq w(t) \leq y_n(t),\ t \in I,\ for\ all\ n \in \mathbb{N}.$$

Taking the limit $n \to +\infty$, we get $x^*(t) \leq w(t) \leq y^*(t),\quad t \in I.$ □

2.3 An example

To illustrate our main results, we present the following example.

Example 2.3.1 *Consider the boundary value problem of fractional differential equation:*

$$\begin{cases} {}^{RL}D^{1/2}x(t) = e^{t\sin^2 x(t)}, \qquad t \in I = [0, \ln 2], \\ x(0) = \frac{1}{3}\int_0^{\ln 2} x(s)ds + \dfrac{1}{2}. \end{cases} \tag{2.10}$$

This problem is a particular case of problem (2.1), with $0 < q = 1/2 < 1$, $T = \ln 2$, $d = \frac{1}{2}$, $\lambda = \frac{1}{3}$ and $0 \leq f(t,x) = e^{t\sin^2 x(t)} \leq e^t$, $t \in I$. It is clear that f is continuous function. Take $x_0(t) = 0$ and $y_0(t) = e^t$ for $t \in I$, then

$$\begin{cases} {}^{RL}D^{1/2}x_0(t) = 0 \leq f(t,x_0(t)) = 1,\ t \in I, \quad x_0(0) = 0 \leq \frac{1}{3}\int_0^{\ln 2} x_0(s)ds + \dfrac{1}{2} = \dfrac{1}{2}, \\ {}^{RL}D^{1/2}y_0(t) = e^t \geq f(t,y_0(t)) = e^{t\sin^2(e^t)},\ t \in I, \quad y_0(0) = 1 \geq \frac{1}{3}\int_0^{\ln 2} y_0(s)ds + \dfrac{1}{2} = \dfrac{5}{6}, \end{cases}$$

So, $x_0(t) = 0$ and $y_0(t) = e^t$ for $t \in I$, are coupled lower and upper solutions of problem (2.10) with $x_0(t) = 0 \leq y_0(t) = e^t$, for $t \in [0,1]$, then assumptions (G_1) and (G_2)

holds.
Let $x, y \in \mathbb{R}$, *with* $x_0(t) = 0 \le x \le y \le y_0(t) = e^t$, *for all* $t \in I$. *then we have:*

$$\begin{aligned} f(t,x) - f(t,y) &= e^{t\sin^2(x)} - e^{t\sin^2(y)} \le 0 \\ &\le \frac{1}{2\sqrt{\pi}}(y-x), \end{aligned}$$

Hence the assumption (G_3) *holds with* $M = \frac{1}{2\sqrt{\pi}} > 0$. *In addition, we have* $MT^q\Gamma(1-q) = \frac{1}{2\sqrt{\pi}}(\ln 2)^{1/2}\sqrt{\pi} \simeq 0.4162 < 1$. *By Theorem 2.2.4, the nonlinear problem* (2.10) *has the extremal solutions (coupled minimal and maximal solutions, respectively)* $(x^*, y^*) \in (\mathfrak{D})^2$ *with* $\mathfrak{D} = [0, e^t]$. *i.e.,* $0 \le x^* \le y^* \le e^t$, $t \in I = [0, \ln 2]$.

Chapter 3

Coupled systems of nonlinear Riemann-Liouville fractional differential equations

In this chapter, by using the monotone iterative technique combined with the method of upper and lower solutions, we investigate the existence of solutions for the following system of nonlinear fractional differential equations:

$$\begin{cases} {}^{RL}D^{\alpha}x(t) = f(t, x(t), y(t)), \quad t \in I = (0, b], \\ {}^{RL}D^{\alpha}y(t) = g(t, y(t), x(t)), \quad t \in I = (0, b], \\ t^{1-\alpha}x(t)\big|_{t=0} = u_0, \quad t^{1-\alpha}y(t)\big|_{t=0} = v_0. \end{cases} \tag{3.1}$$

where, $0 < \alpha \leq 1$, ${}^{RL}D^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order α, $J = [0, b]$, $b > 0$, $u_0, v_0 \in \mathbb{R}$, $u_0 \leq v_0$ and $f, g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. The original results of this chapter are found in [42].

S. Liu. in [30], studied the existence of extremal iteration solution to the following coupled system of nonlinear conformable fractional differential equations:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t), y(t)), \quad t \in [a, b], \\ y^{(\alpha)}(t) = g(t, y(t), x(t)), \quad t \in [a, b], \\ x(a) = \lambda_0, \ y(a) = \beta_0. \end{cases} \tag{3.2}$$

where $f, g \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \lambda_0, \beta_0 \in \mathbb{R}, \lambda_0 \leq \beta_0$, $x^{(\alpha)}$, $y^{(\alpha)}$ are the conformable fractional derivatives with $0 < \alpha \leq 1$.

3.1 Linear problems and comparison results

In this section, we enunciate the existence and uniqueness of solutions for initial linear fractional differential equations and for a linear problem for systems of fractional differential equations.

Consider the set $C_{1-\alpha}(J,\mathbb{R}) = \{f \in C(I,\mathbb{R});\ t^{1-\alpha} f \in C(J,\mathbb{R}),\ \text{with } 0<\alpha<1\}$.

Lemma 3.1.1 *Let $0<\alpha\le 1$, $M\in\mathbb{R}$ and $u_0\in\mathbb{R}$. If $g\in C_{1-\alpha}(J,\mathbb{R})$, then the linear initial value problem:*

$$\begin{cases} {}^{RL}D^{\alpha}x(t)+Mx(t)=g(t), & t\in I=]0,b], \\ t^{1-\alpha}x(t)\big|_{t=0}=u_0, \end{cases} \tag{3.3}$$

has a unique solution $x\in C_{1-\alpha}(J,\mathbb{R})$, and it is given by the following expression:

$$x(t)=\Gamma(\alpha)u_0t^{\alpha-1}E_{\alpha,\alpha}(-Mt^{\alpha})+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-M(t-s)^{\alpha})g(s)ds, \tag{3.4}$$

In particular, when $M=0$, the initial problem (3.3) has the solution

$$x(t)=u_0t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds.$$

where $E_{\alpha,\alpha}(.)$ is the Mittag-Leffler function.

Lemma 3.1.2 *Let $\alpha\in]0,1]$, $(u_0,\ v_0)\in\mathbb{R}^2$, $u_0\le\ v_0$, $M,N\in\mathbb{R}$, $N\ge 0$ and $g_1,g_2\in C_{1-\alpha}(J,\mathbb{R})$. Then the linear system*

$$\begin{cases} {}^{RL}D^{\alpha}x(t)=g_1(t)-M\ x(t)-N\ y(t),\ \text{for } t\in I=]0,b], \\ {}^{RL}D^{\alpha}y(t)=g_2(t)-M\ y(t)-N\ x(t),\ \text{for } t\in I=]0,b], \\ t^{1-\alpha}x(t)\big|_{t=0}=u_0, \quad t^{1-\alpha}y(t)\big|_{t=0}=v_0, \end{cases} \tag{3.5}$$

has a unique system of solutions $(x,y)\in C_{1-\alpha}(J,\mathbb{R})\times C_{1-\alpha}(J,\mathbb{R})$.

Proof. The pair $(x,y)\in C_{1-\alpha}(J,\mathbb{R})\times C_{1-\alpha}(J,\mathbb{R})$ is a solution to system (3.5) if and only if

$$x(t)=\frac{\theta(t)+\vartheta(t)}{2},\ \text{and } y(t)=\frac{\theta(t)-\vartheta(t)}{2}, t\in J.$$

where $\theta(t)$ and $\vartheta(t)$ are the solutions to the following problems:

$$\begin{cases} {}^{RL}D^{\alpha}\theta(t) = (g_1 + g_2)(t) - (M+N)\theta(t), \quad t \in I, \\ t^{1-\alpha}\theta(t)\big|_{t=0} = u_0 + v_0, \end{cases}$$
$$\begin{cases} {}^{RL}D^{\alpha}\vartheta(t) = (g_1 - g_2)(t) - (M-N)\vartheta(t), \quad t \in I, \\ t^{1-\alpha}\vartheta(t)\big|_{t=0} = u_0 - v_0, \end{cases}$$

By Lemma 3.1.1, we have a unique solution $(\theta, \vartheta) \in C_{1-\alpha}(J,\mathbb{R}) \times C_{1-\alpha}(J,\mathbb{R})$, with

$$\theta(t) = \Gamma(\alpha)\theta_0 t^{\alpha-1}E_{\alpha,\alpha}(-Kt^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-K(t-s)^{\alpha})(g_1+g_2)(s)ds,$$
$$\vartheta(t) = \Gamma(\alpha)\vartheta_0 t^{\alpha-1}E_{\alpha,\alpha}(-Lt^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-L(t-s)^{\alpha})(g_1-g_2)(s)ds.$$

where, $\theta_0 = u_0 + v_0$, $\vartheta_0 = u_0 - v_0$, $M + N = K$ and $M - N = L$. In consequence, $(x, y) \in C_{1-\alpha}(J,\mathbb{R}) \times C_{1-\alpha}(J,\mathbb{R})$, are unique too. The proof is finished. □

In the next Lemmas, we discuss comparison results for the linear problem (3.3) and for linear system (3.5).

