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***Monotone Iterative Technique for Systems of Nonlinear
Fractional Differential Equations on Time Scales***

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DEDICATIONS

I dedicate this modest work to my dear parents, to whom i owe everything, for the sacrifices they've made for me and for always encouraging me to give my best.

Also, to my brothers, sister, and entire family for their encouragement, which is the source of my pride and confidence.

MAARADJI Fatima zohra

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NEDJADI zohra

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Abstract

Fractional differential equations play an important role in describing many phenomena and processes in various fields of science such as physics, chemistry, control systems, population dynamics, aerodynamics and electrodynamics, etc. In this work, we present existence of quasi-solutions for the first-order dynamic equations involving integral boundary conditions, and we present the existence of extremal solutions for nonlinear conformable fractional differential equations involving integral boundary conditions. Also, we present the existence of extremal solutions for a coupled system of nonlinear conformable fractional differential equations on time scales with initial conditions. Existence results for these problems are obtained by applying the comparison principle and the monotone iterative technique combined with the method of upper and lower solutions.

Key words and phrases: Calculus on time scales, conformable fractional calculus on time scales, systems of conformable fractional differential equations on time scales, comparison principle, upper and lower solutions, monotone iterative technique.

Résumé

Les équations différentielles fractionnaires jouent un rôle important dans la description de nombreux phénomènes et processus dans divers domaines scientifiques tels que la physique, la chimie, les systèmes de contrôle, la dynamique des populations, l'aérodynamique et l'électrodynamique, etc.

Nous présentons dans ce mémoire, l'existence de quasi-solutions pour des équations dynamiques sur les échelles de temps non linéaires d'ordre un avec condition intégrale aux limites, et l'existence de solutions extrêmes pour des équations différentielles fractionnaires conformes non linéaires avec condition intégrale. Aussi, nous présentons l'existence de solutions extrêmes pour un système couplé d'équations différentielles fractionnaires conformes non linéaires sur les échelles de temps avec conditions initiales. Ces résultats sont obtenus grâce à la technique itérative monotone combinée à la méthode des sous et sur solutions.

Mots Clés: Calcul sur les échelles de temps, calculs fractionnaire conforme sur les échelles de temps, systèmes d'équations différentielles fractionnaires conformes sur les échelles de temps, sous et sur solutions, technique des itérations monotones.

Contents

Contents	1
Introduction	2
1 Preliminaries	4
1.1 Time scales calculus	4
1.2 Conformable Fractional Calculus	5
1.3 Conformable fractional calculus on time scales	7
1.4 Elements of Functional Analysis	9
2 Existence results for a first-order dynamic equations with nonlocal initial conditions on time scales	11
2.1 Linear problems and comparison principles	12
2.2 Main Results	15
2.3 An example	21
3 Existence of solutions for nonlinear conformable fractional differential equations with integral boundary conditions	24
3.1 Linear problem and comparison principle	25
3.2 Main Result	26
3.3 An example	32
4 Extremal solutions to a coupled system of conformable fractional dynamic equations on time scales	34
4.1 Linear problems and comparison principles	35
4.2 Main Result	36
4.3 Examples	42

Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. The theory of time scales was introduced by Stefan Hilger in his PhD thesis [18] in 1988, in order to unify and generalize continuous and discrete analysis.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Fractional differential equations play an important role in describing many phenomena and processes in various fields of science such as physics, chemistry, control systems, population dynamics, aerodynamics and electrodynamics, etc. For examples and details, the reader can see the references [23, 27, 29, 31, 33].

Recently, a new fractional derivative, called the conformable fractional derivative, was introduced by Khalil et al. [22]. For recent results on conformable fractional derivatives we refer the reader to [1, 2, 3, 4, 12, 17]. In particular, Benkhettou et al. [7] introduced a conformable fractional calculus on an arbitrary time scale, which provided a natural extension of the conformable fractional calculus.

In this work, we present existence of quasi-solutions for the first-order dynamic equations involving integral boundary conditions, and we present the existence of extremal solutions for nonlinear conformable fractional differential equations involving integral boundary conditions. Also, we present the existence of extremal solutions for a coupled system of nonlinear conformable fractional differential equations on time scales with initial conditions. Existence results for these problems are obtained by using the monotone iterative technique combined with the method of upper and lower solutions, as presented respectively in the following articles [14, 21, 6]. The purpose of this method is to constructing two monotone iterative sequences, by using γ, δ the lower and upper solutions with $\gamma \leq \delta$, showing the convergence of the constructed sequences, and proving these two sequences approximate the extremal solutions of the given problem. We point out that monotone iterative technique combined with the method of upper and lower solutions has been applied by several authors, see [11, 21, 24, 25, 26, 28].

We have organized this work as follows:

In Chapter 1, we present some definitions and results which are used throughout this work.

In Chapter 2, we prove the existence of quasi-solutions for the following first-order

dynamic equation with nonlocal initial condition (integral boundary condition):

$$\begin{cases} u^\Delta(t) = f(t, u(t)), & t \in I = [a, b]_{\mathbb{T}}, \\ u(a) = \int_a^{\sigma(b)} g(s)u(s)\Delta s, \end{cases}$$

where $a, b \in \mathbb{T}$, $a < b$, $J = [a, \sigma(b)]_{\mathbb{T}}$, $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : J \rightarrow \mathbb{R}$ are continuous functions.

In Chapter 3, we present the existence of extremal solutions for the following nonlinear conformable fractional differential equations with integral boundary conditions:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t)), & t \in]0, 1[, \\ x(0) = -rx(1) + \lambda \int_0^1 x(s)ds, \end{cases}$$

where $0 < \alpha \leq 1$, $r > 0$, $\lambda > 0$, $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at $t \in]0, 1[$ of order α .

In Chapter 4, we investigate the existence of extremal solutions for a coupled system of nonlinear conformable fractional dynamic equations on time scales, by applying the comparison principle and the monotone iterative technique combined with the method of upper and lower solutions:

$$\begin{cases} x_\Delta^{(\alpha)}(t) = f(t, x^\sigma(t), y^\sigma(t)), & t \in I = [a, b]_{\mathbb{T}}, \\ y_\Delta^{(\alpha)}(t) = g(t, y^\sigma(t), x^\sigma(t)), & t \in I = [a, b]_{\mathbb{T}}, \\ x(a) = \lambda_0, \quad y(a) = \beta_0. \end{cases}$$

Here, $\mathbb{T}$ is an arbitrary bounded time scale, $J = [a, \sigma(b)]_{\mathbb{T}}$ with $a, b \in \mathbb{T}$, $0 < a < b$, $\lambda_0, \beta_0 \in \mathbb{R}$, $\lambda_0 \leq \beta_0$, $f, g : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions and $x_\Delta^{(\alpha)}, y_\Delta^{(\alpha)}$ are the conformable fractional derivatives (on time scales) with $\alpha \in (0, 1]$.

Chapter 1

Preliminaries

In this chapter, we present some definitions and results which we will use in this work.

1.1 Time scales calculus

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \text{respectively.}$$

A point $t \in \mathbb{T}$ with $\sigma(t) > t$, $\rho(t) < t$, $t = \sigma(t)$, $t = \rho(t)$, $\rho(t) < t < \sigma(t)$ and $\rho(t) = t = \sigma(t)$ is called right-scattered, left scattered, right dense, left dense, isolated and dense, respectively.

The graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum M , then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$, otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$.

If $f : \mathbb{T} \rightarrow \mathbb{R}$, is a function, then we define the function f^σ by

$$f^\sigma(t) = (f \circ \sigma)(t) = f(\sigma(t)) \quad \text{for all } t \in \mathbb{T}.$$

Definition 1.1.1 [8]. *The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$, write $f \in C_{rd}(\mathbb{T}, \mathbb{R})$.*

Definition 1.1.2 [8](Delta derivative) *Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood U of t such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the delta derivative (Δ -derivative) of f at t and we say that f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

The set of functions $f : \mathbb{T} \rightarrow \mathbb{R}$ which are Δ -differentiable and whose Δ -derivative is rd-continuous is denoted by $C_{rd}^1(\mathbb{T}, \mathbb{R})$.

Definition 1.1.3 [8]. The function $p : \mathbb{T} \rightarrow \mathbb{R}$ is μ -regressive if

$$1 + \mu(t)p(t) \neq 0, \text{ for all } t \in \mathbb{T}^\kappa.$$

The set of all μ -regressive and rd-continuous functions $p : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by $\mathcal{R}_\mu$. We define the set $\mathcal{R}_\mu^+ = \{p \in \mathcal{R}_\mu : 1 + \mu(t)p(t) > 0\}$ for all $t \in \mathbb{T}$.

Definition 1.1.4 [8]. If $p \in \mathcal{R}_\mu$, then we define the delta exponential function e_p by:

$$e_p(t, s) = \exp \left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta \tau \right),$$

for $t, s \in \mathbb{T}$, where the μ -cylinder transformation is as in :

$$\xi_h(z) = \begin{cases} \frac{1}{h} \log(1 + zh); & \text{if } h > 0; \\ z; & \text{if } h = 0. \end{cases}$$

where $\log$ is the principal logarithm function.

Lemma 1.1.5 [8].

(1) If $p \in \mathcal{R}_\mu$ and $s, t, t_0 \in \mathbb{T}$, then

$$e_p(t, t) \equiv 1, e_0(t, s) \equiv 1, e_p(t, s) = \frac{1}{e_p(s, t)}, \text{ and } e_p(t, t_0)e_p(t_0, s) = e_p(t, s).$$

(2) If $p \in \mathcal{R}_\mu^+$ and $t_0 \in \mathbb{T}$, then

$$e_p(t, t_0) > 0, \text{ for all } t \in \mathbb{T}.$$

1.2 Conformable Fractional Calculus

In this section, we introduce some necessary definitions and properties of the conformable fractional calculus which are used in this report and can be found in [1, 19, 22].

Definition 1.2.1 [22] Given a function $f : [0, \infty) \rightarrow \mathbb{R}$ and a real constant $\alpha \in (0, 1]$. The conformable fractional derivative of f of order α is defined by,

$$f^{(\alpha)}(t) := \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \quad (1.1)$$

for all $t > 0$. If $f^{(\alpha)}(t)$ exists and is finite, we say that f is α -differentiable at t .

If f is α -differentiable in some interval $(0, a)$, $a > 0$, and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists, then the conformable fractional derivative of f of order α at $t = 0$ is defined as

$$f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t).$$

Theorem 1.2.2 [22] Let $\alpha \in (0, 1]$ and $f : [0, \infty) \rightarrow \mathbb{R}$ a α -differentiable function at $t_0 > 0$, then f is continuous at t_0 .

Theorem 1.2.3 [22] Let $\alpha \in (0, 1]$ and assume f, g to be α -differentiable at a point $t > 0$. Then,

- (i) $(af + bg)^{(\alpha)} = af^{(\alpha)} + bg^{(\alpha)}$, for all $a, b \in \mathbb{R}$;
- (ii) $(fg)^{(\alpha)} = fg^{(\alpha)} + gf^{(\alpha)}$;
- (iii) $(f/g)^{(\alpha)} = \frac{gf^{(\alpha)} - fg^{(\alpha)}}{g^2}$.
- (iv) If, in addition, f is differentiable at a point $t > 0$, then

$$f^{(\alpha)}(t) = t^{1-\alpha} f'(t).$$

Additionally, conformable fractional derivatives of certain functions as follow:

- 1. $(t^p)^{(\alpha)} = p t^{p-\alpha}$, for all $p \in \mathbb{R}$.
- 2. $(\lambda)^{(\alpha)} = 0$, for all $\lambda \in \mathbb{R}$.
- 3. $(e^{ct})^{(\alpha)} = c t^{1-\alpha} e^{ct}$, for all $c \in \mathbb{R}$.

Definition 1.2.4 (Conformable fractional integral [22]). Let $\alpha \in (0, 1]$ and $f : [a, \infty) \rightarrow \mathbb{R}$. The conformable fractional integral of f of order α from a to t , denoted by $I_\alpha^a(f)(t)$, is defined by

$$I_\alpha^a(f)(t) := \int_a^t f(s) d_\alpha s := \int_a^t f(s) s^{\alpha-1} ds.$$

The considered integral is the usual improper Riemann one.

For $a = 0$ we put $I_\alpha^0(f)(t) = I_\alpha(f)(t)$.

Theorem 1.2.5 [22] If f is a continuous function in the domain of I_α^a then, for all $t \geq a$ we have

$$(I_\alpha^a(f))^{(\alpha)}(t) = f(t).$$

Lemma 1.2.6 [22] Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have

$$I_\alpha^a(f^{(\alpha)})(t) = f(t) - f(a). \quad (1.2)$$

Proposition 1.2.7 [19] Let $0 < a < b$, $f : [a, b] \rightarrow \mathbb{R}$ be continuous function and $0 < \alpha < 1$. Then for all $t \in [a, b]$ we have,

$$|I_\alpha^a(f)(t)| \leq I_\alpha^a|f|(t).$$

1.3 Conformable fractional calculus on time scales

We begin by introducing the notion of conformable fractional derivative of order $\alpha \in]0, 1]$ for function defined on arbitrary time scale $\mathbb{T}$.

Definition 1.3.1 [7] (*Conformable fractional derivative on time scale*) Let $f : \mathbb{T} \rightarrow \mathbb{R}$, $t \in \mathbb{T}^\kappa$, and $\alpha \in]0, 1]$. For $t > 0$, we define $f_\Delta^{(\alpha)}(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a δ -neighborhood $\mathcal{V}_t \subset \mathbb{T}$ (i.e., $\mathcal{V}_t :=]t - \delta, t + \delta[\cap \mathbb{T}$) of t , $\delta > 0$, such that

$$\left| [f(\sigma(t)) - f(s)] t^{1-\alpha} - f_\Delta^{(\alpha)}(t) [\sigma(t) - s] \right| \leq \epsilon |\sigma(t) - s| \text{ for all } s \in \mathcal{V}_t.$$

We call $f_\Delta^{(\alpha)}(t)$ the conformable fractional derivative of f of order α at t , and we define the conformable fractional derivative at 0 as $f_\Delta^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f_\Delta^{(\alpha)}(t)$.

Example 1.3.2 Let $\alpha \in (0, 1]$. Functions $f, g, h : \mathbb{T} \rightarrow \mathbb{R} : f(t) = t$, $p \in \mathbb{R}$, $g(t) \equiv \lambda$, $\lambda \in \mathbb{R}$, and $h(t) = e_p(t, a)$, $p \in \mathcal{R}_\mu$, are conformable fractional derivatives of order α with

$$f_\Delta^{(\alpha)}(t) = t^{1-\alpha}; \quad g_\Delta^{(\alpha)}(t) = 0; \quad h_\Delta^{(\alpha)}(t) = t^{1-\alpha} p e_p(t, a).$$

Remark 1.3.3 (i) If $\alpha = 1$, we have $f_\Delta^{(\alpha)}(t) = f^\Delta(t)$.

(ii) If $\mathbb{T} = \mathbb{R}$, then $f_\Delta^{(\alpha)} = f^{(\alpha)}$ is the conformable fractional derivative of f of order α .

