PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

IBN KHALDOUN UNIVERSITY - TIARET

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
Department : Mathematics

Master's degree in Mathematics

SPECIALTY :
«Mathematics»

OPTION :
« Functional Analysis and its Applications»

Presented by:
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Theme :

Numerical Solution of linear Fractional Integro-Differential Equations

Presented in 29 / 06 / 2025 in Tiaret
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Academic year: 2024/2025

بسم الله الرحمن الرحيم

Dedication

All praise and thanks are due to Allah for His immense blessings and guidance. With all love, appreciation, and gratitude, I dedicate this modest work:

- To my mother and father, all my thanks and love for your sacrifices and your support for me every step of the way.
- To my brother and my sisters for standing by me and offering support at all times.
- To my dear friends, thank you for every moment of support and encouragement, and for your invaluable friendship.
- to my colleague thanks for your efforts and effective cooperation in completing this thesis .
- To those who believed in my abilities and saw the light within me before I ever could thank you .

Meriem Halhal

Dedication

I would like to thank Allah for His boundless grace in achieving this graduation. With the feelings of love and gratitude I dedicate this modest work :

- To my beloved mother and to the soul of my father may Allah bless him.
- To my wife for her unconditioned support and encouragement and to my kids for their inspiration.
- To my family, thank you for being always on there for me.
- With special thanks to my colleague for their invaluable collaboration and support in preparing this thesis.
- To everyone who encouraged me thank you

Brahim Taoussi

Acknowledgements

I begin by thanking Allah, first and foremost

- My deepest gratitude goes to **Mr. Soufiane Benyoussef** , my supervisor, whose profound impact was instrumental in the completion of this thesis. His role extended beyond precise scientific guidance and invaluable advice; it encompassed invaluable support, immense patience, and a deep understanding of every challenge I faced, which strongly motivated me to complete this work.
- I also extend my sincere thanks to **Mr. Laid Elkhiri** for graciously reviewing this research and providing her valuable feedback.
- My profound appreciation goes to **Mr. Abedelmalek Benaissa** for accepting the presidency of the jury for this research work.
- Last but not least, my sincere thanks go to everyone who contributed to this humble work.

Meriem Halhal
Brahim Taoussi

Contents

Introduction

Fractional calculus a generalization of classical integer-order calculus, has garnered significant attention in recent decades. Yet its widespread application in diverse scientific and engineering disciplines is a more recent phenomenon. Unlike classical derivatives and integrals that describe localized changes, fractional operators inherently capture non-local, memory-dependent, and hereditary properties of various phenomena. This unique ability makes them exceptionally well-suited for modeling complex systems in fields such as anomalous diffusion, viscoelasticity, control theory, finance, and biological systems, where traditional integer-order models often fall short.

Among the various forms of fractional equations, Fractional Integro-Differential Equations (FIDEs) stand out as a particularly rich and challenging class. These equations combine fractional derivatives, fractional integrals, and classical integral terms, providing a powerful framework to describe systems exhibiting both memory effects and accumulation processes. The inclusion of the Caputo fractional derivative is particularly significant for practical applications. Unlike the Riemann-Liouville derivative, the Caputo definition allows for the direct use of physically interpretable initial and boundary conditions, aligning more naturally with real-world scenarios and making it a preferred choice for modeling.

Despite their immense modeling potential, finding analytical solutions to most FIDEs is exceptionally difficult, if not impossible. This inherent complexity necessitates the development of robust and efficient numerical methods to approximate their solutions. Over the years, various numerical techniques have emerged, including finite difference methods, finite element methods, and spectral methods. Among these, spectral methods have distinguished themselves by offering high accuracy and rapid convergence rates for problems with smooth solutions. These methods rely on approximating the solution using a series of global basis functions, typically orthogonal polynomials, leading to a system of algebraic equations that can then be solved efficiently.

This thesis focuses on the numerical solution of linear fractional integro-differential equations

Involving ;the Caputo derivative using a spectral collocation method. We specifically leverage Jacobi polynomials, and their important special case, Legendre polynomials, as the basis functions. The collocation approach transforms the FIDE into a system of linear algebraic equations by enforcing the satisfaction of the equation at specific collocation points (e.g., Gauss-Lobatto points). This methodology capitalizes on the high approximation capabilities of orthogonal polynomials and the efficiency of solving linear systems.

The structure of this thesis is organized as follows:

- **Chapter 1** provides a comprehensive theoretical background on fractional calculus, introducing key concepts such as the Gamma, Beta, and Mittag-Leffler special functions. It then delves into the definitions and properties of the Riemann-Liouville and Caputo fractional derivatives, emphasizing the advantages of the latter for our study.

- **Chapter 2** is dedicated to the theory of orthogonal polynomials, with a particular focus
  on Jacobi and Legendre polynomials, discussing their properties and their crucial role in spectral approximations.

- Finally, **Chapter 3** presents the core contribution of this work: the detailed formulation and application of the spectral collocation method for solving linear fractional integro-differential equations with the Caputo derivative. This chapter will include the derivation of necessary operational matrices, the setup of the linear system, and numerical examples to demonstrate the accuracy, efficiency, and convergence of the proposed method.