Lemma 3.1.3 *(Comparison theorem 1) Let $0 < \alpha \le 1$ and $M \in \mathbb{R}$. If $\varphi \in C_{1-\alpha}(J,\mathbb{R})$ satisfy the relations,*

$$\begin{cases} {}^{RL}D^{\alpha}\varphi(t) + M\varphi(t) \ge 0, \quad t \in I \\ t^{1-\alpha}\varphi(t)\big|_{t=0} \ge 0, \end{cases}$$

Then $\varphi(t) \ge 0$ for all $t \in I$.

Proof. 1. The case $M < 0$. In [46, Lemme 2.1], it is shown that if $M \ge 0$, then $\varphi(t) \ge 0$ for all $t \in I$.
2. The case $M < 0$. We put ${}^{RL}D^{\alpha}\varphi(t) + M\varphi(t) = g(t)$ and $t^{1-\alpha}\varphi(t)\big|_{t=0} = u_0 \ge 0$. We are know that $g(t) \ge 0$, for every $t \in I$ and $\varphi \in C_{1-\alpha}(J,\mathbb{R})$ is a solution of the following problem:

$$\begin{cases} {}^{RL}D^{\alpha}\varphi(t) + M\varphi(t) = g(t) \ge 0, \quad t \in I, \\ t^{1-\alpha}\varphi(t)\big|_{t=0} = u_0 \ge 0. \end{cases} \tag{3.6}$$

By Lemma 3.1.1, the expression of $x(t)$ is:

$$\varphi(t) = \Gamma(\alpha)u_0 t^{\alpha-1}E_{\alpha,\alpha}(-Mt^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-M(t-s)^{\alpha})g(s)ds,$$

we can conclude that, $\varphi(t) \ge 0$ for every $t \in I$. □

Lemma 3.1.4 *(Comparison theorem 2). Let $0 < \alpha \leq 1$, $M, N \in \mathbb{R}$ and $N \geq 0$. Assume that $x, y \in C_{1-\alpha}(J, \mathbb{R})$ satisfy*

$$\begin{cases} {}^{RL}D^{\alpha}x(t) \geq -M\ x(t) + N\ y(t), & \textit{for } t \in I, \\ {}^{RL}D^{\alpha}y(t) \geq -M\ y(t) + N\ x(t), & \textit{for } t \in I, \\ t^{1-\alpha}x(t)\big|_{t=0} \geq 0, \quad t^{1-\alpha}y(t)\big|_{t=0} \geq 0. \end{cases} \tag{3.7}$$

Then $x(t) \geq 0$, $y(t) \geq 0$ for all $t \in I$.

Proof. Let $\psi(t) = x(t) + y(t)$, then, by (3.7) we have the following:

$$\begin{cases} {}^{RL}D^{\alpha}\psi(t) \geq -(M - N)\psi(t), & t \in I, \\ t^{1-\alpha}\psi(t)\big|_{t=0} \geq 0, \end{cases} \tag{3.8}$$

Thus, by (3.8) and Lemma 3.1.3, we know that

$$\psi(t) \geq 0, \ \text{ for all } t \in I, \qquad \text{i.e.,} \quad x(t) + y(t) \geq 0, \ \text{ for all } t \in I. \tag{3.9}$$

Next, we show that $x(t) \geq 0$, $y(t) \geq 0$ for all $t \in I$.
In fact, by (3.7) and (3.9), we have that

$$\begin{cases} {}^{RL}D^{\alpha}x(t) + (M + N)x(t) \geq 0, & t^{1-\alpha}x(t)\big|_{t=0} \geq 0 \ \text{ for } t \in I, \\ {}^{RL}D^{\alpha}y(t) + (M + N)y(t) \geq 0, & t^{1-\alpha}y(t)\big|_{t=0} \geq 0 \ \text{ for } t \in I. \end{cases} \tag{3.10}$$

By (3.10) and Lemma 3.1.3, we have $x(t) \geq 0$, $y(t) \geq 0$ for all $t \in I$. The proof is completed. □

3.2 Main Result

In this section, we prove the existence of extremal solutions of nonlinear system (3.1). Let us defining what we mean by a solution of this problem.

Definition 3.2.1 *A solution of problem* (3.1) *will be a pair $(x, y) \in C_{1-\alpha}(J, \mathbb{R}) \times C_{1-\alpha}(J, \mathbb{R})$ for which* (3.1) *is satisfied.*

Next, we introduce the concept of coupled lower and upper solutions of this problem as follows.

Definition 3.2.2 *We say that γ, $\delta \in C_{1-\alpha}(J, \mathbb{R})$ is a pair of coupled lower and upper solutions of the problem* (3.1)*, if $\gamma(t) \leq \delta(t)$ for all $t \in J$ and the following inequalities hold:*

$$\begin{cases} {}^{RL}D^{\alpha}\gamma(t) \leq f(t, \gamma(t), \delta(t)), \ \textit{for } t \in I, \ t^{1-\alpha}\gamma(t)\big|_{t=0} \leq u_0, \\ {}^{RL}D^{\alpha}\delta(t) \geq g(t, \delta(t), \gamma(t)), \quad \textit{for } \ t \in I, \ t^{1-\alpha}\delta(t)\big|_{t=0} \geq v_0. \end{cases} \tag{3.11}$$

We define the sector:

$$[\gamma,\delta]=\{x\in C_{1-\alpha}(J,\mathbb{R}):\ \gamma(t)\leq x(t)\leq\delta(t),\ t\in J=[0,b]\}.$$

We assume the following assumptions:

(C_1) $f,g:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions.

(C_2) There exists $\gamma,\delta\in C_{1-\alpha}(J,\mathbb{R})$, a pair of coupled lower and upper solutions of the problem (3.1).

(C_3) There exist constants $M,N\in\mathbb{R}$ and $N\geq 0$ such that

$$\begin{cases} f(t,x,y)-f(t,\overline{x},\overline{y})\geq -M(x-\overline{x})-N(y-\overline{y}),\\ g(t,x,y)-g(t,\overline{x},\overline{y})\geq -M(x-\overline{x})-N(y-\overline{y}),\end{cases}$$

where $\gamma(t)\leq\overline{x}\leq x\leq\delta(t)$, $\gamma(t)\leq y\leq\overline{y}\leq\delta(t)$ for all $t\in J$, and

$$g(t,y,x)-f(t,x,y)\geq -M(x-y)-N(y-x),$$

where $\gamma(t)\leq x\leq y\leq\delta(t)$ for all $t\in J$.

Now, we can obtain our main theorem.