We introduce the following spaces::

$$C_{rd}^\alpha([a, b]_\mathbb{T}, \mathbb{R}) = \{f \text{ is conformal fractional differentiable of order } \alpha \text{ on } [a, b]_\mathbb{T} \text{ and } f_\Delta^{(\alpha)} \in C_{rd}([a, b]_\mathbb{T}, \mathbb{R})\}.$$

Theorem 1.3.4 [34] Let $\alpha \in]0, 1]$. Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. The following properties hold.

- (i) If f is conformal fractional differentiable of order α at $t > 0$, then f is continuous at t .
- (ii) If f is continuous at t and t is right-scattered, then f is conformable fractional differentiable of order α at t with

$$f_\Delta^{(\alpha)}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha} = t^{1-\alpha} f^\Delta(t).$$

(iii) If t is right-dense, then f is conformable fractional differentiable of order α at t if and only if the limit $\lim_{s \rightarrow t} \frac{f(t) - f(s)}{(t - s)} t^{1-\alpha}$ exists as a finite number. In this case,

$$f_{\Delta}^{(\alpha)}(t) = t^{1-\alpha} f'(t).$$

(iv) If f is conformable fractional differentiable of order α at t , then

$$f(\sigma(t)) = f(t) + (\mu(t)) t^{\alpha-1} f_{\Delta}^{(\alpha)}(t).$$

Theorem 1.3.5 [34] Assume $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are conformable fractional differentiable of order α . Then,

- (i) the sum $f + g$ is conformable fractional differentiable with $(f + g)_{\Delta}^{(\alpha)} = f_{\Delta}^{(\alpha)} + g_{\Delta}^{(\alpha)}$;
- (ii) for any $\lambda \in \mathbb{R}$, λf is conformable fractional differentiable with $(\lambda f)_{\Delta}^{(\alpha)} = \lambda f_{\Delta}^{(\alpha)}$;
- (iii) if f and g are continuous, then the product fg is conformable fractional differentiable with $(fg)_{\Delta}^{(\alpha)} = f_{\Delta}^{(\alpha)} g + (f \circ \sigma) g_{\Delta}^{(\alpha)} = f_{\Delta}^{(\alpha)} (g \circ \sigma) + f g_{\Delta}^{(\alpha)}$;

Now we introduce the α -conformable fractional integral (or α -fractional integral) on time scales.

Definition 1.3.6 [7] Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a regulated function. Then the α -fractional integral of f , $0 < \alpha \leq 1$, is defined by $\int f(t) \Delta^{\alpha} t := \int f(t) t^{\alpha-1} \Delta t$.

Definition 1.3.7 [7] Suppose $f : \mathbb{T} \rightarrow \mathbb{R}$ is a regulated function. Denote the indefinite α -fractional integral of f of order α , $\alpha \in (0, 1]$, as follows: $F(t) = \int f(t) \Delta^{\alpha} t$. Then, for all $a, b \in \mathbb{T}$, we define the Cauchy α -fractional integral by $\int_a^b f(t) \Delta^{\alpha} t = F(b) - F(a)$.

Theorem 1.3.8 [7] Let $\alpha \in (0, 1]$. Then, for any rd-continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$, there exist a function $F : \mathbb{T} \rightarrow \mathbb{R}$ such that $F_{\Delta}^{(\alpha)}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$. Function F is said to be an α -antiderivative of f .

Theorem 1.3.9 [7] If $f : \mathbb{T}^{\kappa} \rightarrow \mathbb{R}$ is a rd-continuous function and $t \in \mathbb{T}^{\kappa}$, then

$$\int_t^{\sigma(t)} f(s) \Delta^{\alpha} s = f(t) \mu(t) t^{\alpha-1}.$$

Theorem 1.3.10 [34] Let $\alpha \in (0, 1]$, $a, b, c \in \mathbb{T}$, $\lambda, \gamma \in \mathbb{R}$, and $f, g : \mathbb{T} \rightarrow \mathbb{R}$ be two rd-continuous functions. Then,

$$(i) \int_a^b [\lambda f(t) + \gamma g(t)] \Delta^{\alpha} t = \lambda \int_a^b f(t) \Delta^{\alpha} t + \gamma \int_a^b g(t) \Delta^{\alpha} t;$$

$$(ii) \int_a^b f(t) \Delta^\alpha t = - \int_b^a f(t) \Delta^\alpha t;$$

$$(iii) \int_a^b f(t) \Delta^\alpha t = \int_a^c f(t) \Delta^\alpha t + \int_c^b f(t) \Delta^\alpha t;$$

$$(iv) \int_a^a f(t) \Delta^\alpha t = 0;$$

(v) if there exist $g : \mathbb{T} \rightarrow \mathbb{R}$ with $\|f(t)\| \leq |g(t)|$ for all $t \in [a, b]$, then

$$\left\| \int_a^b f(t) \Delta^\alpha t \right\| \leq \int_a^b |g(t)| \Delta^\alpha t.$$

1.4 Elements of Functional Analysis

Let $C(J, \mathbb{R})$ be the Banach space of continuous functions from $J = [a, b]$ into $\mathbb{R}$ with the norm

$$\|u\| = \sup\{|u(t)| : t \in J\}.$$

Definition 1.4.1 [32]. Let E, F be Banach spaces and $T : E \rightarrow F$.

- (i) The operator T is said to be bounded if it maps any bounded subset of E into a bounded subset of F .
- (ii) The operator T is called compact if $T(E)$ is relatively compact (i.e., $\overline{T(E)}$ is compact).
- (iii) The operator T is said to be completely continuous if it is continuous and maps any bounded subset of E into a relatively compact subset of F .

Theorem 1.4.2 (Arzela-Ascoli theorem [30]). A subset $\mathcal{F}$ of $C([a, b], \mathbb{R})$ is relatively compact (i.e. $\overline{\mathcal{F}}$ is compact) if and only if the following conditions hold:

1. $\mathcal{F}$ is uniformly bounded i.e, there exists $M > 0$ such that

$$\|f(t)\| < M \text{ for each } t \in [a, b] \text{ and each } f \in \mathcal{F}.$$

2. $\mathcal{F}$ is equicontinuous i.e, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $t_1, t_2 \in [a, b]$, $|t_2 - t_1| \leq \delta$ implies $\|f(t_2) - f(t_1)\| \leq \varepsilon$, for every $f \in \mathcal{F}$.

Theorem 1.4.3 (Arzela-Ascoli Theorem [10](Sequential Version)). If $\{f_n(t)\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.

Theorem 1.4.4 (*Banach's fixed point theorem [16]*) *Let C be a non-empty closed subset of a Banach space X , then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 1.4.5 (*Lebesgue dominated convergence theorem [15]*). *Suppose $f_n : \mathbb{R} \rightarrow [-\infty, +\infty]$ are (Lebesgue) measurable functions such that*

1. $\lim_{n \rightarrow +\infty} f_n(x) = f(x)$.
2. *There is an integrable $g : \mathbb{R} \rightarrow [0, +\infty]$ with $|f_n(x)| \leq g(x)$, for each $x \in \mathbb{R}$.*

Then f is integrable as is f_n for each n , and

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} \lim_{n \rightarrow +\infty} f_n d\mu = \int_{\mathbb{R}} f d\mu.$$

Chapter 2

Existence results for a first-order dynamic equations with nonlocal initial conditions on time scales

In this chapter, we prove the existence of quasi-solutions for the following first-order dynamic equation with nonlocal initial condition (integral boundary condition):

$$\begin{cases} u^\Delta(t) = f(t, u(t)), & t \in I = [a, b]_{\mathbb{T}}, \\ u(a) = \int_a^{\sigma(b)} g(s)u(s)\Delta s, \end{cases} \quad (2.1)$$

where $a, b \in \mathbb{T}$, $a < b$, $J = [a, \sigma(b)]_{\mathbb{T}}$, $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : J \rightarrow \mathbb{R}$ are continuous functions. The existence result of (2.1) is obtained by means of the method of upper and lower solutions and its associated monotone iterative technique. The original results of this chapter are found in [14].

In [13], M. Derhab et al. studied the existence of quasi-solutions for the following problem:

$$\begin{cases} u'(t) = f(t, u(t)), & t \in [0, T], \\ u(0) = \int_0^T g(s)u(s)ds, \end{cases} \quad (2.2)$$

where $T > 0$, $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

2.1 Linear problems and comparison principles

In this section, we study the expression of the solutions of a linear first-order dynamic equation on time scales, with initial condition:

$$\begin{cases} x^\Delta(t) + p(t) x(t) = h(t), & t \in I = [a, b]_{\mathbb{T}}, \\ x(a) = x_0, \end{cases} \quad (2.3)$$

with $-p \in \mathcal{R}_\mu$, $x_0 \in \mathcal{R}$ and $h \in C(I, \mathbb{R})$.

Theorem 2.1.1 *Let $p \in \mathcal{R}_\mu$. For every $h \in C(I, \mathbb{R})$, the problem (2.3) has a unique solution $x \in C(J, \mathbb{R})$ given by:*

$$x(t) = x_0 e_{-p}(t, a) + \int_a^t e_{-p}(t, \sigma(s)) h(s) \Delta s, \quad t \in J = [a, \sigma(b)]_{\mathbb{T}}. \quad (2.4)$$

Proof. Let x be a solution to (2.3). By Theorem 1.3.5, with $\alpha = 1$ consider

$$\begin{aligned} \left[\frac{x(t)}{e_{-p}(t, a)} \right]^\Delta &= \frac{x^\Delta(t) e_{-p}(t, a) - x(t) e_{-p}^\Delta(t, a)}{e_{-p}(t, a) e_{-p}^\sigma(t, t_0)} \\ &= \frac{x^\Delta(t) e_{-p}(t, a) + p(t) x(t) e_{-p}(t, a)}{e_{-p}(t, a) e_{-p}(\sigma(t), a)} \\ &= \frac{x^\Delta(t) + p(t) x(t)}{e_{-p}(\sigma(t), a)} \\ &= \frac{h(t)}{e_{-p}(\sigma(t), a)} \\ &= h(t) e_{-p}(a, \sigma(t)). \end{aligned}$$

and hence integrating the above on $[a, t]_{\mathbb{T}}$ obtain

$$\frac{x(t)}{e_{-p}(t, a)} - \frac{x(a)}{e_{-p}(a, a)} = \int_a^t h(s) e_{-p}(a, \sigma(s)) \Delta s.$$

So,

$$x(t) = x_0 e_{-p}(t, a) + \int_a^t e_{-p}(t, \sigma(s)) h(s) \Delta s. \quad (2.5)$$

□

As a direct consequence of the previous result with $p(t) = M > 0$, for all $t \in I = [a, b]_{\mathbb{T}}$, we deduce the following expression of the initial problem.

Corollary 2.1.2 *The initial problem*

$$\begin{cases} x^\Delta(t) + M x(t) = h(t), & t \in I = [a, b]_{\mathbb{T}}, \\ x(a) = x_0, \end{cases} \quad (2.6)$$

with $M > 0$ and $h \in C(I, \mathbb{R})$, has a unique solution $x \in C(J, \mathbb{R})$ given by:

$$x(t) = x_0 e_{-M}(t, a) + \int_a^t e_{-M}(t, \sigma(s)) h(s) \Delta s, \quad t \in J. \quad (2.7)$$

In the next Lemmas, we discuss comparison results for the linear problem (2.6).

Lemma 2.1.3 *Let $M > 0$, $-M \in \mathcal{R}_\mu^+$ and $x \in C^1(J = [a, \sigma(b)]_{\mathbb{T}}, \mathbb{R})$, such that:*

$$\begin{cases} x^\Delta(t) + M x(t) \leq 0, & t \in [a, b]_{\mathbb{T}}, \\ x(a) \leq 0. \end{cases} \quad (2.8)$$

Then $x(t) \leq 0$, for every $t \in J$.

Proof. we put $x^\Delta(t) + M x(t) = h(t)$ and $x(a) = x_0 \leq 0$. We are know that $h(t) \leq 0$, for every $t \in [a, b]_{\mathbb{T}}$ and

$$\begin{cases} x^\Delta(t) + M x(t) = h(t), & t \in [a, b]_{\mathbb{T}}, \\ x(a) = x_0 \leq 0. \end{cases} \quad (2.9)$$

By Corollary 2.1.2, the expression of $x(t)$ is given by (2.7):

$$x(t) = x_0 e_{-M}(t, a) + \int_a^t e_{-M}(t, \sigma(s)) h(s) \Delta s,$$

we can conclude that, $x(t) \leq 0$ for every $t \in J$. □

Lemma 2.1.4 *Let $M > 0$, $-M \in \mathcal{R}_\mu^+$ and $x \in C^1(J = [a, \sigma(b)]_{\mathbb{T}}, \mathbb{R})$, such that:*

$$\begin{cases} x^\Delta(t) + M x(t) \geq 0, & t \in [a, b]_{\mathbb{T}}, \\ x(a) \geq 0. \end{cases} \quad (2.10)$$

Then $x(t) \geq 0$, for every $t \in J$.

Proof. we put $x^\Delta(t) + M x(t) = h(t)$ and $x(a) = x_0 \geq 0$. We are know that $h(t) \geq 0$, for every $t \in [a, b]_{\mathbb{T}}$ and

$$\begin{cases} x^\Delta(t) + M x(t) = h(t), & t \in [a, b]_{\mathbb{T}}, \\ x(a) = x_0 \geq 0. \end{cases} \quad (2.11)$$

By Corollary 2.1.2, the expression of $x(t)$ is given by (2.7):

$$x(t) = x_0 e_{-M}(t, a) + \int_a^t e_{-M}(t, \sigma(s)) h(s) \Delta s,$$

we can conclude that, $x(t) \geq 0$ for every $t \in J$. $\square$

Now we consider the following linear first-order initial Value problem:

$$\begin{cases} x^\Delta(t) + M x(t) = h(t), & t \in I = [a, b]_{\mathbb{T}}, \\ x(a) = \int_a^c \varphi(s) x(s) \Delta s + \int_c^{\sigma(b)} \varphi(s) x(s) \Delta s, \end{cases} \quad (2.12)$$

where $\varphi : [a, \sigma(b)]_{\mathbb{T}} \rightarrow \mathbb{R}$ is continuous such that

$\varphi(t) \leq 0$, $t \in [a, c]_{\mathbb{T}}$ and $\varphi(t) \geq 0$, $t \in [c, \sigma(b)]_{\mathbb{T}}$ with $a < c < \sigma(b)$ and $c \in [a, \sigma(b)]_{\mathbb{T}}$.

We have the following result:

Lemma 2.1.5 *Let $M > 0$, $-M \in \mathcal{R}_\mu^+$ and $x \in C^1([a, \sigma(b)]_{\mathbb{T}}, \mathbb{R})$, such that:*

$$\begin{cases} x^\Delta(t) + M x(t) \leq 0, & t \in [a, b]_{\mathbb{T}}, \\ x(a) \leq \int_a^c \varphi(s) x(s) \Delta s + \int_c^{\sigma(b)} \varphi(s) x(s) \Delta s, \end{cases} \quad (2.13)$$

and

$$-\int_a^c \varphi(s) e_{-M}(s, a) \Delta s + \int_c^{\sigma(b)} \varphi(s) e_{-M}(s, a) \Delta s < 1. \quad (2.14)$$

Then $x(t) \leq 0$, for all $t \in [a, \sigma(b)]_{\mathbb{T}}$.