Theorem 3.2.3 *Suppose that conditions* (C_1), (C_2) *and* (C_3) *hold. Then, there is* $(x^*,y^*)\in[\gamma,\delta]\times[\gamma,\delta]$ *an extremal solution of the nonlinear problem* (3.1). *Moreover, there exist monotone iterative sequences* $\{x_n\}_{n\in\mathbb{N}},\{y_n\}_{n\in\mathbb{N}}\subset[\gamma,\delta]$ *converging uniformly to* x^*,y^**, respectively,* $(x_n\to x^*,\ y_n\to y^*)$ *on* I*, and*

$$\gamma=x_0\leq x_1\leq\cdots\leqslant x_n\leqslant\cdots\leqslant x^*\leqslant y^*\leqslant\cdots\leqslant y_n\leqslant\cdots\leqslant y_1\leqslant y_0=\delta,\ \text{on } I. \tag{3.12}$$

Proof. Firstly, for all $x_{n-1},y_{n-1}\in C_{1-\alpha}(J,\mathbb{R})$, $n\in\mathbb{N}^*$, we consider the linear system:

$$\begin{cases} {}^{RL}D^{\alpha}x_n(t)=f(t,x_{n-1}(t),y_{n-1}(t))+M\left(x_{n-1}(t)-x_n(t)\right)+N\left(y_{n-1}(t)-y_n(t)\right),t\in I,\\ {}^{RL}D^{\alpha}y_n(t)=g(t,y_{n-1}(t),x_{n-1}(t))+M\left(y_{n-1}(t)-y_n(t)\right)+N\left(x_{n-1}(t)-x_n(t)\right),t\in I,\\ t^{1-\alpha}x_n(t)\big|_{t=0}=u_0,\quad t^{1-\alpha}y_n(t)\big|_{t=0}=v_0.\end{cases} \tag{3.13}$$

By Lemma 3.1.2, the linear system (3.13) has a unique system of solutions in $C_{1-\alpha}(J,\mathbb{R})\times C_{1-\alpha}(J,\mathbb{R})$, Next, we show that $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ satisfy the property:

$$x_{n-1} \leq x_n \leq y_n \leq y_{n-1}, \quad n = 1, 2, 3, ... \tag{3.14}$$

Let $p = x_1 - x_0, q = y_0 - y_1$. From (3.13) and ($C_2$) and ($C_3$), we have

$$\begin{cases} {}^{RL}D^{\alpha}p(t) = ^{RL} D^{\alpha}x_1 + ^{RL} D^{\alpha}x_0 \geq -M\left(x_1(t) - x_0(t)\right) + N\left(y_0(t) - y_1(t)\right), & \text{for } t \in I \\ t^{1-\alpha}p(t)\big|_{t=0} \geq u_0 - u_0 = 0, \\ {}^{RL}D^{\alpha}q(t) = ^{RL} D^{\alpha}y_0 + ^{RL} D^{\alpha}y_1 \geq -M\left(y_0(t) - y_1(t)\right) + N\left(x_1(t) - x_0(t)\right), & \text{for } t \in I \\ t^{1-\alpha}q(t)\big|_{t=0} \geq v_0 - v_0 = 0, \end{cases}$$

i.e.,

$$\begin{cases} {}^{RL}D^{\alpha}p(t) \geq -Mp(t) + Nq(t), \text{ for } t \in I \ t^{1-\alpha}q(t)\big|_{t=0} \geq 0, \\ {}^{RL}D^{\alpha}q(t) \geq -Mq(t) + Np(t), \text{ for } t \in I \ t^{1-\alpha}q(t)\big|_{t=0} \geq 0. \end{cases}$$

Then, by Lemma 3.1.4, we have $p(t) \geq 0,\ q(t) \geq 0$, i.e., $x_1 \geqslant x_0,\ y_1 \leqslant y_0$.

Let $w = y_1 - x_1$. By condition (C_3) and (3.13), we obtain

$$\begin{cases} {}^{RL}D^{\alpha}w(t) & = ^{RL} D^{\alpha}y_1(t) - ^{RL} D^{\alpha}x_1(t) \\ & = g\left(t, y_0(t), x_0(t)\right) + M\left(y_0(t) - y_1(t)\right) + N\left(x_0(t) - x_1(t)\right) \\ & \quad -f\left(t, x_0(t), y_0(t)\right) - M\left(x_0(t) - x_1(t)\right) - N\left(y_0(t) - y_1(t)\right) \\ & \geq -M\left(y_1(t) - x_1(t)\right) + N\left(y_1(t) - x_1(t)\right), \\ & = -(M-N)w(t), \\ t^{1-\alpha}w(t)\big|_{t=0} & = v_0 - u_0 \geq 0. \end{cases}$$

By Lemma 3.1.3, we have $w(t) \geq 0$, i.e., $y_1(t) \geq x_1(t)$ for all $t \in I =]0, b]$. Hence, we have the relation $x_0 \leq x_1 \leq y_1 \leq y_0$.

Now, we assume that $x_{k-1} \leq x_k \leq y_k \leq y_{k-1}$, for some $k \geq 1$ and we prove that (3.14) is true for $k+1$ too. Let $p = x_{k+1} - x_k$, $q = y_k - y_{k+1}$, $w = y_{k+1} - x_{k+1}$.

By condition (C_3) and (3.13), we have that ý

$$\begin{cases} {}^{RL}D^{\alpha}p(t) \geq -Mp(t) + Nq(t), \text{ for } t \in I \ t^{1-\alpha}q(t)\big|_{t=0} = 0, \\ {}^{RL}D^{\alpha}q(t) \geq -Mq(t) + Np(t), \text{ for } t \in I \ t^{1-\alpha}q(t)\big|_{t=0} = 0. \end{cases}$$

and

$$\begin{cases} {}^{RL}D^{\alpha}w(t) \geq -(M-N)w(t), \text{ for } t \in I, \\ t^{1-\alpha}q(t)\big|_{t=0} \geq 0. \end{cases}$$

and so, by Lemmas 3.1.3 and 3.1.4, we have that $x_k \leq x_{k+1} \leq y_{k+1} \leq y_k$. From the above, by induction, it is not difficult to prove that

$$x_0 \leq x_1 \leq ... \leq x_n \leq ... \leq y_n \leq ... \leq y_1 \leq y_0. \tag{3.15}$$

Applying the standard arguments (the sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ are monotone and bounded), we have

$$\underset{n\to\infty}{lim} x_n = x^*, \quad \underset{n\to\infty}{lim} y_n = y^*$$

uniformly on compact subsets of $I =]0, b]$, and the limit functions x^*, y^* satisfy (3.1). Moreover, $(x^*, y^*) \in [\gamma = x_0, \delta = y_0] \times [x_0, y_0]$. Taking the limits in (3.13), we know that (x^*, y^*) is a system of solutions of (3.1) in $[x_0, y_0]\times[x_0, y_0]$. Moreover, (3.12) is true.

Finally, we prove that (3.1) has at most one extremal system of solutions. Assume that $(x, y) \in [x_0, y_0] \times [x_0, y_0]$ is the system of solutions to (3.1), then

$$x_0 = \gamma \leqslant x, \quad y \leqslant y_0 = \delta$$

and

$$\begin{cases} {}^{RL}D^{\alpha}x(t) = f(t, x(t), y(t)), \quad t \in I = (0, b], \\ {}^{RL}D^{\alpha}y(t) = g(t, y(t), x(t)), \quad t \in I = (0, b], \\ t^{1-\alpha}x(t)\big|_{t=0} = u_0, \quad t^{1-\alpha}y(t)\big|_{t=0} = v_0. \end{cases} \tag{3.16}$$

For some $k \in \mathbb{N}$, assume that the following relation holds

$$x_k(t) \leqslant x(t), \quad y(t) \leqslant y_k(t), \quad t \in [a, b].$$

Let $u(\mathrm{t}) = \mathrm{x(t)} - \mathrm{x_{k+1}(t)}, \quad v(\mathrm{t}) = \mathrm{y_{k+1}(t)} - \mathrm{y(t)}$. According to (3.13) and (C_3), we have

$$\begin{cases} {}^{RL}D^{\alpha}u(t) & = {}^{RL}D^{\alpha}x(t) - {}^{RL}D^{\alpha}x_{k+1}(t) \\ & = f(t, x(t), y(t)) - f(t, x_k(t), y_k(t)) - M(x_k(t) - x_{k+1}(t)) \\ & \quad -N(y_k(t) - y_{k+1}(t)), \\ & \geq -M(x(t) - x_k(t)) - N(y(t) - y_k(t)) - M(x_k(t) - x_{k+1}(t)) \\ & \quad -N(y_k(t) - y_{k+1}(t)) \\ & = -M(x(t) - x_{k+1}(t)) + N(y_{k+1}(t) - y(t)). \end{cases}$$

and

$$\begin{cases} {}^{RL}D^{\alpha}v(t) & = {}^{RL}D^{\alpha}y_{k+1}(t) - {}^{RL}D^{\alpha}y(t) \\ & = g\left(t, y_k(t), x_k(t)\right) + M\left(y_k(t) - y_{k+1}(t)\right) + N\left(x_k(t) - x_{k+1}(t)\right) \\ & \quad -g(t, y(t), x(t)) \\ & \geq -M\left(y_k(t) - y(t)\right) - N\left(x_k(t) - x(t)\right) + M\left(y_k(t) - y_{k+1}(t)\right) \\ & \quad +N\left(x_k(t) - x_{k+1}(t)\right) \\ & = -M\left(y_{k+1}(t) - y(t)\right) + N\left(x(t) - x_{k+1}(t)\right), \end{cases}$$

we can get

$$\begin{cases} {}^{RL}D^{\alpha}u(t) \geq -Mu(t) + Nv(t), \text{ for } t \in I \ t^{1-\alpha}u(t)\big|_{t=0} \geq 0, \\ {}^{RL}D^{\alpha}v(t) \geq -Mv(t) + Nu(t), \text{ for } t \in I \ t^{1-\alpha}v(t)\big|_{t=0} \geq 0. \end{cases}$$