Proof. we put $x^\Delta(t) + M x(t) = h(t)$. We are know that $h(t) \leq 0$, for every $t \in [a, b]_{\mathbb{T}}$. By Corollary 2.1.2, we have

$$x(t) = x(a) e_{-M}(t, a) + \int_a^t e_{-M}(t, \sigma(s)) h(s) \Delta s,$$

we can conclude that,

$$x(t) \leq x(a) e_{-M}(t, a) \text{ for every } t \in J. \quad (2.15)$$

Then by the second inequality in (2.13), one has

$$x(a) \leq -x(a) \int_a^c \varphi(s) e_{-M}(s, a) \Delta s + x(a) \int_c^{\sigma(b)} \varphi(s) e_{-M}(s, a) \Delta s.$$

That

$$x(a) \left(1 + \int_a^c \varphi(s) e_{-M}(s, a) \Delta s - \int_c^{\sigma(b)} \varphi(s) e_{-M}(s, a) \Delta s \right) \leq 0.$$

Then by this last inequality and (2.14), we obtain:

$$x(a) \leq 0,$$

and consequently by (2.15), it follows that

$$x(t) \leq 0, \text{ for all } t \in [a, \sigma(b)]_{\mathbb{T}}.$$

□

2.2 Main Results

In this section, we prove the existence of quasi-solutions for the first-order dynamic equation involving integral boundary condition (2.1). Let us define what we mean by a solution of this problem.

Definition 2.2.1 *A solution of problem (2.1) will be a function $x \in C^1([a, \sigma(b)]_{\mathbb{T}}, \mathbb{R})$ for which (2.1) is satisfied.*

We introduce the notion of quasi-solutions of this problem.

Definition 2.2.2 *The pair of functions $(x_*, x^*) \in (C^1([a, \sigma(b)]_{\mathbb{T}}, \mathbb{R}))^2$ are called quasi-solutions of the problem (2.1), if the following inequalities hold:*

$$\begin{cases} x_*^\Delta(t) = f(t, x_*(t)), t \in I = [a, b]_{\mathbb{T}}, x_*(a) = \int_a^c g(s) x^*(s) \Delta s + \int_c^{\sigma(b)} g(s) x_*(s) \Delta s, \\ x^{*\Delta}(t) = f(t, x^*(t)), t \in I = [a, b]_{\mathbb{T}}, x^*(a) = \int_a^c g(s) x_*(s) \Delta s + \int_c^{\sigma(b)} g(s) x^*(s) \Delta s. \end{cases} \quad (2.16)$$

Next, we introduce the concept of coupled lower and upper solutions of this problem as follows.

Definition 2.2.3 We say that $(\underline{x}, \bar{x}) \in (C^1([a, \sigma(b)]_{\mathbb{T}}, \mathbb{R}))^2$ is a pair of coupled lower and upper solutions of the problem (2.1), if $\underline{x} \leq \bar{x}$ in $J = [a, \sigma(b)]_{\mathbb{T}}$ and the following inequalities hold:

$$\begin{cases} \underline{x}^\Delta(t) \leq f(t, \underline{x}(t)), t \in I = [a, b]_{\mathbb{T}}, \underline{x}(a) \leq \int_a^c g(s) \bar{x}(s) \Delta s + \int_c^{\sigma(b)} g(s) \underline{x}(s) \Delta s, \\ \bar{x}^\Delta(t) \geq f(t, \bar{x}(t)), t \in I = [a, b]_{\mathbb{T}}, \bar{x}(a) \geq \int_a^c g(s) \underline{x}(s) \Delta s + \int_c^{\sigma(b)} g(s) \bar{x}(s) \Delta s. \end{cases} \quad (2.17)$$

We assume the following hypothesis:

(H_1) : $f : I = [a, b]_{\mathbb{T}} \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : J = [a, \sigma(b)]_{\mathbb{T}} \rightarrow \mathbb{R}$ are continuous functions, with $g(t) \leq 0, t \in [a, c]_{\mathbb{T}}$ and $g(t) \geq 0, t \in [c, \sigma(b)]_{\mathbb{T}}$ with $a < c < \sigma(b)$ and $c \in J$.

(H_2) : There exists $(\underline{x}, \bar{x}) \in (C^1(J, \mathbb{R}))^2$, a pair of coupled lower and upper solutions of the problem (2.1) such that $\underline{x} \leq \bar{x}$ in J .

(H_3) : There exists a real constant $M > 0$ such that $x \mapsto f(t, x) + Mx$ is increasing in x on $[\underline{x}, \bar{x}]$. i.e.,

$$f(t, x_2) - f(t, x_1) \geq -M(x_2 - x_1), \text{ where } \underline{x} \leq x_1 \leq x_2 \leq \bar{x}, \text{ for all } t \in J.$$

We define the sector $[\underline{x}, \bar{x}] = \{x \in C^1(J, \mathbb{R}) : \underline{x}(t) \leq x(t) \leq \bar{x}(t), t \in J\}$.

We have the following result.

Theorem 2.2.4 Assume that (H_1) , (H_2) and (H_3) hold. Then the problem (2.1) admits a pair of quasi-solutions (x_*, x^*) such that

$$\underline{x} \leq x_* \leq x^* \leq \bar{x} \text{ in } J.$$

Proof. We take $x_0 = \underline{x}$, $x_1 = \bar{x}$, and define the sequence $(x_n)_{n \geq 0}$ in the following way:

$$\begin{cases} x_{n+2}^\Delta(t) + Mx_{n+2}(t) = y_n(t), t \in [a, b]_{\mathbb{T}}, \\ x_{n+2}(a) = \int_a^c g(s) x_{n+1}(s) \Delta s + \int_c^{\sigma(b)} g(s) x_n(s) \Delta s, \end{cases} \quad (2.18)$$

where

$$y_n(t) = f(t, x_n(t)) + Mx_n(t). \quad (2.19)$$

Claim 1: For all $n \in \mathbb{N}$, we have

$$x_{2n} \leq x_{2n+2} \leq x_{2n+3} \leq x_{2n+1} \text{ in } [a, \sigma(b)]_{\mathbb{T}}.$$

Let

$$w_0(t) := x_2(t) - x_0(t), \quad t \in [a, \sigma(b)]_{\mathbb{T}}.$$

By (2.18) and using the Definition 2.2.3, we have

$$\begin{cases} w_0^\Delta(t) + Mw_0(t) \geq 0, t \in [a, b]_{\mathbb{T}}, \\ w_0(a) \geq 0. \end{cases}$$

Then by Lemma 2.1.4, we have

$$w_0(t) \geq 0 \text{ for all } t \in [a, \sigma(b)]_{\mathbb{T}}.$$

That is

$$x_0 \leq x_2 \text{ in } [a, \sigma(b)]_{\mathbb{T}}. \quad (2.20)$$

Similarly, we can prove that

$$x_3 \leq x_1 \text{ in } [a, \sigma(b)]_{\mathbb{T}}. \quad (2.21)$$

Now, we put by definition

$$w_1(t) = x_2(t) - x_1(t), \quad t \in [a, \sigma(b)]_{\mathbb{T}},$$

and

$$z_1(t) = x_1(t) - x_0(t), \quad t \in [a, \sigma(b)]_{\mathbb{T}}.$$

By (2.18), we have

$$\begin{cases} w_1^\Delta(t) + Mw_1(t) \leq y_0(t) - y_1(t), \quad t \in [a, b]_{\mathbb{T}}, \\ w_1(a) \leq \int_a^c g(s)z_1(s)\Delta s - \int_c^{\sigma(b)} g(s)z_1(s)\Delta s. \end{cases}$$

Since $x_0 = \underline{x} \leq \bar{x} = x_1$ in $[a, \sigma(b)]_{\mathbb{T}}$ and using the hypothesis (H_3) , we obtain

$$\begin{cases} w_1^\Delta(t) - Mw_1(t) \leq 0, \quad t \in [a, b]_{\mathbb{T}}, \\ w_1(a) \leq 0. \end{cases}$$

Then by Lemma 2.1.3, we have

$$w_1(t) \leq 0, \text{ for all } t \in [a, \sigma(b)]_{\mathbb{T}}.$$

That is

$$x_2 \leq x_1 \text{ in } [a, \sigma(b)]_{\mathbb{T}}. \quad (2.22)$$

Now we are going to prove that

$$x_2 \leq x_3 \text{ in } [a, \sigma(b)]_{\mathbb{T}}.$$

For this we put by definition

$$w_3(t) = x_2(t) - x_3(t), t \in [a, \sigma(b)]_{\mathbb{T}}.$$

By (2.18), we have

$$\begin{cases} w_3^\Delta(t) + Mw_3(t) = y_0(t) - y_1(t), t \in [a, b]_{\mathbb{T}}, \\ w_3(a) = - \int_a^c g(s)w_1(s)\Delta s - \int_c^{\sigma(b)} g(s)z_1(s)\Delta s. \end{cases}$$

Since $x_0 \leq x_2 \leq x_1$ in $[a, \sigma(b)]_{\mathbb{T}}$ and using the hypothesis (H_3) , we obtain

$$\begin{cases} w_3^\Delta(t) + Mw_3(t) \leq 0, t \in [a, b]_{\mathbb{T}}, \\ w_3(a) \leq 0. \end{cases}$$

Then by Lemma 2.1.3, we have

$$w_3(t) \leq 0, \text{ for all } t \in [a, \sigma(b)]_{\mathbb{T}}.$$

That is

$$x_2 \leq x_3 \text{ in } [a, \sigma(b)]_{\mathbb{T}}. \tag{2.23}$$

Then by (2.20) – (2.23), we have

$$x_0 \leq x_2 \leq x_3 \leq x_1 \text{ in } [a, \sigma(b)]_{\mathbb{T}}.$$

Assume for fixed $n \geq 1$, we have

$$x_{2n} \leq x_{2n+2} \leq x_{2n+3} \leq x_{2n+1} \text{ in } [a, \sigma(b)]_{\mathbb{T}},$$

and we show that

$$x_{2n+2} \leq x_{2n+4} \leq x_{2n+5} \leq x_{2n+3} \text{ in } [a, \sigma(b)]_{\mathbb{T}}.$$

We put by definition

$$w_{n+1}(t) := x_{2n+4}(t) - x_{2n+2}(t), t \in [a, \sigma(b)]_{\mathbb{T}},$$

and

$$z_{n+1}(t) := x_{2n+3}(t) - x_{2n+1}(t), t \in [a, \sigma(b)]_{\mathbb{T}}.$$

By (2.18), we have

$$\begin{cases} w_{n+1}^\Delta(t) + Mw_{n+1}(t) = y_{2n+2}(t) - y_{2n}(t), t \in [a, b]_{\mathbb{T}}, \\ w_{n+1}(a) = \int_a^c g(s)z_{n+1}(s)\Delta s + \int_c^{\sigma(b)} g(s)w_n(s)\Delta s. \end{cases}$$

Since by the hypothesis of recurrence, we have $x_{2n} \leq x_{2n+2} \leq x_{2n+3} \leq x_{2n+1}$ in $[a, \sigma(b)]_{\mathbb{T}}$ and by using the hypothesis (H_3) , we obtain

$$\begin{cases} w_{n+1}^\Delta(t) + Mw_{n+1}(t) \geq 0, t \in [a, b]_{\mathbb{T}}, \\ w_{n+1}(a) \geq 0. \end{cases}$$

Then by Lemma 2.1.4, we have

$$w_{n+1}(t) \geq 0, \text{ for all } t \in [a, \sigma(b)]_{\mathbb{T}}.$$

That is

$$x_{2n+2} \leq x_{2n+4}, \text{ in } [a, \sigma(b)]_{\mathbb{T}}. \quad (2.24)$$

Similarly, we can prove that

$$x_{2n+5} \leq x_{2n+3}, \text{ in } [a, \sigma(b)]_{\mathbb{T}}. \quad (2.25)$$

$$x_{2n+4} \leq x_{2n+3}, \text{ in } [a, \sigma(b)]_{\mathbb{T}}. \quad (2.26)$$

and

$$x_{2n+4} \leq x_{2n+5}, \text{ in } [a, \sigma(b)]_{\mathbb{T}}. \quad (2.27)$$

Then by (2.24) – (2.27) one has

$$x_{2n+2} \leq x_{2n+4} \leq x_{2n+5} \leq x_{2n+3} \text{ in } [a, \sigma(b)]_{\mathbb{T}}.$$

Hence for all $n \in \mathbb{N}$, we have

$$x_{2n} \leq x_{2n+2} \leq x_{2n+3} \leq x_{2n+1} \text{ in } [a, \sigma(b)]_{\mathbb{T}}.$$

Claim 2: The sequences of functions $(x_{2n})_{n \in \mathbb{N}}$ and $(x_{2n+1})_{n \in \mathbb{N}}$ converge to quasi-solutions of (2.1).

By Claim 1, the sequences $(x_{2n})_{n \in \mathbb{N}}$ and $(x_{2n+1})_{n \in \mathbb{N}}$ are monotone and bounded, hence the sequences of functions $(x_{2n})_{n \in \mathbb{N}}$ and $(x_{2n+1})_{n \in \mathbb{N}}$ converge to x_* and x^* , i.e

$$\lim_{n \rightarrow \infty} x_{2n} = x_*, \quad \lim_{n \rightarrow \infty} x_{2n+1} = x^*.$$

Next, we prove that (x_*, x^*) is a pair of quasi-solutions to (2.1).

Let $n \in \mathbb{N}^*$ and $t \in [a, b]_{\mathbb{T}}$, we have

$$x_{2n}(t) = \int_a^c g(s)x_{2n-1}(s)\Delta s + \int_c^{\sigma(b)} g(s)x_{2n-2}(s)\Delta s + \int_a^t \tilde{y}_n(s)\Delta s$$

and

$$x_{2n+1}(t) = \int_0^c g(s)x_{2n}(s)\Delta s + \int_c^{\sigma(b)} g(s)x_{2n-1}(s)\Delta s + \int_a^t \widehat{y}_n(s)ds$$

where

$$\widetilde{y}_n(s) := f(s, x_{2n-2}(s)) + M(x_{2n-2}(s) - x_{2n}(s)),$$

and

$$\widehat{y}_n(s) := f(s, x_{2n-1}(s)) + M(x_{2n-1}(s) - x_{2n+1}(s)).$$

Now, as n tends to $+\infty$, we obtain

$$\widetilde{y}_n(s) \rightarrow f(s, x_*(s)),$$

and

$$\widehat{y}_n(s) \rightarrow f(s, x^*(s)).$$

Also, we have

$$\exists c_1 > 0, \forall n \in \mathbb{N}, \forall s \in [a, b]_{\mathbb{T}}, |\widetilde{y}_n(s)| \leq c_1,$$

and

$$\exists c_2 > 0, \forall n \in \mathbb{N}, \forall s \in [a, b]_{\mathbb{T}}, |\widehat{y}_n(s)| \leq c_2.$$

Hence, the dominated convergence theorem of Lebesgue implies that

$$x_*(t) = \int_a^c g(s)x^*(s)\Delta s + \int_c^{\sigma(b)} g(s)x_*(s)\Delta s + \int_a^t f(s, x_*(s))\Delta s, \quad (2.28)$$

and

$$x^*(t) = \int_a^c g(s)x_*(s)\Delta s + \int_c^{\sigma(b)} g(s)x^*(s)\Delta s + \int_a^t f(s, x^*(s))\Delta s. \quad (2.29)$$

Now we are going to prove that (x_*, x^*) is a pair of quasi-solutions of (2.1).