Then, by Lemma 3.1.3, we have $u(t) \geq 0,\ v(t) \geq 0$, i.e.,

$$x_{k+1}(t) \leqslant x(t),\ y(t) \leqslant y_{k+1}(t),\ t \in I.$$

By the induction arguments, the following relation holds

$$x_n(t) \leqslant x(t), \quad y(t) \leqslant y_n(t), \quad on\ I \quad for\ all\ n \in \mathbb{N}. \tag{3.17}$$

Taking the limit as $n \to \infty$ in (3.17), we get that $x^* \leqslant x,\ y \leqslant y^*$.
Hence, $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$ is the extremal system of solutions to (3.1).
So the proof is finished. □

3.3 An example

We present an example where we apply Theorem 3.2.3.

Example 3.3.1 *Consider the system of nonlinear fractional differential equations:*

$$\begin{cases} {}^{RL}D^{\alpha}x(t) = 2t^3[t - x(t)]^3 - t^4y^2(t), & t \in I =]0, 1], \\ {}^{RL}D^{\alpha}y(t) = 2t^3[t - y(t)]^3 - t^4x^2(t), & t \in I =]0, 1], \\ t^{1-\alpha}x(t)\big|_{t=0} = 0, \quad t^{1-\alpha}y(t)\big|_{t=0} = 0, \end{cases} \tag{3.18}$$

where $J = [0, 1],\ f(t, x, y) = 2t^3[t - x(t)]^3 - t^4y^2(t),\ g(t, y, x) = 2t^3[t - y(t)]^3 - t^4x^2(t)$ *and* ${}^{RL}D^{\alpha}x$ *denotes the Riemann-Liouville fractional derivative of* x *of order* α.

It is clear that f, g *are continuous functions. Take* $\gamma(t) = 0$ *and* $\delta(t) = t$ *for* $t \in [0,1]$, *then*

$$^{RL}D^{\alpha}\gamma(t) = 0 \leq f(t, \gamma(t), \delta(t)) = t^6 \text{ for } t \in]0,1], \quad t^{1-\alpha}\gamma(t)\big|_{t=0} = 0 \leq 0,$$

and

$$^{RL}D^{\alpha}\delta(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \geq g(t, \delta(t), \gamma(t)) = 0 \text{ for } t \in]0,1], \quad t^{1-\alpha}\delta(t)\big|_{t=0} = 0 \geq 0.$$

So, γ *and* δ, *are lower and upper solutions of problem* (3.18), *respectively with* $\gamma(t) = 0 \leq \delta(t) = t$ *for* $t \in [0,1]$, *then assumptions* (C_1) *and* (C_2) *holds.*
Let $x, \overline{x}, y, \overline{y} \in \mathbb{R}$, *then we have:*

$$\begin{aligned} f(t,x,y) - f(t,\overline{x},\overline{y}) &= 2t^3\left((t-x)^3 - (t-\overline{x})^3\right) - t^4(y^2 - \overline{y}^2) \\ &\geq -2t^3(x-\overline{x})\left((t-x)^2 + (t-x)(t-\overline{x}) + (t-\overline{x})^2\right) - (y-\overline{y})(y+\overline{y}) \\ &\geq -6(x-\overline{x}) \\ &\geq -6(x-\overline{x}) - 0(y-\overline{y}), \end{aligned}$$

$$g(t,x,y) - g(t,\overline{x},\overline{y}) \geq -6.(x-\overline{x}) - 0(y-\overline{y}),$$

with $\gamma(t) \leq \overline{x} \leq x \leq \delta(t)$, $\gamma(t) \leq y \leq \overline{y} \leq \delta(t)$ *for all* $t \in J$, *and we have*

$$g(t,y,x) - f(t,x,y) \geq 6(x-y) + 0.(y-x),$$

with $\gamma(t) \leq x \leq y \leq \delta(t)$, *for all* $t \in I$.

Hence the assumption (C_3) *holds with* $M = 6$ *and* $N = 0$. *By Theorem 3.2.3, the nonlinear system* (3.18) *has the extremal solution* $(x^*, y^*) \in C_{1-\alpha}([0,1], \mathbb{R}) \times C_{1-\alpha}([0,1], \mathbb{R})$, *such that* $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$ *on* $[0,1]$, *which can be obtained by taking limits from the iterative sequences:*

$$x_{n+1}(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-6(t-s)^{\alpha}) \left(2s^3(s - x_n(s))^3 + 6x_n(s) - s^4 y_n^2(s)\right) ds, \quad t \in j,$$
$$y_{n+1}(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-6(t-s)^{\alpha}) \left(2s^3(s - y_n(s))^3 + 6y_n(s) - s^4 x_n^2(s)\right) ds, \quad t \in J.$$

Chapter 4

Coupled systems of fractional differential equations with ψ-Caputo fractional derivatives

In this chapter, we investigate the existence of extremal solutions for the following coupled systems of nonlinear fractional differential equations involving the $\psi-$Caputo derivative with initial conditions, by using the comparison principle and the monotone iterative technique combined with the method of upper and lower solutions:

$$\begin{cases} {}^{C}D_{a+}^{\alpha;\psi}x(t) = h_1(t, x(t), y(t)), \quad t \in J = [a, b], \\ {}^{C}D_{a+}^{\alpha;\psi}y(t) = h_2(t, y(t), x(t)), \quad t \in J = [a, b], \\ x(a) = x_a, \ y(a) = y_a. \end{cases} \tag{4.1}$$

where $h_1, h_2 \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $x_a, y_a \in \mathbb{R}$, $x_a \leq y_a$ and ${}^{C}D_{a+}^{\alpha;\psi}$ is the $\psi-$Caputo fractional derivative of order $0 < \alpha \leq 1$. The original results of this chapter are found in [21].

C. Derbazi. in [20], studied the existence of extremal iteration solution to the following nonlinear initial value problem of fractional differential equations involving the $\psi-$Caputo derivative:

$$\begin{cases} {}^{C}D_{a+}^{\alpha;\psi}x(t) = f(t, x(t)), \quad t \in J = [a, b], \\ x(a) = x_a. \end{cases} \tag{4.2}$$

where $f \in C([a, b] \times \mathbb{R}, \mathbb{R})$, $x_a \in \mathbb{R}$, and ${}^{C}D_{a+}^{\alpha;\psi}$ is the $\psi-$Caputo fractional derivative of order $0 < \alpha \leq 1$.

4.1 Linear problems and comparison results

In this section, we study the expression of the solutions of a linear $\psi-$Caputo fractional differential equations and of a linear system of $\psi-$Caputo differential equations with initial value conditions.

Lemma 4.1.1 *Let* $0 < \alpha \leq 1$, $\lambda \in \mathbb{R}$ *and* $x_a \in \mathbb{R}$. *If* $h \in C([a,b],\mathbb{R})$, *then the linear problem:*

$$\begin{cases} {}^{C}D^{\alpha;\psi}_{a+}x(t) + \lambda x(t) = h(t), \qquad t \in J = [a,b], \\ x(a) = x_a, \end{cases} \tag{4.3}$$

has a unique solution $x \in C(J,\mathbb{R})$, *and it is given by the following expression:*

$$\begin{aligned} x(t) &= x_a\, E_{\alpha,1}\left(-\lambda(\psi(t)-\psi(a))^{\alpha}\right) \\ &\quad + \int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(\psi(t)-\psi(s))^{\alpha}\right) h(s)ds. \end{aligned} \tag{4.4}$$

In particular, when $\lambda = 0$, *the initial problem (4.3) has the unique solution*

$$x(t) = x_a + \int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} h(s)ds.$$

where $E_{\alpha,\beta}(.)$ *is the two-parametric Mittag-Leffer function.*

Lemma 4.1.2 *Let* $\alpha \in]0,1]$, $M, N \in \mathbb{R}$, *and* $h, g \in C(J,\mathbb{R})$. *Then the linear system*

$$\begin{cases} {}^{C}D^{\alpha;\psi}_{a+}x(t) + \lambda\ x(t) + \mu\ y(t) = h(t),\ \text{for } t \in J = [a,b], \\ {}^{C}D^{\alpha;\psi}_{a+}y(t) = g(t) - M\ y(t) - N\ x(t),\ \text{for } t \in J = [a,b], \\ x(a) = x_a, \quad y(a) = y_a, \end{cases} \tag{4.5}$$

has a unique system of solutions $(x,y) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$.