First, it is not difficult to see that

$$\begin{cases} x_*(a) = \int_a^c g(s)x^*(s)\Delta s + \int_c^{\sigma(b)} g(s)x_*(s)\Delta s, \\ x^*(a) = \int_a^c g(s)x_*(s)\Delta s + \int_c^{\sigma(b)} g(s)x^*(s)\Delta s. \end{cases} \quad (2.30)$$

On the other hand since f is continuous, $\underline{x} \leq x_* \leq \bar{x}$ and $\underline{x}$ and $\bar{x}$ are continuous, then there exists a constant $K_1 > 0$ such that for all $s \in [a, b]_{\mathbb{T}}$, we have

$$|f(s, x_*(s))| \leq K_1. \quad (2.31)$$

Then by (2.28) and (2.31), one has

$$\forall t_1 \in [a, \sigma(b)]_{\mathbb{T}}, \forall t_2 \in [a, \sigma(b)]_{\mathbb{T}}, |x_*(t_1) - x_*(t_2)| \leq K |t_1 - t_2|.$$

This implies that x_* is continuous on $[a, \sigma(b)]_{\mathbb{T}}$ and consequently by (2.28) it follows that x_* is Δ -differentiable on $[a, b]_{\mathbb{T}}$.

Similarly, we can prove that the functions x^* is Δ -differentiable on $[a, b]_{\mathbb{T}}$ and consequently by (2.28) and (2.29), it follows that

$$\begin{cases} x_*^\Delta(t) = f(t, x_*(t)), t \in [a, b]_{\mathbb{T}}, \\ x^{*\Delta}(t) = f(t, x^*(t)), t \in [a, b]_{\mathbb{T}}, \end{cases}$$

and by (2.30), it follows that (x_*, x^*) is a pair of quasi-solutions of (2.1).

The proof of Theorem 2.2.4 is complete. $\square$

2.3 An example

To illustrate our main results, we present the following example.

Example 2.3.1 *We consider the following boundary value problem:*

$$\begin{cases} x^{(\Delta)}(t) = 4t \sin(x(t)) - 11x(t) + e^{-t}, & t \in I = [0, 10]_{\mathbb{T}}, \\ x(0) = \frac{1}{35} \int_0^{\sigma(10)} (s - 5)x(s) \Delta s. \end{cases} \quad (2.32)$$

This problem is a particular case of problem (2.1), with, $a = 0, b = 10, I = [0, 10]_{\mathbb{T}}, J = [0, \sigma(10)]_{\mathbb{T}}, f(t, x) = 4t \sin(x) - 11x + e^{-t}, t \in I$ and $g(t) = \frac{1}{35}(t - 5), t \in J$.

It is clear that f and g are continuous functions, with:

$$g(t) \leq 0, t \in [0, 5]_{\mathbb{T}} \text{ and } g(t) \geq 0, t \in [5, \sigma(10)]_{\mathbb{T}}.$$

We put $(\underline{x}(t), \bar{x}(t)) = (-L, L)$, where $L > 0$ and $t \in J = [0, \sigma(10)]_{\mathbb{T}}$. $(\underline{x}, \bar{x})$ is a lower-upper solutions of (2.32) if we have

$$\begin{cases} \underline{x}^{(\Delta)}(t) \leq f(t, \underline{x}(t)), & t \in [0, 10], \\ \underline{x}(0) \leq \frac{1}{35} \int_0^5 (s - 5)\bar{x}(s) \Delta s + \frac{1}{35} \int_5^{\sigma(10)} (s - 5)\underline{x}(s) \Delta s, \\ \bar{x}^{(\Delta)}(t) \geq f(t, \bar{x}(t)), & t \in [0, 10], \\ \bar{x}(0) \geq \frac{1}{35} \int_0^5 (s - 5)\underline{x}(s) \Delta s + \frac{1}{35} \int_5^{\sigma(10)} (s - 5)\bar{x}(s) \Delta s. \end{cases}$$

That is

$$\begin{cases} 0 \leq 4t \sin(-L) + 11L + e^{-t}, & t \in [0, 10], \\ -L \leq \frac{L}{35} \int_0^5 (s-5) \Delta s - \frac{L}{35} \int_5^{\sigma(10)} (s-5) \Delta s, \\ 0 \geq 4t \sin(L) - 11L + e^{-t}, & t \in [0, 10], \\ L \geq -\frac{L}{35} \int_0^5 (s-5) \Delta s + \frac{L}{35} \int_5^{\sigma(10)} (s-5) \Delta s. \end{cases}$$

Then if we choose for example $L \geq 1$, we obtain $(\underline{x}, \bar{x})$ is a lower and upper solutions of (2.32) respectively, then assumptions (H_1) and (H_2) holds.

Let $x_1, x_2 \in \mathbb{R}$ with $\underline{x}(t) \leq x_1 \leq x_2 \leq \bar{x}(t)$, $t \in J$, then we have

$$\begin{aligned} f(t, x_2) - f(t, x_1) &= 4t (\sin(x_2) - \sin(x_1)) - 11(x_2 - x_1) \\ &\geq -4t(x_2 - x_1) - 11(x_2 - x_1) \\ &\geq -4t(x_2 - x_1) - 11(x_2 - x_1) \\ &\geq -51(x_2 - x_1). \end{aligned}$$

Hence the assumption (H_3) holds with $M = 51$.

By Theorem 2.2.4, problem (2.32) admits a pair of quasi-solutions $(x_*, x^*) \in [-L, L] \times [-L, L]$ on $[0, \sigma(10)]_{\mathbb{T}}$, which can be obtained by taking limits from the iterative sequences:

$$\begin{aligned} x_{2n}(t) &= \frac{1}{35} \int_0^5 (s-5) x_{2n-1}(s) \Delta s + \frac{1}{35} \int_5^{\sigma(10)} (s-5) x_{2n-2}(s) \Delta s + \int_0^t \tilde{y}_n(s) \Delta s, \\ t &\in J = [0, \sigma(10)]_{\mathbb{T}}, \end{aligned}$$

$$\begin{aligned} x_{2n+1}(t) &= \frac{1}{35} \int_0^5 (s-5) x_{2n}(s) \Delta s + \frac{1}{35} \int_5^{\sigma(10)} (s-5) x_{2n-1}(s) \Delta s + \int_0^t \hat{y}_n(s) ds, \\ t &\in J, \end{aligned}$$

where

$$\tilde{y}_n(s) := f(s, x_{2n-2}(s)) + 51(x_{2n-2}(s) - x_{2n}(s)),$$

and

$$\hat{y}_n(s) := f(s, x_{2n-1}(s)) + 51(x_{2n-1}(s) - x_{2n+1}(s)).$$

Case If $\mathbb{T} = \mathbb{R}$.

In this case, problem (2.32) is equivalent to the following:

$$\begin{cases} x'(t) = 4t \sin(x(t)) - 11x(t) + e^{-t}, & t \in I = [0, 10], \\ x(0) = \frac{1}{35} \int_0^{10} (s-5)x(s) ds. \end{cases} \quad (2.33)$$

We choose $L \geq 1$, we obtain $(-L, L)$ is a lower and upper solutions of (2.33) respectively, with

$$\begin{cases} 0 \leq 4t \sin(-L) + 11L + e^{-t}, & t \in [0, 10], \\ -L \leq \frac{L}{35} \int_0^5 (s-5)ds - \frac{L}{35} \int_5^{10} (s-5)ds = -0.71429L, \\ 0 \geq 4t \sin(L) - 11L + e^{-t}, & t \in [0, 10], \\ L \geq -\frac{L}{35} \int_0^5 (s-5)ds + \frac{L}{35} \int_5^{10} (s-5)ds = 0.71429L. \end{cases}$$

Assumptions $(H_1) - (H_3)$ are satisfied.

By Theorem 2.2.4, problem (2.33) admits a pair of quasi-solutions $(x_*, x^*) \in [-L, L] \times [-L, L]$ on $[0, 10]$, which can be obtained by taking limits from the iterative sequences:

$$x_{2n}(t) = \frac{1}{35} \int_0^5 (s-5)x_{2n-1}(s)ds + \frac{1}{35} \int_5^{\sigma(10)} (s-5)x_{2n-2}(s)\Delta s + \int_0^t \tilde{y}_n(s)ds, \quad t \in [0, 10],$$

and

$$x_{2n+1}(t) = \frac{1}{35} \int_0^5 (s-5)x_{2n}(s)\Delta s + \frac{1}{35} \int_5^{\sigma(10)} (s-5)x_{2n-1}(s)\Delta s + \int_0^t \hat{y}_n(s)ds, \quad t \in I.$$

Chapter 3

Existence of solutions for nonlinear conformable fractional differential equations with integral boundary conditions

In this chapter, we prove the existence of extremal solutions for the following nonlinear conformable fractional differential equations with integral boundary conditions:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t)), & t \in]0, 1[, \\ x(0) = -rx(1) + \lambda \int_0^1 x(s)ds, \end{cases} \quad (3.1)$$

where $0 < \alpha \leq 1$, $r > 0$, $\lambda > 0$, $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at $t \in]0, 1[$ of order α . The existence of solutions for this problem is proved by using the monotone iterative technique and the method of coupled upper and lower solution. The original results of this chapter are found in [21].

In [20] T. Jankowski studied the existence of extremal solutions to the following nonlinear ordinary differential equations with integral boundary conditions:

$$\begin{cases} x'(t) = f(t, x(t)), & t \in I = [0, T], \quad T > 0 \\ x(0) = \lambda \int_0^T x(t)dt + d, \end{cases} \quad (3.2)$$

where $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $d \in \mathbb{R}$ and $\lambda = -1$ or $\lambda = 1$.

M. Benchohra et al. in [5], studied the existence of solutions to the following

nonlinear fractional differential equations with integral boundary conditions:

$$\begin{cases} {}^c D^\alpha y(t) = f(t, y(t)), & t \in J = [0, T], \\ y(0) + \mu \int_0^T y(s) ds = y(T), \end{cases} \quad (3.3)$$

where $0 < \alpha \leq 1$, $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function satisfying some assumptions, ${}^c D^\alpha$ is the Caputo fractional derivative and $\mu \in \mathbb{R}^*$.

3.1 Linear problem and comparison principle

In this section, we study the expression of the solutions of a linear conformable fractional differential equation with initial value conditions.

Lemma 3.1.1 *Let $0 < \alpha \leq 1$, $M > 0$, $m \in \mathbb{R}$ and $g \in C([0, 1], \mathbb{R})$. The function $y : [0, 1] \rightarrow \mathbb{R}$ defined by*

$$y(t) = me^{-\frac{M}{\alpha}t^\alpha} + \int_0^t s^{\alpha-1} g(s) e^{\frac{M}{\alpha}(s^\alpha - t^\alpha)} ds. \quad (3.4)$$

is a solution of the initial value problem:

$$\begin{cases} y^{(\alpha)}(t) + My(t) = g(t), & t \in]0, 1], \\ y(0) = m. \end{cases} \quad (3.5)$$

Proof. Assume that $y(t)$ is given by (3.4), then y is differentiable for $t > 0$, therefore we have

$$\begin{aligned} y^{(\alpha)}(t) &= t^{1-\alpha} y'(t) = t^{1-\alpha} \left(me^{-\frac{M}{\alpha}t^\alpha} + e^{-\frac{M}{\alpha}t^\alpha} \int_0^t s^{\alpha-1} g(s) e^{\frac{M}{\alpha}s^\alpha} ds \right)' \\ &= t^{1-\alpha} \left(-\frac{M}{\alpha} \alpha t^{\alpha-1} me^{-\frac{M}{\alpha}t^\alpha} - Me^{-\frac{M}{\alpha}t^\alpha} t^{\alpha-1} \int_0^t s^{\alpha-1} g(s) e^{\frac{M}{\alpha}s^\alpha} ds + e^{-\frac{M}{\alpha}t^\alpha} t^{\alpha-1} g(t) e^{\frac{M}{\alpha}t^\alpha} \right) \\ &= t^{1-\alpha} t^{\alpha-1} \left(-Mme^{-\frac{M}{\alpha}t^\alpha} - Me^{-\frac{M}{\alpha}t^\alpha} \int_0^t s^{\alpha-1} g(s) e^{\frac{M}{\alpha}s^\alpha} ds + g(t) \right) \\ &= -M \left(me^{-\frac{M}{\alpha}t^\alpha} + \int_0^t s^{\alpha-1} g(s) e^{\frac{M}{\alpha}(s^\alpha - t^\alpha)} ds \right) + g(t) \\ &= -My(t) + g(t), \end{aligned}$$

from Theorem 1.2.3, and $y(t)$ subject to the condition

$$y(0) = m.$$

□

In the next Lemmas, we discuss comparison results for the linear problem (3.5)

Lemma 3.1.2 *Let $y \in C([0, 1], \mathbb{R})$ satisfy*

$$\begin{cases} y^{(\alpha)}(t) + My(t) \leq 0, & t \in [0, 1], \\ y(0) \leq 0, \end{cases}$$

where $0 < \alpha \leq 1$, $M > 0$, then $y(t) \leq 0$, for all $t \in [0, 1]$.

Proof. we put $y^{(\alpha)}(t) + My(t) = g(t)$ and $y(0) = m \leq 0$. We know that $g(t) \leq 0$, for every $t \in I = [0, 1]$ and

$$\begin{cases} y^{(\alpha)}(t) + My(t) = g(t), & t \in [0, 1], \\ y(0) = m \leq 0. \end{cases} \quad (3.6)$$

By Lemma 3.1.1, the expression of $y(t)$ is:

$$y(t) = me^{-\frac{M}{\alpha}t^\alpha} + \int_0^t s^{\alpha-1} g(s) e^{\frac{M}{\alpha}(s^\alpha - t^\alpha)} ds$$

we can conclude that, $y(t) \leq 0$ for every $t \in [0, 1]$. □

3.2 Main Result

In this section, we prove the existence of extremal solutions for problem (3.1). Let us defining what we mean by a solution of this problem.

Definition 3.2.1 *A solution of problem (3.1) will be a function $x \in C([0, 1], \mathbb{R})$ for which (3.1) is satisfied.*

We introduce the notion of coupled solutions of this problem.

Definition 3.2.2 *The function pair $(y, z) \in (C([0, 1], \mathbb{R}))^2$ is said to be coupled solutions of (3.1), if the following inequalities hold:*

$$\begin{cases} y^{(\alpha)}(t) = f(t, y(t)), & t \in]0, 1[, \quad y(0) + rz(1) = \lambda \int_0^1 y(s) ds, \\ z^{(\alpha)}(t) = f(t, z(t)), & t \in]0, 1[, \quad z(0) + ry(1) = \lambda \int_0^1 z(s) ds. \end{cases} \quad (3.7)$$

Let $\gamma, \delta \in C([0, 1], \mathbb{R})$, then (γ, δ) is said to be minimum and maximum coupled solutions (are extremal solutions) of (3.1), respectively, if (γ, δ) are coupled solutions of (3.1), and $\gamma(t) \leq y(t), z(t) \leq \delta(t)$ for any coupled solution (y, z) of (3.1).

Next, we introduce the concept of coupled lower and upper solutions of this problem as follows.

Definition 3.2.3 We say that $(y_0, z_0) \in (C([0, 1], \mathbb{R}))^2$ is a pair of coupled lower and upper solutions of the problem (3.1), respectively, if $y_0(t) \leq z_0(t)$ for all $t \in [0, 1]$ and the following inequalities hold:

$$\begin{cases} y_0^{(\alpha)}(t) \leq f(t, y_0(t)), & t \in]0, 1[, \quad y_0(0) + rz_0(1) \leq \lambda \int_0^1 y_0(s)ds, \\ z_0^{(\alpha)}(t) \geq f(t, z_0(t)), & t \in]0, 1[, \quad z_0(0) + ry_0(1) \geq \lambda \int_0^1 z_0(s)ds. \end{cases} \quad (3.8)$$

We define the sector:

$$\mathcal{D} = [y_0, z_0] = \{x \in C([0, 1], \mathbb{R}) : y_0(t) \leq x(t) \leq z_0(t), \quad t \in I = [0, 1]\}.$$

We assume the following hypothesis:

(H₁) $f : I = [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous function.

(H₂) There exists $(y_0, z_0) \in (C([0, 1], \mathbb{R}))^2$ a pair of coupled lower and upper solutions of (3.1), with $y_0(t) \leq z_0(t)$ for $t \in I = [0, 1]$.

(H₃) There exist $M > 0$ such that

$$f(t, x_1) - f(t, x_2) \geq -M(x_1 - x_2), \quad (3.9)$$

where $y_0(t) \leq x_2 \leq x_1 \leq z_0(t)$, for all $t \in I$.

Now, We have the following results.

Theorem 3.2.4 Assume that (H₁), (H₂) and (H₃) hold. If we take $y_0(t), z_0(t)$ as initial elements, the iterative sequences defined by:

$$\begin{cases} y_n(t) = \left(\lambda \int_0^1 y_{n-1}(s)ds - rz_{n-1}(1) \right) e^{\frac{-M}{\alpha}t^\alpha} + \int_0^t s^{\alpha-1} f_{y_{n-1}}(s) e^{\frac{M}{\alpha}(s^\alpha - t^\alpha)} ds, & t \in I, \\ z_n(t) = \left(\lambda \int_0^1 z_{n-1}(s)ds - ry_{n-1}(1) \right) e^{\frac{-M}{\alpha}t^\alpha} + \int_0^t s^{\alpha-1} f_{z_{n-1}}(s) e^{\frac{M}{\alpha}(s^\alpha - t^\alpha)} ds, & t \in I, \end{cases} \quad (3.10)$$

are $\{y_n(t)\}$ and $\{z_n(t)\}$, then

- 1) $y_n(t) \rightarrow y^*(t)$ and $z_n(t) \rightarrow z^*(t)$ converging uniformly and $y^*, z^* \in \mathcal{D}$;
- 2) (y^*, z^*) are coupled minimal and maximal solutions of (3.1) respectively in $\mathcal{D}$;
- 3) If $x(t)$ is the solution of (3.1) in $\mathcal{D}$, then we have $y^* \leq x \leq z^*$; i.e.,

$$y^*(t) \leq x(t) \leq z^*(t), \quad \text{for } t \in I = [0, 1].$$

Proof. 1). There is a unique solution to the boundary value problem as follows

$$\begin{cases} y^{(\alpha)}(t) = f(t, u(t)) - M(y(t) - u(t)), \quad t \in]0, 1[, \text{ and } y(0) + rv(1) = \lambda \int_0^1 u(s)ds \\ z^{(\alpha)}(t) = f(t, v(t)) - M(z(t) - v(t)), \quad t \in]0, 1[, \quad z(0) + ru(1) = \lambda \int_0^1 v(s)ds. \end{cases}$$

which is given by

$$\begin{cases} y(t) = \left(\lambda \int_0^1 u(s)ds - rv(1) \right) e^{\frac{-M}{\alpha}t^\alpha} + \int_0^t s^{\alpha-1} f_u(s) e^{\frac{M}{\alpha}(s^\alpha - t^\alpha)} ds, \quad t \in I, \\ z(t) = \left(\lambda \int_0^1 v(s)ds - ru(1) \right) e^{\frac{-M}{\alpha}t^\alpha} + \int_0^t s^{\alpha-1} f_v(s) e^{\frac{M}{\alpha}(s^\alpha - t^\alpha)} ds, \quad t \in I, \end{cases}$$

for $u, v \in \mathcal{D}$, and $u \leq v$ from Theorem 1.2.3 and Lemma 3.1.1, where $f_v(t) = f(t, v(t)) + Mv(t)$, $f_u(t) = f(t, u(t)) + Mu(t)$.

Define operator $\mathcal{T} : \mathcal{D} \times \mathcal{D} \rightarrow C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R})$

$$\mathcal{T}(u, v)(t) = (\mathcal{T}_1(u, v)(t), \mathcal{T}_2(u, v)(t)),$$

where operators $\mathcal{T}_1, \mathcal{T}_2$ are given by

$$\begin{cases} \mathcal{T}_1(u, v)(t) = \left(\lambda \int_0^1 u(s)ds - rv(1) \right) e^{\frac{-M}{\alpha}t^\alpha} + \int_0^t s^{\alpha-1} f_u(s) e^{\frac{M}{\alpha}(s^\alpha - t^\alpha)} ds, \quad t \in I, \\ \mathcal{T}_2(u, v)(t) = \left(\lambda \int_0^1 v(s)ds - ru(1) \right) e^{\frac{-M}{\alpha}t^\alpha} + \int_0^t s^{\alpha-1} f_v(s) e^{\frac{M}{\alpha}(s^\alpha - t^\alpha)} ds, \quad t \in I, \end{cases}$$

respectively. Then the fixed point of operator $\mathcal{T}$ in $\mathcal{D} \times \mathcal{D}$ means the coupled solutions of (3.1).

Let $y_1 = \mathcal{T}_1(y_0, z_0)$, $z_1 = \mathcal{T}_2(y_0, z_0)$.

Here we prove that $y_0 \leq y_1$; $z_1 \leq z_0$, $y_1 \leq z_1$, and y_1, z_1 are coupled lower and upper solutions of (3.1).

Whereas

$$\begin{cases} y_1^{(\alpha)}(t) = f(t, y_0(t)) - M(y_1(t) - y_0(t)), \quad t \in]0, 1[, \text{ and } y_1(0) + rz_0(1) = \lambda \int_0^1 y_0(s)ds. \\ z_1^{(\alpha)}(t) = f(t, z_0(t)) - M(z_1(t) - z_0(t)), \quad t \in]0, 1[, \quad z_1(0) + ry_0(1) = \lambda \int_0^1 z_0(s)ds. \end{cases} \quad (3.11)$$

And y_0, z_0 are coupled lower and upper solutions of (3.1), then we have

$$\begin{cases} \left(y_0^{(\alpha)}(t) - y_1^{(\alpha)}(t) \right) + M (y_0(t) - y_1(t)) \leq 0, & t \in]0, 1[, \text{ and } y_0(0) - y_1(0) \leq 0, \\ \left(z_1^{(\alpha)}(t) - z_0^{(\alpha)}(t) \right) + M (z_1(t) - z_0(t)) \leq 0, & t \in]0, 1[, \text{ and } z_1(0) - z_0(0) \leq 0. \end{cases} \quad (3.12)$$

Let $p = y_0 - y_1$ and $q = z_1 - z_0$, then (3.12) is equivalent to the following:

$$\begin{cases} p^{(\alpha)}(t) + M p(t) \leq 0, & t \in]0, 1[, \text{ and } p(0) \leq 0, \\ q^{(\alpha)}(t) + M q(t) \leq 0, & t \in]0, 1[, \text{ and } q(0) \leq 0. \end{cases}$$

By Lemma 3.1.2, we know that $p(t) \leq 0, q(t) \leq 0$, for all $t \in I$, i.e.,

$$y_0(t) \leq y_1(t), \quad z_1(t) \leq z_0(t), \quad t \in [0, 1].$$

So we can easily get that

$$\begin{cases} y_1^{(\alpha)}(t) = f(t, y_0(t)) - M (y_1(t) - y_0(t)) \leq f(t, y_1(t)), & t \in]0, 1[, \\ y_1(0) + r z_1(1) \leq \lambda \int_0^1 y_1(s) ds. \\ z_1^{(\alpha)}(t) = f(t, z_0(t)) - M (z_1(t) - z_0(t)) \geq f(t, z_1(t)), & t \in]0, 1[, \\ z_1(0) + r y_1(1) \geq \lambda \int_0^1 z_1(s) ds. \end{cases}$$

From formula (3.9) and (3.11). i.e., (y_1, z_1) are coupled lower and upper solutions of (3.1). We also get that

$$\begin{cases} y_1^{(\alpha)}(t) - z_1^{(\alpha)}(t) \leq -M (y_1(t) - z_1(t)), & t \in]0, 1[, \\ y_1(0) - z_1(0) = \lambda \int_0^1 (y_0(s) - z_0(s)) ds + r(y_0(1) - z_0(1)) \leq 0, \end{cases} \quad (3.13)$$

from formula (3.11) and $y_0 \leq z_0$.

Similarly, Let $w = y_1 - z_1$, then (3.13) is equivalent to the following:

$$\begin{cases} w^{(\alpha)}(t) + M w(t) \leq 0, & t \in]0, 1[, \\ w(0) \leq 0. \end{cases}$$

By Lemma 3.1.2, we know that $w(t) \leq 0$, for all $t \in I$, i.e.,

$$y_1(t) \leq z_1(t), \quad t \in [0, 1].$$

Let

$$y_n = \mathcal{T}_1(y_{n-1}, z_{n-1}), \quad z_n = \mathcal{T}_2(y_{n-1}, z_{n-1}),$$

then from formula (3.10), we have that y_n and z_n are coupled lower and upper solutions of (3.1) for any $n \geq 2$, which is similar to the proof above. And

$$y_{n-1} \leq y_n \leq z_n \leq z_{n-1}.$$

In summary, we have

$$y_0(t) \leq y_1(t) \leq \dots \leq y_n(t) \leq \dots \leq z_n(t) \leq \dots \leq z_1(t) \leq z_0(t), \quad \text{for } t \in [0, 1].$$

Therefore, sequences $\{y_n(t)\}$ and $\{z_n(t)\}$ are uniformly bounded, i.e.,

$$\|y_n\| \leq M_0, \quad \|z_n\| \leq M_0,$$

for $n = 0, 1, 2, \dots$ and $M_0 > 0$. Because f is continuous, we have

$$|f_{y_{n-1}}(t)| = |f(t, y_{n-1}(t)) + M y_{n-1}(t)| \leq M_1,$$

for $t \in [0, 1]$, $n = 1, 2, 3, \dots$, and $M_1 > 0$.

In addition, because that functions $e^{\frac{M}{\alpha}(s^\alpha - t^\alpha)}$ and $e^{-\frac{M}{\alpha}t^\alpha}$ are continuous, we have

$$\begin{aligned} & |y_n(t_2) - y_n(t_1)| \\ &= \left| \left(\lambda \int_0^1 y_{n-1}(s) ds - r z_{n-1}(1) \right) \left(e^{-\frac{M}{\alpha}t_2^\alpha} - e^{-\frac{M}{\alpha}t_1^\alpha} \right) \right. \\ &\quad \left. + \int_0^{t_2} s^{\alpha-1} f_{y_{n-1}}(s) e^{\frac{M}{\alpha}(s^\alpha - t_2^\alpha)} ds - \int_0^{t_1} s^{\alpha-1} f_{y_{n-1}}(s) e^{\frac{M}{\alpha}(s^\alpha - t_1^\alpha)} ds \right| \\ &\leq \left| \left(\lambda \int_0^1 y_{n-1}(s) ds - r z_{n-1}(1) \right) \right| \left| e^{-\frac{M}{\alpha}t_2^\alpha} - e^{-\frac{M}{\alpha}t_1^\alpha} \right| \\ &\quad + \left| \int_0^{t_1} s^{\alpha-1} f_{y_{n-1}}(s) \left(e^{\frac{M}{\alpha}(s^\alpha - t_2^\alpha)} - e^{\frac{M}{\alpha}(s^\alpha - t_1^\alpha)} \right) ds \right| + \left| \int_{t_1}^{t_2} s^{\alpha-1} f_{y_{n-1}}(s) e^{\frac{M}{\alpha}(s^\alpha - t_2^\alpha)} ds \right| \\ &\leq \left| \left(\lambda \int_0^1 y_{n-1}(s) ds - r z_{n-1}(1) \right) \right| \left| \left(e^{-\frac{M}{\alpha}t_2} - e^{-\frac{M}{\alpha}t_1} \right) \right| \\ &\quad + \left| \int_0^{t_1} s^{\alpha-1} f_{y_{n-1}}(s) \right| \left| e^{\frac{M}{\alpha}(s^\alpha - t_2^\alpha)} - e^{\frac{M}{\alpha}(s^\alpha - t_1^\alpha)} \right| ds + \int_{t_1}^{t_2} \left| s^{\alpha-1} f_{y_{n-1}}(s) e^{\frac{M}{\alpha}(s^\alpha - t_2^\alpha)} \right| ds \quad \text{benchoua1} \\ &\rightarrow 0 \end{aligned}$$

if $0 \leq t_1 < t_2 \leq 1$ and $t_2 \rightarrow t_1$. Hence, the sequence $\{y_n(t)\}$ is equicontinuous. We can also obtain that $\{z_n(t)\}$ is equicontinuous similarly.

In summary, by the Ascoli-Arzelà Theorem, we can prove that the sequences $\{y_n\}$ and $\{z_n\}$ are convergent because of the monotonicity of sequences, i.e., there exist two

functions y^*, z^* such that

$$\lim_{n \rightarrow \infty} \|y_n - y^*\|_\infty = 0, \quad \lim_{n \rightarrow \infty} \|z_n - z^*\|_\infty = 0.$$

and $y^*, z^* \in \mathcal{D}$. Next we take limits on both sides of (3.10), then from Lebesgue Dominated Convergence Theorem, we have

$$\mathcal{T}(y^*, z^*) = (\mathcal{T}_1(y^*, z^*)(t), (\mathcal{T}_2(y^*, z^*)) = (y^*, z^*)(t),$$

if $n \rightarrow \infty$, i.e., (y^*, z^*) are coupled solutions of (3.1).

2). Here, we prove that (y^*, z^*) are coupled minimal and maximal solutions of (3.1) respectively in $\mathcal{D}$.

Assume that (x_1, x_2) are a set of coupled solutions of (3.1), then the above problem is equivalent to prove that

$$y^* \leq x_1, \quad x_2 \leq z^*.$$

Whereas $x_1, x_2 \in \mathcal{D}$, therefore $y_0 \leq x_1 \leq x_2 \leq z_0$. Assume that $y_k \leq x_1 \leq x_2 \leq z_k$ for $k > 1$, here we prove that $y_{k+1} \leq x_1, x_2 \leq z_{k+1}$.