Proof. Let $(x,y) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$ with

$$x(t) = \frac{\theta(t)+\vartheta(t)}{2},\ \text{and } y(t) = \frac{\theta(t)-\vartheta(t)}{2}, t \in J.$$

Using (4.5), we have:

$$\begin{cases} {}^{C}D^{\alpha;\psi}_{a+}\theta(t) + (\lambda+\mu)\theta(t) = (h+g)(t), \qquad t \in J = [a,b], \\ \theta(a) = \theta_a = x_a + y_a, \end{cases} \tag{4.6}$$

and

$$\begin{cases} {}^{C}D_{a+}^{\alpha;\psi}\vartheta(t) + (\lambda+\mu)\vartheta(t) = (h+g)(t), & t \in J = [a,b], \\ \vartheta(a) = \vartheta_a = x_a - y_a, \end{cases} \tag{4.7}$$

From Lemma , we know that equations (4.6) and (4.7) have a unique solution $\theta \in C(J,\mathbb{R})$ **and** $\vartheta \in C(J,\mathbb{R})$, respectively, which can be expressed as follows:

$$\begin{aligned} \theta(t) &= \theta_a\, E_{\alpha,1}\left(-(\lambda+\mu)(\psi(t)-\psi(a))^{\alpha}\right) \\ &\quad + \int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-(\lambda+\mu)(\psi(t)-\psi(s))^{\alpha}\right)(h(s)+g(s))ds, \end{aligned}$$

and

$$\begin{aligned} \vartheta(t) &= \vartheta_a\, E_{\alpha,1}\left(-(\lambda-\mu)(\psi(t)-\psi(a))^{\alpha}\right) \\ &\quad + \int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-(\lambda-\mu)(\psi(t)-\psi(s))^{\alpha}\right)(h(s)-g(s))ds, \end{aligned}$$

Consequently, the linear system (4.5) has a unique solution (θ, ϑ). □

In the next Lemmas, we discuss comparison results for linear problem (4.3) and for linear system (4.5).

Lemma 4.1.3 *(Comparison result 1). Let $0 < \alpha \leq 1$, $\lambda \in \mathbb{R}$. If $\varphi \in C(J,\mathbb{R})$ satisfies,*

$$\begin{cases} {}^{C}D_{a+}^{\alpha;\psi}\varphi(t) \geq -\lambda\varphi(t), & t \in J = [a,b], \\ \varphi(a) \geq 0, \end{cases} \tag{4.8}$$

then $\varphi(t) \geqslant 0$ for all $t \in J = [a,b]$.

Proof. we put ${}^{C}D_{a+}^{\alpha;\psi}\varphi(t) + \lambda\varphi(t) = g(t)$ and $\varphi(a) = \varphi_a \geq 0$. We are know that $\varphi_a \geq 0$, $g(t) \geq 0$, for every $t \in J = [a,b]$ and

$${}^{C}D_{a+}^{\alpha;\psi}\varphi(t) + \lambda\varphi(t) = g(t), \quad \varphi(a) = \varphi_a.$$

By Lemma 4.1, the expression of $x(t)$ is:

$$\begin{aligned} \varphi(t) &= \varphi_a\, E_{\alpha,1}\left(-\lambda(\psi(t)-\psi(a))^{\alpha}\right) \\ &\quad + \int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda(\psi(t)-\psi(s))^{\alpha}\right) g(s)ds. \end{aligned}$$

We have $E_{\alpha,1}(z) \geq 0$ and $E_{\alpha,\alpha}(z)$, for all $0 < \alpha \leq 1$ and $z \in \mathbb{R}$.

Then, we can conclude that, $\varphi(t) \geq 0$ for every $t \in J = [a,b]$. □

Lemma 4.1.4 *(Comparison theorem 2). Let $0 < \alpha \leq 1$, $M, N \in \mathbb{R}$ and $N \geq 0$. If $x, y \in C(J, \mathbb{R})$ satisfy*

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} x(t) \geq -M\ x(t) + N\ y(t), & \text{for } t \in J = [a, b], \\ {}^C D_{a+}^{\alpha;\psi} y(t) \geq -M\ y(t) + N\ x(t), & \text{for } t \in J, \\ x(a) \geq 0, \quad y(a) \geq 0. \end{cases} \tag{4.9}$$

Then $x(t) \geq 0$, $y(t) \geq 0$ for all $t \in J$.

Proof. Let $\varphi(t) = x(t) + y(t)$, $t \in J$. Then, by (4.9) we have the following:

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} \varphi(t) \geq -(M - N)\varphi(t), & t \in J, \\ \varphi(a) = x(a) + y(a) \geq 0, \end{cases} \tag{4.10}$$

Thus, by (4.10) and Lemma 4.1.3, we know that

$$\varphi(t) \geq 0, \quad \text{for all } t \in J, \qquad \text{i.e.,} \quad x(t) + y(t) \geq 0, \quad \text{for all } t \in J. \tag{4.11}$$

Next, we show that $x(t) \geq 0$, $y(t) \geq 0$ for all $t \in J$.
In fact, by (4.9) and (4.11), we have that

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} x(t) + (M + N)x(t) \geq 0, & x(a) \geq 0 \ \text{ for } t \in J, \\ {}^C D_{a+}^{\alpha;\psi} y(t) + (M + N)y(t) \geq 0, & y(a) \geq 0 \ \text{ for } t \in J. \end{cases} \tag{4.12}$$

It follows from inequalities (4.12) and Lemma 4.1.3 that:

$$x(t) \geq 0 \quad \text{and} \quad y(t) \geq 0, \quad t \in J.$$

□

4.2 Main Result

In this section, we apply the monotone iterative procedure and the method of upper and lower solutions to prove the existence of extremal solutions to the problem (4.1). Let us defining what we mean by a solution of this problem.

Definition 4.2.1 *A solution of problem (4.1) will be a pair of functions $(x, y) \in C(J, \mathbb{R}) \times C(J, \mathbb{R})$ that satisfies the system*

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} x(t) = h_1(t, x(t), y(t)), & t \in J, \\ {}^C D_{a+}^{\alpha;\psi} y(t) = h_2(t, y(t), x(t)), & t \in J, \end{cases}$$

with the initial conditions $x(a) = x_a$, and $y(a) = y_a$.

Next, we introduce the concept of coupled lower and upper solutions of this problem as follows.

Definition 4.2.2 *We say that* γ, $\delta \in C(J,\mathbb{R})$ *is a pair of coupled lower and upper solutions of the problem* (4.1)*, respectively, if* $\gamma(t) \le \delta(t)$ *for all* $t \in J$ *and the following inequalities hold:*

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi}\gamma(t) \le h_1(t,\gamma(t),\delta(t)), \quad \textit{for} \ \ t \in J, \\ \\ \gamma(a) \le x_a, \end{cases} \tag{4.13}$$

and

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi}\delta(t) \ge h_2(t,\delta(t),\gamma(t)), \quad \textit{for} \ \ t \in J, \\ \\ \delta(a) \ge y_a. \end{cases} \tag{4.14}$$

We define the sector:

$$[\gamma,\delta] = \{x \in C(J,\mathbb{R}) : \ \gamma(t) \le x(t) \le \delta(t), \ t \in J = [a,b]\}.$$

We assume the following hypothesis:

(F_1) $h_1, h_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

(F_2) There exists $\gamma, \delta \in C(J,\mathbb{R})$, a pair of coupled lower and upper solutions of the problem (4.1), respectively.