Consider that

$$\begin{cases} y_{k+1}^{(\alpha)}(t) = f(t, y_k(t)) - M(z_k(t) - y_k(t)), & t \in]0, 1[, \\ y_{k+1}(0) + r_1 z_k(1) = \lambda \int_0^1 y_k(s) ds. \\ z_{k+1}^{(\alpha)}(t) = f(t, z_k(t)) - M(z_{k+1}(t) - z_k(t)), & t \in]0, 1[, \\ z_{k+1}(0) + r_1 y_k(1) = \lambda \int_0^1 z_k(s) ds. \end{cases}$$

And from Definition 3.2.2, we have that

$$\begin{cases} x_1^{(\alpha)}(t) = f(t, x_1(t)), & t \in]0, 1[, \text{ and } x_1(0) + r x_2(1) = \lambda \int_0^1 x_1(s) ds. \\ x_2^{(\alpha)}(t) = f(t, x_2(t)), & t \in]0, 1[, \text{ and } x_2(0) + r x_1(1) = \lambda \int_0^1 x_2(s) ds. \end{cases}$$

Then from (3.9), we get that

$$\begin{cases} x_1^{(\alpha)}(t) - y_{k+1}^{(\alpha)}(t) + M(x_1(t) - y_{k+1}(t)) \geq 0, & t \in]0, 1[, \text{ and } x_1(0) - y_{k+1}(0) \geq 0. \\ z_{k+1}^{(\alpha)}(t) - x_2^{(\alpha)}(t) + M(z_{k+1}(t) - x_2(t)) \geq 0, & t \in]0, 1[, \text{ and } z_{k+1}(0) - x_2(0) \geq 0. \end{cases} \quad (3.14)$$

Let $\varphi = y_{k+1} - x_1$, and $\psi = x_2 - z_{k+1}$, then (3.14) is equivalent to the following:

$$\begin{cases} \varphi^{(\alpha)}(t) + M \varphi(t) \leq 0, & t \in]0, 1[, \text{ and } \varphi(0) \leq 0, \\ \psi^{(\alpha)}(t) + M \psi(t) \leq 0, & t \in]0, 1[, \text{ and } \psi(0) \leq 0. \end{cases}$$

By Lemma 3.1.2, we know that $\varphi(t) \leq 0, \psi \leq 0$, for all $t \in I$, i.e.,

$$y_{k+1}(t) \leq x_1(t), \quad x_2(t) \leq z_{k+1}(t), \quad t \in [0, 1],$$

By Mathematical Induction, we can get

$$y_n \leq x_1, \leq x_2 \leq z_n.$$

for $n = 1, 2, 3, \dots$. In addition, because of the convergence of iterative sequences, we have

$$y^*(t) \leq x_1(t), \quad x_2(t) \leq z^*(t),$$

if $n \rightarrow \infty$, i.e.,

$$y^*(t) \leq x_1(t), \quad x_2(t) \leq z^*(t),$$

for $t \in [0, 1]$. Therefore, (y^*, z^*) are coupled minimal and maximal solutions of (3.1) respectively in $\mathcal{D}$ from Definition 3.2.2.

3). Here we prove that if $x(t)$ is the solution of (3.1) in $\mathcal{D}$, then $y^* \leq x \leq z^*$. In conclusion 2) above, let $x_1(t) = x(t) = x_2(t)$, because that x is the solution of (3.1) in D , therefore, (x_1, x_2) are a set of coupled solutions of (3.1). Obviously, x subject to

$$y^* \leq x \leq z^*.$$

Thus, the proof is finished.

3.3 An example

To illustrate our main results, we present the following example.

Example 3.3.1 Consider the boundary value problem of conformable fractional differential equation:

$$\begin{cases} x^{(\frac{1}{2})}(t) = t(1 - x^2(t)) - 3x(t), & t \in]0, 1[, \\ x(0) = -\frac{1}{3}x(1) + \frac{1}{2} \int_0^1 x(s)ds. \end{cases} \quad (3.15)$$

This problem is a particular case of problem (3.1), with $\alpha = \frac{1}{2}$, $f(t, x) = t(1 - x^2(t)) - 3x(t)$, $0 < r = \frac{1}{3} < 1$ and $0 < \lambda = \frac{1}{2}$.

It is clear that f is continuous function. Take $y_0(t) = -1$ and $z_0(t) = 1$ for $t \in [0, 1]$, then

$$\begin{cases} y_0^{(\frac{1}{2})}(t) = 0 \leq f(t, y_0(t)) = 3, & t \in]0, 1[, \quad y_0(0) + rz_0(1) = -\frac{2}{3} \leq \lambda \int_0^1 y_0(s)ds = -\frac{1}{2}, \\ z_0^{(\frac{1}{2})}(t) = 0 \geq f(t, z_0(t)) = -3, & t \in]0, 1[, \quad z_0(0) + ry_0(1) = \frac{2}{3} \geq \lambda \int_0^1 z_0(s)ds = \frac{1}{2}. \end{cases}$$

So, $y_0(t) = -1$ and $z_0(t) = 1$ for $t \in [0, 1]$, are coupled lower and upper solutions of problem (3.15) with $y_0(t) = -1 \leq z_0(t) = 1$, for $t \in [0, 1]$, then assumptions (H_1) and

(H_2) holds.

Let $x_1, x_2 \in \mathbb{R}$, then we have:

$$\begin{aligned} f(t, x_1) - f(t, x_2) &= t(1 - x_1^2) - 3x_1 - t(1 - x_2^2) + 3x_2 \\ &= -(t(x_1 + x_2) + 3)(x_1 - x_2) \\ &\geq -5(x_1 - x_2), \end{aligned}$$

with $-1 = y_0(t) \leq x_2 \leq x_1 \leq y_0(t) = 1$, for all $t \in [0, 1]$.

Hence the assumption (H_3) holds with $M = 5 > 0$. By Theorem 3.2.4, the nonlinear problem (3.15) has coupled minimal and maximal solutions, respectively (the extremal solutions) $(y^*, z^*) \in (\mathcal{D})^2$ with $\mathcal{D} = [-1, 1] = \{x \in C([0, 1], \mathbb{R}) \mid -1 \leq x(t) \leq 1\}$. i.e., $(x^*, y^*) \in [-1, 1] \times [-1, 1]$ on $I = [0, 1]$, which can be obtained by taking limits from the iterative sequences:

$$\begin{cases} y_n(t) = \left(\frac{1}{2} \int_0^1 y_{n-1}(s) ds - \frac{1}{3} z_{n-1}(1) \right) e^{-10t^{1/2}} + \int_0^t s^{-1/2} f_{y_{n-1}}(s) e^{10(s^{1/2}-t^{1/2})} ds, & t \in I, \\ z_n(t) = \left(\frac{1}{2} \int_0^1 z_{n-1}(s) ds - \frac{1}{3} y_{n-1}(1) \right) e^{-10t^{1/2}} + \int_0^t s^{-1/2} f_{z_{n-1}}(s) e^{10(s^{1/2}-t^{1/2})} ds, & t \in I, \end{cases}$$

where

$$\begin{cases} f_{y_{n-1}} = f(t, y_{n-1}) + 5y_{n-1} = t(1 - y_{n-1}^2) + 2y_{n-1}, \\ f_{z_{n-1}} = f(t, z_{n-1}) + 5z_{n-1} = t(1 - z_{n-1}^2) + 2z_{n-1}. \end{cases}$$

Chapter 4

Extremal solutions to a coupled system of conformable fractional dynamic equations on time scales

In this chapter, we investigate the existence of extremal solutions for a coupled system of nonlinear conformable fractional dynamic equations on time scales, by applying the comparison principle and the monotone iterative technique combined with the method of upper and lower solutions:

$$\begin{cases} x_{\Delta}^{(\alpha)}(t) = f(t, x^{\sigma}(t), y^{\sigma}(t)), & t \in I = [a, b]_{\mathbb{T}}, \\ y_{\Delta}^{(\alpha)}(t) = g(t, y^{\sigma}(t), x^{\sigma}(t)), & t \in I = [a, b]_{\mathbb{T}}, \\ x(a) = \lambda_0, \quad y(a) = \beta_0. \end{cases} \quad (4.1)$$

Here, $\mathbb{T}$ is an arbitrary bounded time scale, $J = [a, \sigma(b)]_{\mathbb{T}}$ with $a, b \in \mathbb{T}$, $0 < a < b$, $\lambda_0, \beta_0 \in \mathbb{R}$, $\lambda_0 \leq \beta_0$, $f, g : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions and $x_{\Delta}^{(\alpha)}, y_{\Delta}^{(\alpha)}$ are the conformable fractional derivatives (on time scales) with $\alpha \in (0, 1]$. The original results of this chapter are found in [6].

S. Liu et al. in [25], studied the existence of extremal iteration solution to the following coupled system of conformable nonlinear fractional differential equations:

$$\begin{cases} x^{(\alpha)}(t) = f(t, x(t), y(t)), & t \in [a, b], \\ y^{(\alpha)}(t) = g(t, y(t), x(t)), & t \in [a, b], \\ x(a) = x_0^*, \quad y(a) = y_0^*, \end{cases} \quad (4.2)$$

where, $x_0^*, y_0^* \in \mathbb{R}$, $x_0^* \leq y_0^*$, $f, g \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $x^{(\alpha)}, y^{(\alpha)}$ are the conformable fractional derivatives with $\alpha \in (0, 1]$.

4.1 Linear problems and comparison principles

In this section, we study the expression of the solutions of a linear conformable fractional dynamic equation of order $\alpha \in (0, 1]$, with initial value conditions.

Lemma 4.1.1 *The initial problem*

$$\begin{cases} x_{\Delta}^{(\alpha)}(t) - t^{1-\alpha} p x(\sigma(t)) = g(t), & t \in I = [a, b]_{\mathbb{T}}; \\ x(a) = \lambda_0, \end{cases} \quad (4.3)$$

with $-p \in \mathcal{R}_{\mu}$, $\lambda_0 \in \mathbb{R}$, and $g \in C_{rd}^{\alpha}(I, \mathbb{R})$, has a unique solution $x \in C_{rd}^{\alpha}(J, \mathbb{R})$, given by the following expression

$$x(t) := \int_{[a, t]_{\mathbb{T}}} e_{-p}(s, t) g(s) \Delta^{\alpha} s + \lambda_0 e_{-p}(a, t), \quad t \in J = [a, \sigma(b)]_{\mathbb{T}}. \quad (4.4)$$

Proof. Let x be a solution to (4.3). By Theorem 1.3.5, consider

$$\begin{aligned} \left[x(t) e_{-p}(t, a) \right]_{\Delta}^{(\alpha)} &= x_{\Delta}^{(\alpha)}(t) e_{-p}(t, a) - p(t) t^{1-\alpha} e_{-p}(t, a) x(\sigma(t)), \\ &= e_{-p}(t, a) g(t). \end{aligned}$$

and hence integrating the above on $[a, t]_{\mathbb{T}}$ obtain

$$x(t) e_{-p}(t, a) - x(a) = \int_{[a, t]_{\mathbb{T}}} e_{-p}(s, a) g(s) \Delta^{\alpha} s. \quad (4.5)$$

So,

$$\begin{aligned} x(t) &= e_{-p}(a, t) \left(x(a) + \int_{[a, t]_{\mathbb{T}}} e_{-p}(s, a) g(s) \Delta^{\alpha} s \right) \\ &= \int_{[a, t]_{\mathbb{T}}} e_{-p}(s, t) g(s) \Delta^{\alpha} s + \lambda_0 e_{-p}(a, t). \end{aligned}$$

□

As a direct consequence of expression (4.4), we deduce the following comparison result for the linear problem (4.3):

Lemma 4.1.2 (*Comparison principle*). Let $x \in C_{rd}^{\alpha}(J, \mathbb{R})$, then the following comparison principles hold for every $-p \in \mathcal{R}_{\mu}^{+}$:

(i) If

$$\begin{cases} x_{\Delta}^{(\alpha)}(t) - t^{1-\alpha} p x(\sigma(t)) \geq 0, & t \in I; \\ x(a) \geq 0, \end{cases}$$

then $x \geq 0$ on J .

(ii) If

$$\begin{cases} x_{\Delta}^{(\alpha)}(t) - t^{1-\alpha} p x(\sigma(t)) \leq 0, & t \in I; \\ x(a) \leq 0, \end{cases}$$

then $x \leq 0$ on J .

Proof.

- (i) We put $x_{\Delta}^{(\alpha)}(t) - t^{1-\alpha} p x(\sigma(t)) = g(t)$ and $x(a) = \lambda_0 \geq 0$. We are know that $g(t) \geq 0$, $e_{-p}(\cdot, t) > 0$ for every $t \in I = [a, b]_{\mathbb{T}}$ and

$$\begin{cases} x_{\Delta}^{(\alpha)}(t) - t^{1-\alpha} p x(\sigma(t)) = g(t), & t \in [a, b]_{\mathbb{T}}, \\ x(a) = \lambda_0 \geq 0. \end{cases} \quad (4.6)$$

By Lemma 4.1.1, the expression of $x(t)$ is:

$$x(t) = \int_{[a, t]_{\mathbb{T}}} e_{-p}(s, t) g(s) \Delta^{\alpha} s + \lambda_0 e_{-p}(a, t)$$

we can conclude that, $x(t) \geq 0$ for every $t \in I = [a, b]_{\mathbb{T}}$.

- (ii) We put $x_{\Delta}^{(\alpha)}(t) - t^{1-\alpha} p x(\sigma(t)) = g(t)$ and $x(a) = \lambda_0 \leq 0$. We are know that $g(t) \leq 0$, $e_{-p}(\cdot, t) > 0$ for every $t \in I = [a, b]_{\mathbb{T}}$ and

$$\begin{cases} x_{\Delta}^{(\alpha)}(t) - t^{1-\alpha} p x(\sigma(t)) = g(t), & t \in [a, b]_{\mathbb{T}}, \\ x(a) = \lambda_0 \leq 0. \end{cases} \quad (4.7)$$

By Lemma 4.1.1, the expression of $x(t)$ is:

$$x(t) = \int_{[a, t]_{\mathbb{T}}} e_{-p}(s, t) g(s) \Delta^{\alpha} s + \lambda_0 e_{-p}(a, t)$$

we can conclude that, $x(t) \leq 0$ for every $t \in I = [a, b]_{\mathbb{T}}$. □

4.2 Main Result

In this section, we prove the existence of extremal solutions for problem (4.1). Let us defining what we mean by a solution of this problem.

Definition 4.2.1 A solution of problem (4.1) will be a pair $(x, y) \in C_{rd}^{\alpha}(J, \mathbb{R}) \times C_{rd}^{\alpha}(J, \mathbb{R})$ for which (4.1) is satisfied.

Next, we introduce the concept of coupled lower and upper solutions of this problem as follows.

Definition 4.2.2 *We say that $\gamma, \delta \in C_{rd}^\alpha(J, \mathbb{R})$ is a pair of coupled lower and upper solutions of the problem (4.1), if $\gamma \leq \delta$ in J and the following inequalities hold:*

$$\begin{cases} \gamma_\Delta^{(\alpha)}(t) \leq f(t, \gamma^\sigma(t), \delta^\sigma(t)), & \text{for } t \in I, \gamma(a) \leq \lambda_0, \\ \delta_\Delta^{(\alpha)}(t) \geq g(t, \delta^\sigma(t), \gamma^\sigma(t)), & \text{for } t \in I, \delta(a) \geq \beta_0. \end{cases} \quad (4.8)$$

We assume the following hypothesis:

(H₁) $f, g : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

(H₂) There exists $\gamma, \delta \in C_{rd}^\alpha(J, \mathbb{R})$, a pair of coupled lower and upper solutions of the problem (4.1).

(H₃) There exist constants $p, q \in \mathbb{R}$ with $-p, -q \in \mathcal{R}_\mu^+$ and $q \leq 0$ such that

$$\begin{cases} f(t, x, y) - f(t, \bar{x}, \bar{y}) \geq t^{1-\alpha}p(x - \bar{x}) + t^{1-\alpha}q(y - \bar{y}), \\ g(t, \bar{y}, \bar{x}) - g(t, y, x) \geq t^{1-\alpha}p(\bar{y} - y) + t^{1-\alpha}q(\bar{x} - x), \end{cases}$$

where $\gamma^\sigma(t) \leq \bar{x} \leq x \leq \delta^\sigma(t)$, $\gamma^\sigma(t) \leq y \leq \bar{y} \leq \delta^\sigma(t)$ for all $t \in I$, and

$$g(t, y, x) - f(t, x, y) \geq t^{1-\alpha}p(y - x) + t^{1-\alpha}q(x - y),$$

where $\gamma^\sigma(t) \leq x \leq y \leq \delta^\sigma(t)$ for all $t \in I$.

To study the nonlinear system (4.1), we first consider the associated linear system:

$$\begin{cases} x_\Delta^{(\alpha)}(t) = h_1(t) + t^{1-\alpha}p x(\sigma(t)) + t^{1-\alpha}q y(\sigma(t)), & \text{for } t \in I = [a, b]_\mathbb{T}, \\ y_\Delta^{(\alpha)}(t) = h_2(t) + t^{1-\alpha}p y(\sigma(t)) + t^{1-\alpha}q x(\sigma(t)), & \text{for } t \in I = [a, b]_\mathbb{T}, \\ x(a) = \lambda_0, \quad y(a) = \beta_0, \end{cases} \quad (4.9)$$

where $\alpha \in (0, 1]$, $(\lambda_0, \beta_0) \in \mathbb{R}^2$, $\lambda_0 \leq \beta_0$, $-p, -q \in \mathcal{R}_\mu$, $q \leq 0$ and $h_1, h_2 \in C(I, \mathbb{R})$.

Lemma 4.2.3 *The linear system (4.9) has a unique solution $(x, y) \in C_{rd}^\alpha(J, \mathbb{R}) \times C_{rd}^\alpha(J, \mathbb{R})$, with*

$$x(t) = \frac{z(t) + w(t)}{2}, \quad y(t) = \frac{z(t) - w(t)}{2}, \quad t \in J = [a, \sigma(b)]_\mathbb{T},$$

where

$$z(t) := \int_{[a,t]_{\mathbb{T}}} e_{-(p+q)}(s,t)(h_1 + h_2)(s)\Delta^\alpha s + (\lambda_0 + \beta_0)e_{-(p+q)}(a,t), \quad t \in J,$$

and

$$w(t) := \int_{[a,t]_{\mathbb{T}}} e_{-(p-q)}(s,t)(h_1 - h_2)(s)\Delta^\alpha s + (\lambda_0 - \beta_0)e_{-(p-q)}(a,t), \quad t \in J.$$

Proof. The pair $(x, y) \in C_{rd}^\alpha(J, \mathbb{R}) \times C_{rd}^\alpha(J, \mathbb{R})$ is a solution to system (4.9) if and only if

$$x(t) = \frac{z(t) + w(t)}{2}, \quad y(t) = \frac{z(t) - w(t)}{2}, \quad \text{for every } t \in J = [a, \sigma(b)]_{\mathbb{T}},$$

where $z(t)$ and $w(t)$ are the solutions to the following problems:

$$\begin{cases} z_\Delta^{(\alpha)}(t) = (h_1(t) + h_2(t)) + t^{1-\alpha}(p+q) z(\sigma(t)), & \text{for } t \in I = [a, b]_{\mathbb{T}}, \\ z(a) = \lambda_0 + \beta_0, \end{cases}$$

and

$$\begin{cases} w_\Delta^{(\alpha)}(t) = (h_1(t) - h_2(t)) + t^{1-\alpha}(p-q) w(\sigma(t)), & \text{for } t \in I = [a, b]_{\mathbb{T}}, \\ w(a) = \lambda_0 - \beta_0. \end{cases}$$

By Lemma 4.1.1, we have

$$z(t) := \int_{[a,t]_{\mathbb{T}}} e_{-(p+q)}(s,t)(h_1 + h_2)(s)\Delta^\alpha s + (\lambda_0 + \beta_0)e_{-(p+q)}(a,t), \quad t \in J, \quad (4.10)$$

$$w(t) := \int_{[a,t]_{\mathbb{T}}} e_{-(p-q)}(s,t)(h_1 - h_2)(s)\Delta^\alpha s + (\lambda_0 - \beta_0)e_{-(p-q)}(a,t), \quad t \in J. \quad (4.11)$$

The proof is finished. $\square$

Lemma 4.2.4 (*Comparison principle 2*). Let $(x, y) \in C_{rd}^\alpha(J, \mathbb{R}) \times C_{rd}^\alpha(J, \mathbb{R})$ satisfy

$$\begin{cases} x_\Delta^{(\alpha)}(t) - t^{1-\alpha}p x(\sigma(t)) + t^{1-\alpha}q y(\sigma(t)) \geq 0, & \text{for } t \in I = [a, b]_{\mathbb{T}}, \\ y_\Delta^{(\alpha)}(t) - t^{1-\alpha}p y(\sigma(t)) + t^{1-\alpha}q x(\sigma(t)) \geq 0, & \text{for } t \in I = [a, b]_{\mathbb{T}}, \\ x(a) \geq 0, \quad y(a) \geq 0, \end{cases} \quad (4.12)$$

where $\alpha \in (0, 1]$, $-p, -q \in \mathcal{R}_\mu^+$ and $q \leq 0$. Then $x(t) \geq 0$, $y(t) \geq 0$ for all $t \in J$.

Proof. Let $w(t) = x(t) + y(t)$, then (4.12) is equivalent to the following:

$$\begin{cases} w_{\Delta}^{(\alpha)}(t) - t^{1-\alpha}(p - q) w(\sigma(t)) \geq 0, & \text{for } t \in I = [a, b]_{\mathbb{T}}, \\ w(a) \geq 0. \end{cases}$$

By Lemma 4.1.2, we know that $w(t) \geq 0$, for all $t \in J$, i.e., $x(t) + y(t) \geq 0$, for all $t \in J$,

So.

$$\begin{cases} x_{\Delta}^{(\alpha)}(t) - t^{1-\alpha}(p + q) x(\sigma(t)) \geq 0, & \text{for } t \in I = [a, b]_{\mathbb{T}}, \\ y_{\Delta}^{(\alpha)}(t) - t^{1-\alpha}(p + q) y(\sigma(t)) \geq 0, & \text{for } t \in I = [a, b]_{\mathbb{T}}, \\ x(a) \geq 0, y(a) \geq 0. \end{cases}$$

By Lemma 4.1.2, we have $x(t) \geq 0$ and $y(t) \geq 0$ for all $t \in J$. The proof is completed. $\square$

The obtained result is the following.

Theorem 4.2.5 *Assume that (H_1) , (H_2) and (H_3) hold. Then (4.1) has an extremal system of solutions $(x^*(t), y^*(t)) \in [\gamma(t), \delta(t)] \times [\gamma(t), \delta(t)]$, and there exist two monotone sequences $\{y_n\}_{n \in \mathbb{N}}$ and $\{z_n\}_{n \in \mathbb{N}}$ converging uniformly to $x^*(t), y^*(t)$, respectively, where $y_n(t), z_n(t) \in [\gamma(t), \delta(t)]$, such that*

$$\gamma =: y_0 \leq y_1 \leq \dots \leq y_n \leq \dots \leq z_n \leq \dots \leq z_1 \leq z_0 := \delta, \quad \text{on } J \text{ for all } n \in \mathbb{N}.$$

Proof. Firstly, for all $y_n, z_n \in C_{rd}^{\alpha}(J, \mathbb{R})$, we consider the linear system:

$$\begin{cases} y_{n+1\Delta}^{(\alpha)}(t) = f(t, y_n^{\sigma}(t), z_n^{\sigma}(t)) - t^{1-\alpha}p(y_n^{\sigma}(t) - y_{n+1}^{\sigma}(t)) - t^{1-\alpha}q(z_n^{\sigma}(t) - z_{n+1}^{\sigma}(t)), \\ \quad \text{for } t \in I, \\ z_{n+1\Delta}^{(\alpha)}(t) = g(t, z_n^{\sigma}(t), y_n^{\sigma}(t)) - t^{1-\alpha}p(z_n^{\sigma}(t) - z_{n+1}^{\sigma}(t)) - t^{1-\alpha}q(y_n^{\sigma}(t) - y_{n+1}^{\sigma}(t)), \\ \quad \text{for } t \in I, \\ y_{n+1}(a) = \lambda_0, z_{n+1}(a) = \beta_0, \end{cases}$$

i.e.,

$$\begin{cases} y_{n+1\Delta}^{(\alpha)}(t) = (f(t, y_n^{\sigma}(t), z_n^{\sigma}(t)) - t^{1-\alpha}(py_n^{\sigma}(t) + qz_n^{\sigma}(t))) + t^{1-\alpha}(py_{n+1}^{\sigma}(t) + qz_{n+1}^{\sigma}(t)), \\ \quad \text{for } t \in I, \\ z_{n+1\Delta}^{(\alpha)}(t) = (g(t, z_n^{\sigma}(t), y_n^{\sigma}(t)) - t^{1-\alpha}(pz_n^{\sigma}(t) + qy_n^{\sigma}(t))) + t^{1-\alpha}(pz_{n+1}^{\sigma}(t) + qy_{n+1}^{\sigma}(t)), \\ \quad \text{for } t \in I, \\ y_{n+1}(a) = \lambda_0, z_{n+1}(a) = \beta_0. \end{cases} \tag{4.13}$$

By Lemma 4.2.3, the linear system (4.13) has a unique solution $(y_{n+1}, z_{n+1}) \in C_{rd}^\alpha(J, \mathbb{R}) \times C_{rd}^\alpha(J, \mathbb{R})$, with

$$y_{n+1}(t) = \frac{v_{n+1}(t) + w_{n+1}(t)}{2}, \quad z_{n+1}(t) = \frac{v_{n+1}(t) - w_{n+1}(t)}{2}, \quad \text{for every } t \in J = [a, \sigma(b)]_{\mathbb{T}},$$

where

$$\begin{aligned} v_{n+1}(t) = & \int_{[a,t]_{\mathbb{T}}} e_{-(p+q)}(s, t) \left[f(s, y_n^\sigma(s), z_n^\sigma(s)) + g(s, z_n^\sigma(s), y_n^\sigma(s)) \right. \\ & \left. - s^{1-\alpha}(p+q)(y_n^\sigma(s) + z_n^\sigma(s)) \right] \Delta^\alpha s + (\lambda_0 + \beta_0)e_{-(p+q)}(a, t), \quad t \in J, \end{aligned}$$

and

$$\begin{aligned} w_{n+1}(t) = & \int_{[a,t]_{\mathbb{T}}} e_{-(p-q)}(s, t) \left[f(s, y_n^\sigma(s), z_n^\sigma(s)) - g(s, z_n^\sigma(s), y_n^\sigma(s)) \right. \\ & \left. - s^{1-\alpha}(p-q)(y_n^\sigma(s) - z_n^\sigma(s)) \right] \Delta^\alpha s + (\lambda_0 - \beta_0)e_{-(p-q)}(a, t), \quad t \in J. \end{aligned}$$

Secondly, we shall prove that

$$y_n \leq y_{n+1} \leq z_{n+1} \leq z_n, \quad \text{on } J \text{ for all } n \in \mathbb{N}.$$

Let $v := y_1 - y_0 = y_1 - \gamma$, $w := z_0 - z_1 = \delta - z_1$. According to (4.13) and (H_1) – (H_2) , we have

$$\begin{cases} v_\Delta^{(\alpha)}(t) \geq -t^{1-\alpha}p(y_0(\sigma(t)) - y_1(\sigma(t))) - t^{1-\alpha}q(z_0(\sigma(t)) - z_1(\sigma(t))), & \text{for } t \in I, \\ v(a) \geq \lambda_0 - \lambda_0 = 0, \\ w_\Delta^{(\alpha)}(t) \geq t^{1-\alpha}p(z_0(\sigma(t)) - z_1(\sigma(t))) + t^{1-\alpha}q(y_0(\sigma(t)) - y_1(\sigma(t))), & \text{for } t \in I, \\ w(a) \geq \beta_0 - \beta_0 = 0, \end{cases}$$

i.e.,

$$\begin{cases} v_\Delta^{(\alpha)}(t) \geq t^{1-\alpha}pv(\sigma(t)) - t^{1-\alpha}qw(\sigma(t)), & \text{for } t \in I, \quad v(a) \geq 0, \\ w_\Delta^{(\alpha)}(t) \geq t^{1-\alpha}pw(\sigma(t)) - t^{1-\alpha}qv(\sigma(t)), & \text{for } t \in I, \quad w(a) \geq 0. \end{cases}$$

Then, by Lemma 4.2.4, we have $v(t) \geq 0, w(t) \geq 0$, i.e., $y_1 \geq y_0, z_0 \geq z_1$. Let $\xi := z_1 - y_1$. According to (4.13) and (H_3) , we have

$$\begin{aligned} \xi_\Delta^{(\alpha)}(t) &= z_{1\Delta}^{(\alpha)}(t) - y_{1\Delta}^{(\alpha)}(t) \\ &= g(t, z_0^\sigma(t), y_0^\sigma(t)) - t^{1-\alpha}(pz_0^\sigma(t) + qy_0^\sigma(t)) + t^{1-\alpha}(pz_1^\sigma(t) + qy_1^\sigma(t)) \\ &\quad - f(t, y_0^\sigma(t), z_0^\sigma(t)) + t^{1-\alpha}(py_0^\sigma(t) + qz_0^\sigma(t)) - t^{1-\alpha}(py_1^\sigma(t) + qz_1^\sigma(t)) \\ &\geq t^{1-\alpha}p(z_1^\sigma(t) - y_1^\sigma(t)) - t^{1-\alpha}q(z_1^\sigma(t) - y_1^\sigma(t)) = t^{1-\alpha}(p-q)\xi^\sigma(t). \end{aligned}$$

So.

$$\begin{cases} \xi_{\Delta}^{(\alpha)}(t) \geq t^{1-\alpha}(p-q)\xi^{\sigma}(t), & \text{for } t \in I, \\ \xi(a) = \beta_0 - \lambda_0 \geq 0. \end{cases} \quad (4.14)$$

By Lemma 4.1.2, we have $\xi(t) \geq 0$, i.e., $z_1(t) \geq y_1(t)$ for all $t \in J$.

By mathematical induction, we can prove that

$$y_n \leq y_{n+1} \leq z_{n+1} \leq z_n, \quad \text{on } J \text{ for all } n \in \mathbb{N}.$$

Thirdly, the sequences $\{y_n\}_{n \in \mathbb{N}}$ and $\{z_n\}_{n \in \mathbb{N}}$ are monotone and bounded, hence

$$\lim_{n \rightarrow \infty} y_n = x^*, \quad \lim_{n \rightarrow \infty} z_n = y^*,$$

(x^*, y^*) is an extremal system of solutions to (4.1).

Finally, we prove that (4.1) has at most one extremal system of solutions.