(F_3) There exist constants $M \in \mathbb{R}$ and $N \ge 0$ such that

$$\begin{cases} h_1(t,x,y) - h_1(t,\overline{x},\overline{y}) \ge -M(x-\overline{x}) - N(y-\overline{y}), \\ \\ h_2(t,y,x) - h_2(t,\overline{y},\overline{x}) \le -M(y-\overline{y}) - N(x-\overline{x}), \end{cases}$$

where $\gamma(t) \le \overline{x} \le x \le \delta(t)$, $\gamma(t) \le y \le \overline{y} \le \delta(t)$ for all $t \in J$, and

$$h_2(t,y,x) - h_1(t,x,y) \ge -M(y-x) - N(x-y),$$

where $\gamma(t) \le x \le y \le \delta(t)$ for all $t \in J$.

We need the following lemma.

Lemma 4.2.3 *Assume that* $\{w_n(t)\}$ *be a family of continuous functions on* J *satisfying*

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} w_n(t) = f(t,w_n(t)), \qquad t \in J, \\ \\ w_n(a) = w_a, \end{cases} \tag{4.15}$$

for $n \in \mathbb{N}^*$*, and where* $|f(t,w_n(t))| \le K$ *(with* $K > 0$ *independent of* n*) for* $t \in J$*. Then, the family* $\{w_n(t)\}$ *is equicontinuous on* J*.*

Proof. According to Lemma 4.1, the integral representation of (4.15) is given by

$$w_n(t) = w_a + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} f(s, w_n(s)) ds. \tag{4.16}$$

For all $t_1, t_2 \in J$ with $a \leq t_1 \leq t_2 \leq b$, from (4.16) we have

$$\begin{aligned}
|w_n(t_2) - & \; w_n(t_1)| \\
&= \frac{1}{\Gamma(\alpha)} \left| \int_a^{t_2} \psi'(s) \left(\psi(t_2) - \psi(s)\right)^{\alpha-1} f(s, w_n(s)) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \psi'(s) \left(\psi(t_1) - \psi(s)\right)^{\alpha-1} f(s, w_n(s)) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \psi'(s) \left[\left(\psi(t_1) - \psi(s)\right)^{\alpha-1} - \left(\psi(t_2) - \psi(s)\right)^{\alpha-1} \right] |f(s, w_n(s))| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s) \left(\psi(t_2) - \psi(s)\right)^{\alpha-1} |f(s, w_n(s))| ds \\
&\leq \frac{K}{\alpha\Gamma(\alpha)} \left[\left[(\psi(t_2) - \psi(s))^{\alpha} - (\psi(t_1) - \psi(s))^{\alpha} \right]_a^{t_1} - \left[(\psi(t_2) - \psi(s))^{\alpha} \right]_{t_1}^{t_2} \right] \\
&\leq \frac{K}{\Gamma(\alpha+1)} \left[(\psi(t_1) - \psi(a))^{\alpha} + 2\left(\psi(t_2) - \psi(t_1)\right)^{\alpha} - (\psi(t_2) - \psi(a))^{\alpha} \right] \\
&\leq \frac{2K}{\Gamma(\alpha+1)} \left(\psi(t_2) - \psi(t_1)\right)^{\alpha}.
\end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero independently of $\{w_n(t)\}$. Thus, the family $\{w_n(t)\}$ is equicontinuous on J. □

Now, we can obtain our main theorem.

Theorem 4.2.4 *Assume that* (F_1), (F_2) *and* (F_3) *hold. Then the system* (4.1) *has an extremal system of solutions* $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$, *and there exist two monotone iterative sequences* $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ *converging uniformly to* x^*, y^*, *respectively, where* $x_n, y_n \in [\gamma, \delta]$, *are defined by*

$$x_{n+1}(t) = \frac{p_{n+1}(t) + q_{n+1}(t)}{2}, \quad y_{n+1}(t) = \frac{p_{n+1}(t) - q_{n+1}(t)}{2}, \text{for all } t \in J = [a, b] \tag{4.17}$$

with

$$\begin{aligned}
p_{n+1}(t) &= (x_a + y_a) E_{\alpha,1}\left(-(M+N)(\psi(t) - \psi(a))^{\alpha}\right) \\
&\quad + \int_a^t \psi'(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-(M+N)(\psi(t) - \psi(s))^{\alpha}\right) \\
&\quad \times \left(h_1(s, x_n(s), y_n) + h_2(s, y_n(s), x_n) + (M+N)(x_n(s) + y_n(s))\right) ds,
\end{aligned} \tag{4.18}$$

$$q_{n+1}(t) = (x_a - y_a)\, E_{\alpha,1}\left(-(M-N)(\psi(t)-\psi(a))^{\alpha}\right) + \int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} E_{\alpha,\alpha}\left(-(M-N)(\psi(t)-\psi(s))^{\alpha}\right) \times \left(h_1(s,x_n(s),y_n) + h_2(s,y_n(s),x_n) + (M-N)(x_n(s)-y_n(s))\right) ds, \tag{4.19}$$

and

$$\gamma(t) = x_0(t) \leq x_1(t) \leq \cdots \leq x_n(t) \leq \cdots \leq y_n(t) \leq \cdots \leq y_1(t) \leq y_0(t) = \delta(t), \;\; t \in J. \tag{4.20}$$

Proof. Firstly, for any $x_0 = \gamma(t), y_0 = \delta(t) \in C(J,\mathbb{R})$, we consider the linear system:

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} x_{n+1}(t) = h_1(t, x_n(t), y_n(t)) - M\left(x_{n+1}(t) - x_n(t)\right) - N\left(y_{n+1}(t) - y_n(t)\right), t \in J, \\ {}^C D_{a+}^{\alpha;\psi} y_{n+1}(t) = h_2(t, y_n(t), x_n(t)) - M\left(y_{n+1}(t) - y_n(t)\right) - N\left(x_{n+1}(t) - x_n(t)\right), t \in J, \\ x_{n+1}(a) = x_a, \quad y_{n+1}(a) = y_a. \end{cases} \tag{4.21}$$

By Lemma 4.1, the linear system (4.21) has a unique system of solutions in $C(J,\mathbb{R}) \times C(J,\mathbb{R})$, which is defined by 4.17. We complete the proof of the theorem through the following three steps:

Step 1: The sequences $\{x_n(t)\}$ and $\{y_n(t)\}$ satisfy the relation

$$x_n(t) \leqslant x_{n+1}(t) \leqslant y_{n+1}(t) \leqslant y_n(t), \quad t \in J, \; for\ all\ n \in \mathbb{N}. \tag{4.22}$$

Let $\theta(t) = x_1(t) - x_0(t), \vartheta(t) = y_0(t) - y_1(t)$. According to (4.21) and $(F_1) - (F_2)$, we have

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} \theta(t) = {}^C D_{a+}^{\alpha;\psi} x_1(t) - {}^C D_{a+}^{\alpha;\psi} x_0(t) \geq -M(x_1(t) - x_0(t)) - N(y_1(t) - y_0(t)), \quad t \in J \\ \theta(a) = x_1(a) - x_0(a) \geq x_a - x_a = 0, \\ {}^C D_{a+}^{\alpha;\psi} \vartheta(t) = {}^C D_{a+}^{\alpha;\psi} y_0(t) - {}^C D_{a+}^{\alpha;\psi} y_1(t) \geq -M(y_0(t) - y_1(t)) + N(x_1(t) - x_0(t)), \quad t \in J \\ \vartheta(a) = y_0(a) - y_1(a) \geq y_a - y_a = 0, \end{cases}$$

i.e.,

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} \theta(t) \geq -M\theta(t) + N\vartheta(t), \quad \theta(a) \geq 0, \text{ for } t \in J \\ {}^C D_{a+}^{\alpha;\psi} \vartheta(t) \geq -M\vartheta(t) + N\theta(t), \quad \vartheta(a) \geq 0, \text{ for } t \in J, \end{cases}$$

Then, by Lemma 4.1.4, we have $\theta(t) \geqslant 0,\ \vartheta(t) \geqslant 0$, i.e., $x_1 \geqslant x_0,\ y_1 \leqslant y_0$.