Assume that $(x, y) \in [\gamma = y_0, \delta = z_0] \times [y_0, z_0]$ is the system of solutions to (4.1), then

$$y_0 = \gamma \leq x, \quad y \leq z_0 = \delta.$$

For some $k \in \mathbb{N}$, assume that the following relation holds

$$y_k(t) \leq x(t), \quad y(t) \leq z_k(t), \quad t \in J.$$

Let $u(t) = x(t) - y_{k+1}(t)$, $\vartheta(t) = z_{k+1}(t) - y(t)$. According to (4.13) and (H_3) , we have

$$\begin{aligned} u_{\Delta}^{(\alpha)}(t) &= x_{\Delta}^{(\alpha)}(t) - y_{k+1\Delta}^{(\alpha)}(t) \\ &= f(t, x^{\sigma}(t), y^{\sigma}(t)) - f(t, y_k^{\sigma}(t), z_k^{\sigma}(t)) + t^{1-\alpha}p(y_k^{\sigma}(t) - y_{k+1}^{\sigma}(t)) \\ &\quad + t^{1-\alpha}q(z_k^{\sigma}(t) - z_{k+1}^{\sigma}(t)) \\ &\geq t^{1-\alpha}p(x^{\sigma}(t) - y_k^{\sigma}(t)) + t^{1-\alpha}q(y^{\sigma}(t) - z_k^{\sigma}(t)) + t^{1-\alpha}p(y_k^{\sigma}(t) - y_{k+1}^{\sigma}(t)) \\ &\quad + t^{1-\alpha}q(z_k^{\sigma}(t) - z_{k+1}^{\sigma}(t)) \\ &= t^{1-\alpha}p(x^{\sigma}(t) - y_{k+1}^{\sigma}(t)) + t^{1-\alpha}q(y^{\sigma}(t) - z_{k+1}^{\sigma}(t)), \end{aligned}$$

and

$$\begin{aligned} \vartheta_{\Delta}^{(\alpha)}(t) &= z_{k+1\Delta}^{(\alpha)}(t) - y_{\Delta}^{(\alpha)}(t) \\ &= g(t, z_k^{\sigma}(t), y_k^{\sigma}(t)) - g(t, y^{\sigma}(t), x^{\sigma}(t)) - t^{1-\alpha}p(z_k^{\sigma}(t) - z_{k+1}^{\sigma}(t)) \\ &\quad - t^{1-\alpha}q(y_k^{\sigma}(t) - y_{k+1}^{\sigma}(t)) \\ &\geq t^{1-\alpha}p(z_k^{\sigma}(t) - y^{\sigma}(t)) + t^{1-\alpha}q(y_k^{\sigma}(t) - x^{\sigma}(t)) - t^{1-\alpha}p(z_k^{\sigma}(t) - z_{k+1}^{\sigma}(t)) \\ &\quad - t^{1-\alpha}q(y_k^{\sigma}(t) - y_{k+1}^{\sigma}(t)) \\ &= t^{1-\alpha}p(z_{k+1}^{\sigma}(t) - y^{\sigma}(t)) - t^{1-\alpha}q(x^{\sigma}(t) - y_{k+1}^{\sigma}(t)), \end{aligned}$$

we can get

$$\begin{cases} u_{\Delta}^{(\alpha)}(t) \geq t^{1-\alpha}pu(\sigma(t)) - t^{1-\alpha}q\vartheta(\sigma(t)), & \text{for } t \in I, \quad u(a) \geq 0, \\ \vartheta_{\Delta}^{(\alpha)}(t) \geq t^{1-\alpha}p\vartheta(\sigma(t)) - t^{1-\alpha}qu(\sigma(t)), & \text{for } t \in I, \quad \vartheta(a) \geq 0. \end{cases}$$

Then, by Lemma 4.2.4, we have $u(t) \geq 0$, $\vartheta(t) \geq 0$, i.e., $y_{k+1}(t) \leq x(t)$, $y(t) \leq z_{k+1}(t)$, $t \in J$,

By the induction arguments, the following relation holds

$$y_n(t) \leq x(t), \quad y(t) \leq z_n(t), \quad t \in J.$$

Taking the limit as $n \rightarrow \infty$, we get that $x^* \leq x$, $y \leq y^*$ on J . Hence, $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$ is the extremal system of solutions to (4.1). So the proof is finished. $\square$

4.3 Examples

We present the following two examples, where we apply Theorem 4.2.5.

Example 4.3.1 Consider the system of nonlinear conformable fractional dynamic equations:

$$\begin{cases} x_{\Delta}^{(\frac{1}{3})}(t) = \frac{t(2 - x(\sigma(t)))^2 - y^2(\sigma(t))}{\sqrt[3]{t}}, & t \in I = [1, 2]_{\mathbb{T}}, \\ y_{\Delta}^{(\frac{1}{3})}(t) = t^{\frac{2}{3}}(2 - y(\sigma(t)))^3 - t^{-\frac{1}{3}}x^2(\sigma(t)), & t \in I = [1, 2]_{\mathbb{T}}, \\ x(1) = 0, \quad y(1) = 0.5, \end{cases} \quad (4.15)$$

where $\alpha = \frac{1}{3}$, $f(t, x, y) = \frac{t(2 - x)^2 - y^2}{\sqrt[3]{t}}$ and $g(t, y, x) = t^{\frac{2}{3}}(2 - y)^3 - t^{-\frac{1}{3}}x^2$.

It is clear that f, g are continuous functions. Take $\gamma(t) = 0 \leq \delta(t) = 2$ for $t \in [1, \sigma(2)]_{\mathbb{T}}$, then

$$\gamma_{\Delta}^{(\frac{1}{3})}(t) = 0 \leq f(t, \gamma^{\sigma}(t), \delta^{\sigma}(t)) = \frac{4(t-1)}{\sqrt[3]{t}} \text{ for } t \in [1, 2]_{\mathbb{T}}, \quad \gamma(1) = 0 \leq 0,$$

and

$$\delta_{\Delta}^{(\frac{1}{3})}(t) = 0 \geq g(t, \delta^{\sigma}(t), \gamma^{\sigma}(t)) = 0 \text{ for } t \in [1, 2]_{\mathbb{T}}, \quad \delta(1) = 2 \geq 0.5,$$

then assumptions (H_1) and (H_2) holds.

Let $x, \bar{x}, y, \bar{y} \in \mathbb{R}$, then we have:

$$f(t, x, y) - f(t, \bar{x}, \bar{y}) = t^{1-\frac{1}{3}} ((2-x)^2 - (2-\bar{x})^2) - \frac{1}{\sqrt[3]{t}}(y^2 - \bar{y}^2)$$

$$\begin{aligned}
&\geq t^{1-\frac{1}{3}} (-4(x - \bar{x}) + x^2 - \bar{x}^2) \\
&\geq -4t^{1-\frac{1}{3}}(x - \bar{x}) \\
&\geq -12t^{1-\frac{1}{3}}(x - \bar{x}) + 0.t^{1-\frac{1}{3}}(y - \bar{y}),
\end{aligned}$$

$$\begin{aligned}
g(t, \bar{y}, \bar{x}) - g(t, y, x) &= t^{\frac{2}{3}} ((2 - \bar{y})^3 - (2 - y)^3) - t^{-\frac{1}{3}}(\bar{x}^3 - x^3) \\
&\geq t^{\frac{2}{3}} (-4(\bar{y} - y) + 2(\bar{y}^2 - y^2) - (\bar{y}^3 - y^3)) \\
&\geq -12t^{1-\frac{1}{3}}(\bar{y} - y) + 0.t^{1-\frac{1}{3}}(\bar{x} - x),
\end{aligned}$$

with $\gamma^\sigma(t) \leq \bar{x} \leq x \leq \delta^\sigma(t)$, $\gamma^\sigma(t) \leq y \leq \bar{y} \leq \delta^\sigma(t)$ for all $t \in I$, and we have

$$\begin{aligned}
g(t, y, x) - f(t, x, y) &= t^{\frac{2}{3}} ((2 - y)^3 - (2 - x)^2) + t^{-\frac{1}{3}}(y^2 - x^2) \\
&\geq t^{\frac{2}{3}} (-4(y - x) + (4 - x^2 + 2y^2) - y^3) \\
&\geq -12t^{1-\frac{1}{3}}(y - x) + 0.t^{1-\frac{1}{3}}(x - y).
\end{aligned}$$

with $\gamma^\sigma(t) \leq x \leq y \leq \delta^\sigma(t)$, for all $t \in I$.

Hence the assumption (H_3) holds with $p = -12$ and $q = 0$. By Theorem 4.2.5, the non-linear system (4.15) has the extremal solution $(x^*, y^*) \in C_{rd}^{\frac{1}{3}}([1, \sigma(2)]_{\mathbb{T}}) \times C_{rd}^{\frac{1}{3}}([1, \sigma(2)]_{\mathbb{T}})$, such that $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$ on $[1, \sigma(2)]_{\mathbb{T}}$, which can be obtained by taking limits from the iterative sequences:

$$\begin{aligned}
x_{n+1}(t) &= \int_{[1, t]_{\mathbb{T}}} s^{-\frac{2}{3}} e_{12}(s, t) \left[\frac{t(2 - x_n(\sigma(t)))^2 - y_n^2(\sigma(t))}{\sqrt[3]{t}} + 12(x_n^\sigma(s)) \right] \Delta s, \\
t &\in J = [1, \sigma(2)]_{\mathbb{T}},
\end{aligned}$$

$$\begin{aligned}
y_{n+1}(t) &= 0.5e_{12}(1, t) + \int_{[1, t]_{\mathbb{T}}} s^{-\frac{2}{3}} e_{12}(s, t) \left[t^{\frac{2}{3}}(2 - y_n(\sigma(t)))^3 - t^{-\frac{1}{3}}x_n^2(\sigma(t)) + 12(y_n^\sigma(s)) \right] \Delta s, \\
t &\in J,
\end{aligned}$$

Example 4.3.2 Consider the system of nonlinear conformable fractional differential equations:

$$\begin{cases} x^{(\frac{1}{3})}(t) = \frac{t(2 - x(t))^2 - y^2(t)}{\sqrt[3]{t}}, & t \in I = [1, 2], \\ y^{(\frac{1}{3})}(t) = t^{\frac{2}{3}}(2 - y(t))^3 - t^{-\frac{1}{3}}x^2(t), & t \in I = [1, 2], \\ x(1) = 0, \quad y(1) = 0.5, \end{cases} \quad (4.16)$$

where $\alpha = \frac{1}{3}$, $f(t, x, y) = \frac{t(2 - x)^2 - y^2}{\sqrt[3]{t}}$ and $g(t, y, x) = t^{\frac{2}{3}}(2 - y)^3 - t^{-\frac{1}{3}}x^2$.

It is clear that f, g are continuous functions. Take $\gamma(t) = 0$ and $\delta(t) = 2$ for $t \in [1, 2]$, then

$$\gamma^{(\frac{1}{3})}(t) = 0 \leq f(t, \gamma(t), \delta(t)) = \frac{4(t - 1)}{\sqrt[3]{t}} \text{ for } t \in [1, 2], \quad \gamma(1) = 0 \leq 0,$$

and

$$\delta^{(\frac{1}{3})}(t) = 0 \geq g(t, \delta(t), \gamma(t)) = 0 \text{ for } t \in [1, 2], \quad \delta(1) = 2 \geq 0.5.$$

So, γ and δ , are lower and upper solutions of problem (4.15), respectively with $\gamma(t) = 0 \leq \delta(t) = 2$ for $t \in [1, 2]$, then assumptions (H_1) and (H_2) holds.

Let $x, \bar{x}, y, \bar{y} \in \mathbb{R}$, then we have:

$$\begin{aligned} f(t, x, y) - f(t, \bar{x}, \bar{y}) &= t^{1-\frac{1}{3}} \left((2-x)^2 - (2-\bar{x})^2 \right) - \frac{1}{\sqrt[3]{t}} (y^2 - \bar{y}^2) \\ &\geq t^{1-\frac{1}{3}} \left(-4(x - \bar{x}) + x^2 - \bar{x}^2 \right) - (y^2 - \bar{y}^2) \\ &\geq -4t^{\frac{2}{3}}(x - \bar{x}) \\ &\geq -24(x - \bar{x}) - 0(y - \bar{y}), \end{aligned}$$

$$\begin{aligned} g(t, \bar{y}, \bar{x}) - g(t, y, x) &= t^{\frac{2}{3}} \left((2-\bar{y})^3 - (2-y)^3 \right) - t^{-\frac{1}{3}} (\bar{x}^2 - x^2) \\ &\geq -t^{\frac{2}{3}}(\bar{y} - y) \left((2-\bar{y})^2 + (2-\bar{y})(2-y) + (2-y)^2 \right) \\ &\geq -12t^{\frac{2}{3}}(\bar{y} - y), \\ &\geq -24(\bar{y} - y) - 0.(\bar{x} - x), \end{aligned}$$

with $\gamma(t) \leq \bar{x} \leq x \leq \delta(t)$, $\gamma(t) \leq y \leq \bar{y} \leq \delta(t)$ for all $t \in I$, and we have

$$\begin{aligned} g(t, y, x) - f(t, x, y) &= t^{\frac{2}{3}} \left((2-y)^3 - (2-x)^2 \right) + t^{-\frac{1}{3}} (y^2 - x^2) \\ &\geq -t^{\frac{2}{3}}(y - x) \left((2-y)^2 + (2-y)(2-x) + (2-x)^2 \right) \\ &\geq -12t^{1-\frac{1}{3}}(y - x) \\ &\geq -24(y - x) - 0.(x - y). \end{aligned}$$

with $\gamma(t) \leq x \leq y \leq \delta(t)$, for all $t \in I$.

Hence the assumption (H_3) holds with $p = -24$ and $q = 0$. By Theorem 4.2.5, the nonlinear system (4.16) has the extremal solution $(x^*, y^*) \in C^{\frac{1}{3}}([1, 2]) \times C^{\frac{1}{3}}([1, 2])$, such that $(x^*, y^*) \in [\gamma, \delta] \times [\gamma, \delta]$ on $[1, 2]$, which can be obtained by taking limits from the iterative sequences:

$$\begin{aligned} x_{n+1}(t) &= \int_1^t s^{-\frac{2}{3}} e^{72(s^{\frac{1}{3}} - t^{\frac{1}{3}})} \left[\frac{s(2 - x_n(s))^2 - y_n^2(s)}{\sqrt[3]{s}} + 24(x_n(s)) \right] ds, \quad t \in I, \\ y_{n+1}(t) &= \int_1^t s^{-\frac{2}{3}} e^{72(s^{\frac{1}{3}} - t^{\frac{1}{3}})} \left[s^{\frac{2}{3}}(2 - y_n(s))^3 - s^{-\frac{1}{3}}x_n^2(s) + 24y_n(s) \right] ds + 0.5e^{72(1-t^{\frac{1}{3}})}, \quad t \in I. \end{aligned}$$

Conclusion

In this work, we have considered the existence of quasi-solutions for the first-order dynamic equations involving integral boundary conditions, and we present the existence of extremal solutions for nonlinear conformable fractional differential equations involving integral boundary conditions. Also, we present the existence of extremal solutions for a coupled system of nonlinear conformable fractional differential equations on time scales with initial conditions.

These results were obtained by applying the comparison principle and the monotone iterative technique combined with the method of upper and lower solutions.

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