Let $\varphi(t) = y_1(t) - x_1(t)$. According to (4.21) and (F_3), we have

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi}\varphi(t) & = {}^C D_{a+}^{\alpha;\psi} y_1(t) - {}^C D_{a+}^{\alpha;\psi} x_1(t) \\ & = h_2\left(t, y_0(t), x_0(t)\right) - h_1\left(t, x_0(t), y_0(t)\right) - M\left(y_1(t) - y_0(t)\right) \\ & \quad -N\left(x_1(t) - x_0(t)\right) + M\left(x_1(t) - x_0(t)\right) + N\left(y_1(t) - y_0(t)\right) \\ & \geqslant -M\left(y_1(t) - x_1(t)\right) + N\left(y_1(t) - x_1(t)\right) = -(M-N)\left(y_1(t) - x_1(t)\right), \\ \varphi(a) & = y_1(a) - x_1(a) = y_a - x_a \geq 0. \end{cases}$$

i.e.,

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi}\varphi(t)(t) \geq -(M-N)\varphi(t), & \text{for } t \in J \\ \varphi(a) \geq 0. \end{cases}$$

By Lemma 4.1.3, we have $\varphi(t) \geqslant 0$, i.e., $y_1(t) \geqslant x_1(t)$ for all $t \in J = [a, b]$.

Next, we show that $x_1(t)$ and $y_1(t)$ satisfy inequalities (4.13) and (4.14), respectively. Since $x_0(t)$ and $y_0(t)$ are respective solutions of (4.13) and (4.14), it follows that

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} x_1(t) & = h_1\left(t, x_0(t), y_0(t)\right) - M\left(x_1(t) - x_0(t)\right) - N\left(y_1(t) - y_0(t)\right) \\ & \quad \leq h_1\left(t, x_1(t), y_1(t)\right) \\ x_1(a) & \leq x_a, \end{cases}$$

and

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} y_1(t) & = h_2\left(t, y_0(t), x_0(t)\right) - M\left(y_1(t) - y_0(t)\right) - N\left(x_1(t) - x_0(t)\right) \\ & \quad \geq h_2\left(t, y_1(t), x_1(t)\right) \\ y_1(a) & \leq y_a, \end{cases}$$

Therefore, $x_1(t)$ and $y_1(t)$ satisfy the inequalities (4.13) and (4.14), respectively. By the above arguments and mathematical induction, the relation (4.22) holds, i.e.,

$$x_n(t) \leqslant x_{n+1}(t) \leqslant y_{n+1}(t) \leqslant y_n(t), \quad t \in J, \ for \ all \ n \in \mathbb{N}.$$

Step 2: The sequences $\{x_n\}$ and $\{y_n\}$ converge uniformly to their limit functions x^* and y^*, respectively. By (4.20), the sequences $\{x_n\}$ and $\{y_n\}$ are uniformly bounded on J. From Lemma 4.2.3, the sequences $\{x_n\}$ and $\{y_n\}$ are equicontinuous on J. Hence by the Ascoli-Arzelà Theorem, there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ that converge uniformly to x^* and y^*, respectively, on J. This, together with the monotonicity of the sequences $\{x_n\}$ and $\{y_n\}$, implies

$$\lim_{n\to\infty} x_n = x^*, \quad and \quad \lim_{n\to\infty} y_n = y^*,$$

uniformly on $t \in J$, and the limit functions x^* and y^* satisfy the problem (4.1).

Step 3: System (4.1) has an extremal solution.
Assume that $(x(t), y(t)) \in [x_0(t), y_0(t)] \times [x_0(t), y_{0(}t)]$ be any solutions of system (4.1). That is,

$$\begin{cases} {}^C D^{\alpha;\psi}_{a+} x(t) = h_1(t, x(t), y(t)), & x(a) = x_a, \quad t \in J, \\ {}^C D^{\alpha;\psi}_{a+} y(t) = h_2(t, y(t), x(t)), & y(a) = y_a, \quad t \in J. \end{cases}$$

We need to prove that $x^* \leq x$ and $y \leq y^*$, we do so by using induction. Clearly, $\gamma(t) = x_0(t) \leq x(t)$ and $y(t) \leq y_0(t) = \delta(t),\ t \in J.$. Assume that for some $n \in \mathbb{N}$,

$$x_n(t) \leqslant x(t),\ and\ y(t) \leqslant y_n(t), \quad t \in J. \tag{4.23}$$

Let $p(\mathrm{t}) = \mathrm{x(t)} - \mathrm{x_{n+1}(t)}, \quad q(\mathrm{t}) = \mathrm{y_{n+1}(t)} - \mathrm{y(t)}$. According to (4.21) and (F_3), we have

$$\begin{cases} {}^C D^{\alpha;\psi}_{a+} p(t) & = {}^C D^{\alpha;\psi}_{a+} x(t) - {}^C D^{\alpha;\psi}_{a+} x_{n+1}(t) \\ & = h_1(t, x(t), y(t)) - h_1(t, x_n(t), y_n(t)) + M\left(x_{n+1}(t) - x_n(t)\right) \\ & \qquad + N\left(y_{n+1}(t) - y_n(t)\right), \\ & \geq -M\left(x(t) - x_n(t)\right) - N\left(y(t) - y_n(t)\right) + M\left(x_{n+1}(t) - x_n(t)\right) \\ & \qquad + N\left(y_{n+1}(t) - y_n(t)\right) \\ & = -M\left(x(t) - x_{n+1}(t)\right) - N\left(y(t) - y_{n+1}(t)\right) = -Mp(t) + Nq(t), \\ p(a) & = x(a) - x_{n+1}(a) = x_a - x_a = 0. \end{cases}$$

and

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} q(t) & = {}^C D_{a+}^{\alpha;\psi} y_{n+1}(t) - {}^C D_{a+}^{\alpha;\psi} y(t) \\ & = h_2(t, y_n(t), x_n(t)) - h_2(t, y(t), x(t)) - M\left(y_{n+1}(t) - y_n(t)\right) \\ & \quad -N\left(x_{n+1}(t) - x_n(t)\right), \\ & \geq M\left(y(t) - y_n(t)\right) + N\left(x(t) - x_n(t)\right) - M\left(y_{n+1}(t) - y_n(t)\right) \\ & \quad -N\left(x_{n+1}(t) - x_n(t)\right) \\ & = -M\left(y_{n+1}(t) - y(t)\right) + N\left(x(t) - x_{n+1}(t)\right) = -Mq(t) + Np(t), \\ q(a) & = y_{n+1}(a) - y(a) = y_a - y_a = 0. \end{cases}$$

we can get

$$\begin{cases} {}^C D_{a+}^{\alpha;\psi} p(t) \geq -Mp(t) + Nq(t), \text{ for } t \in J \quad p(a) \geq 0, \\ {}^C D_{a+}^{\alpha;\psi} q(t) \geq -Mq(t) + Np(t), \text{ for } t \in J \quad q(a) \geq 0. \end{cases}$$

Then, by Lemma 4.1.4, we have $p(t) \geqslant 0,\ q(t) \geqslant 0$, i.e.,

$$x_{n+1}(t) \leqslant x(t),\ y(t) \leqslant y_{n+1}(t),\ t \in J = [a, b].$$

By the induction arguments, the relation (4.23) holds, .i.e.,

$$x_n(t) \leqslant x(t), \quad y(t) \leqslant y_n(t), \quad on\ J \quad for\ all\ n \in \mathbb{N}.$$

Taking the limit as $n \to \infty$ on both sides of (4.23), we get that

$$x^* \leqslant x,\ y \leqslant y^*.$$

Hence, $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$ is the extremal system of solutions to (4.1).
So the proof is finished. □

4.3 Examples

We present two examples, where we apply Theorem 4.2.4.

Example 4.3.1 *Consider the system of nonlinear fractional differential equations:*

$$\begin{cases} {}^C D_{0^+}^{1/2;\psi} x(t) = (t - x(t))^2 - \frac{1}{2} t y(t), \quad x(0) = 0,\ t \in J = [0, 1], \\ {}^C D_{0^+}^{1/2;\psi} y(t) = (t - y(t))^2 - \frac{1}{2} t x(t), \quad y(0) = 0,\ t \in J = [0, 1], \end{cases} \tag{4.24}$$

This problem is a particular case of (4.1)*, with* $\alpha = 1/2$*,* $a = 0, b = 1$*,* $x_a = y_a = 0$*,* $h_1(t,x,y) = (t-x)^2 - \frac{1}{2}ty$*,* $h_2(t,y,x) = (t-y)^2 - \frac{1}{2}tx$ *and* $\psi(t) = t$*.*
We have ${}^C D_{0^+}^{1/2;\psi} = {}^C D_{0^+}^{1/2}$ *Caputo fractional derivative.*
It is clear that h_1, h_2 *are continuous functions. Take* $\gamma(t) = x_0(t) = 0$ *and* $\delta(t) = y_0(t) = t$ *for* $t \in [0,1]$*, then*

$$ {}^C D_{0^+}^{1/2} x_0(t) = 0 \leq h_1(t, x_0(t), y_0(t)) = \frac{1}{2}t^2 \; for \; t \in [0,1], \quad x_0(0) = 0 \leq 0, $$

and

$$ {}^C D_{0^+}^{1/2} y_0(t) = 2\frac{\sqrt{t}}{\sqrt{\pi}} \geq h_2(t, y_0(t), x_0(t)) = 0 \; for \; t \in [0,1], \quad y_0(0) = 0 \geq 0. $$

So, x_0 *and* y_0*, are lower and upper solutions of problem* (4.24)*, respectively with* $x_0(t) = 0 \leq y_0(t) = t$ *for* $t \in [0,1]$*, then assumptions* (F_1) *and* (F_2) *holds.*
Let $x, \overline{x}, y, \overline{y} \in \mathbb{R}$*, then we have:*

$$ \begin{aligned} h_1(t,x,y) - h_1(t,\overline{x},\overline{y}) &= (t-x)^2 - (t-\overline{x})^2 - \frac{1}{2}t(y-\overline{y}) \\ &\geq (x-\overline{x})(-2t + x + \overline{x}) - 0.(y-\overline{y}) \\ &\geq -2(x-\overline{x}) - 0.(y-\overline{y}), \end{aligned} $$

$$ \begin{aligned} h_2(t,y,x) - h_2(t,\overline{y},\overline{x}) &= (t-y)^2 - (t-\overline{y})^2 - \frac{1}{2}t(x-\overline{x}) \\ &\leq (y-\overline{y})(-2t + y + \overline{y}) - 0.(x-\overline{x}) \\ &\leq -2(y-\overline{y}) - 0.(x-\overline{x}), \end{aligned} $$

with $0 = x_0(t) \leq \overline{x} \leq x \leq y_0(t) = t \leq 1$*,* $0 = x_0(t) \leq y \leq \overline{y} \leq y_0(t) = t \leq 1$ *for all* $t \in J$*, and we have*

$$ \begin{aligned} h_2(t,y,x) - h_1(t,x,y) &= (t-y)^2 - (t-x)^2 - \frac{1}{2}t(x-y) \\ &\geq (y-x)\,(-2t + y + x) - 0.(x-y) \\ &\geq -2(y-x) - 0.(x-y). \end{aligned} $$

with $0 = x_0(t) \leq x \leq y \leq y_0(t) = t \leq 1$*, for all* $t \in J$*.*
Hence the assumption (F_3) *holds with* $M = 2$ *and* $N = 0$*. By Theorem 4.2.4, the nonlinear system* (4.24) *has the extremal solution* $(x^*, y^*) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$*, such that* $(x^*, y^*) \in [0,t] \times [0,t]$ *on* $[0,1]$*, which can be obtained by taking limits from the iterative sequences:*

$$ x_{n+1}(t) = \int_0^t (t-s)^{-1/2} E_{1/2,1/2}(-2\sqrt{t-s}) \left((s - x_n(s))^2 - s y_n(s) + 2x_n(s)\right) ds, \; t \in J, \; n \geq 0 $$

$$ y_{n+1}(t) = \int_0^t (t-s)^{-1/2} E_{1/2,1/2}(-2\sqrt{t-s}) \left((s - y_n(s))^2 - s x_n(s) + 2y_n(s)\right) ds, \; t \in J, \; n \geq 0. $$

Example 4.3.2 *Consider the system of nonlinear fractional differential equations:*

$$\begin{cases} {}^C D_{1^+}^{1/2;\psi} x(t) = 2\left(\ln^2(t) - x^2(t)\right) - \ln(t)y(t), \quad x(1) = 0,\ t \in J = [1, e], \\ {}^C D_{1^+}^{1/2;\psi} y(t) = 2\left(\ln^2(t) - y^2(t)\right) - \ln(t)x(t), \quad y(1) = 0,\ t \in J = [1, e], \end{cases} \tag{4.25}$$

This problem is a particular case of (4.1)*, with* $\alpha = 1/2$*,* $a = 1, b = e$*,* $x_a = y_a = 0$*,* $h_1(t, x, y) = 2\left(\ln^2(t) - x^2\right) - \ln(t)y$*,* $h_2(t, y, x) = 2\left(\ln^2(t) - y^2\right) - \ln(t)x$ *and* $\psi(t) = \ln(t)$*. It is clear that* h_1, h_2 *are continuous functions.*
Take $x_0(t) = 0$ *and* $y_0(t) = ln(t)$ *for* $t \in [1, e]$*, then*

$${}^C D_{1^+}^{1/2;\psi} x_0(t) = 0 \leq h_1(t, x_0(t), y_0(t)) = \ln(t) \ for\ t \in [1, e], \quad x_0(1) = 0 \leq 0,$$

and

$${}^C D_{0^+}^{1/2} y_0(t) = \frac{2}{\sqrt{\pi}}\sqrt{\ln(t)} \geq h_2(t, y_0(t), x_0(t)) = 0 \ for\ t \in [1, e], \quad y_0(1) = 0 \geq 0.$$

So, x_0 *and* y_0*, are lower and upper solutions of problem* (4.25)*, respectively with* $x_0(t) = 0 \leq y_0(t) = \ln(t)$ *for* $t \in [1, e]$*, then assumptions* (F_1) *and* (F_2) *holds.*
Let $x, \overline{x}, y, \overline{y} \in \mathbb{R}$*, then we have:*

$$\begin{aligned} h_1(t, x, y) - h_1(t, \overline{x}, \overline{y}) &= -2(x - \overline{x})(x + \overline{x}) - \ln(t)(y - \overline{y}) \\ &\geq -4(x - \overline{x}) - 0.(y - \overline{y}), \end{aligned}$$

$$\begin{aligned} h_2(t, y, x) - h_2(t, \overline{y}, \overline{x}) &= -2(y - \overline{y})(y + \overline{y}) - \ln(t)(x - \overline{x}) \\ &\leq -4(y - \overline{y}) - 0.(x - \overline{x}), \end{aligned}$$

with $0 = x_0(t) \leq \overline{x} \leq x \leq y_0(t) = \ln(t) \leq 1$*,* $0 = x_0(t) \leq y \leq \overline{y} \leq y_0(t) = \ln(t) \leq 1$ *for all* $t \in J$*, and we have*

$$\begin{aligned} h_2(t, y, x) - h_1(t, x, y) &= -2(y - x)(y + x) - \ln(t)(x - y) \\ &\geq -4(y - x) - 0.(x - y). \end{aligned}$$

with $0 = x_0(t) \leq x \leq y \leq y_0(t) = \ln(t) \leq 1$*, for all* $t \in J$*.*
Hence the assumption (F_3) *holds with* $M = 4$ *and* $N = 0$*. By Theorem 4.2.4, the nonlinear system* (4.25) *has the extremal solution* $(x^*, y^*) \in C([1, e], \mathbb{R}) \times C([1, e], \mathbb{R})$*, such that* $(x^*, y^*) \in [0, \ln(t)] \times [0, \ln(t)]$ *on* $[1, e]$*, which can be obtained by taking limits from the iterative sequences:*

$$\begin{aligned} x_{n+1}(t) = &\int_1^t (\ln(t) - \ln(s))^{-1/2} E_{1/2,1/2}(-4\sqrt{\ln(t) - \ln(s)}) \left(2(\ln^2(s) - x_n^2(s))\right. \\ &\left. - \ln(s)y_n(s) + 4x_n(s)\right) \frac{ds}{s}, \quad n \geq 0 \\ y_{n+1}(t) = &\int_1^t (\ln(t) - \ln(s))^{-1/2} E_{1/2,1/2}(-4\sqrt{\ln(t) - \ln(s)}) \left(2(\ln^2(s) - y_n^2(s))\right. \\ &\left. - \ln(s)x_n(s) + 4y_n(s)\right) \frac{ds}{s}, \quad n \geq 0. \end{aligned}$$

Conclusion

Conclusion In this work, we have considered the existence of extremal solutions for nonlinear Riemann-Liouville fractional differential equation involving integral boundary condition, and for a coupled system of nonlinear Riemann-Liouville fractional differential equations with initial conditions. Also, we present the existence of extremal solutions for a coupled system of nonlinear $\psi-$Caputo fractional differential equations with initial conditions.

These results will be obtained by using the monotone iterative technique combined with the method of upper and lower solutions.

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