Chapter 1

Fractional derivatives

In this chapter we mainly concentrate on introducing typical fraction derivatives;
Fractional derivatives are a mathematical concept that extands traditional derivatives to include derivatives of non-integer orders;
such a shalf-derivatives or quarter-derivatives.They allow for a more flexible way of calculating changes in functions ;going beyond the conventional limits of integer derivatives .
Fractional derivatives are used in various fields ; such as physics ; engineering; neuroscience; and mathematical modeling of systems exhbiting nonlinear behavior or memory effects .
There are several types of fractional derivatives including :
Riemann-Liouville ; Caputo

1.1 Special Function

A quick tour of the mathematical definitions associated with this concept can simplify and clarify the understanding of the definitions and the use of fractional calculus. Among these definitions, we will briefly look at the Gamma function, the Beta function, The Error function and The Complementary Error function.

1.1.1 Gamma Function

One of the fundamental functions of fractional calculus is Gamma function. This function generalizes the factorial $n!$ and allows n to be a non - integer number.

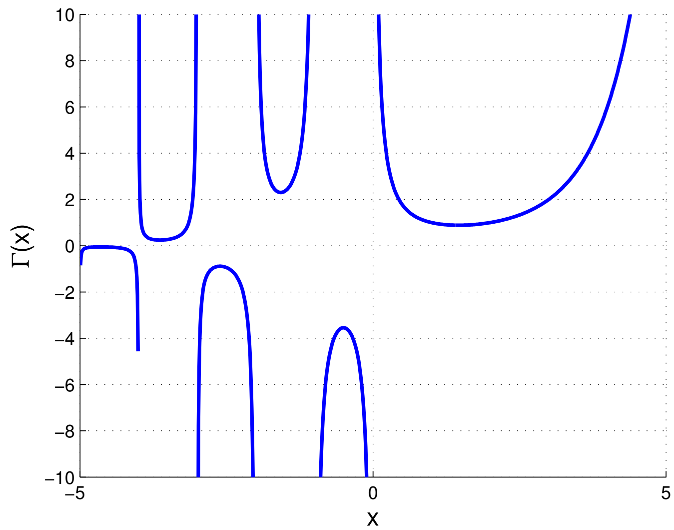

Figure 1.1: Gamma Function Plot on [-5, 5]

Definition 1.1.1 *[1]*

The gamma function $\Gamma(z)$ *is defined by the integral*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \tag{1.1}$$

Properties 1.1.1 *[2]*

The basic properties of the Gamma function are:

1 *The* $\Gamma(z)$ *function is continuous for* $z > 0$.

2 *One of the fundamental properties of the Gamma function is that it satisfies the following functional equation:*

$$\Gamma(z+1) = z\Gamma(z). \tag{1.2}$$

which can be easily proven by including:

$$\begin{aligned}
\Gamma(z) &= \int_0^{+\infty} e^{-t} t^{z-1} dt && (1.3)\\
&= \left[\frac{e^{-t}t^z}{z}\right]_0^{+\infty} + \frac{1}{z}\int_0^{+\infty} e^{-t}t^z dt && (1.4)\\
&= 0 + \frac{1}{z}\int_0^{+\infty} e^{-t} t^{(z+1)-1} dt && (1.5)\\
&= \frac{1}{z}\Gamma(z+1), && (1.6)
\end{aligned}$$

then

$$z\Gamma(z) = \Gamma(z+1).$$

3 ***Factorial Property*** *Obviously,* $\Gamma(1) = 1$*, and using the recurrence relation, we get values for* $z = 1, 2, 3, \ldots$*:*

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1! \tag{1.7}$$
$$\Gamma(3) = 2 \cdot \Gamma(2) = 2! \tag{1.8}$$
$$\Gamma(4) = 3 \cdot \Gamma(3) = 3! \tag{1.9}$$
$$\vdots \tag{1.10}$$
$$\Gamma(n+1) = n\Gamma(n) = n!. \tag{1.11}$$

4 ***Gamma Function at Zero***

$$\Gamma(0) = \infty. \tag{1.12}$$

5 ***Gamma Function for Negative Integers*** *For* $z = -n = -1, -2, -3, \ldots$*, the result is:*

$$\Gamma(-n) = \frac{\Gamma(-n+1)}{-n} \tag{1.13}$$
$$= \frac{\Gamma(-n+2)}{(-n)(-n+1)} = \frac{\Gamma(-n+3)}{(-n)(-n+1)(-n+2)} = \cdots = \frac{\Gamma(0)}{(-1)^n n!} \tag{1.14}$$
$$= \begin{cases} -\infty, & \text{if } n \text{ is even,} \\ +\infty, & \text{if } n \text{ is odd.} \end{cases} \tag{1.15}$$

6 ***Binomial Relation***

$$\frac{\Gamma(z+1)}{\Gamma(y+1)\Gamma(z-y+1)} = \binom{z}{y}. \tag{1.16}$$

7 ***Special Values*** *The following special values for the Gamma function are known:*

- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
- $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$.
- $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1 + \frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$.
- $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(2 + \frac{1}{2}\right) = \frac{4}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}$.

8 ***Reflection Property*** *The reflection property of the Gamma function is given by:*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad 0 \leq z \leq 1. \tag{1.17}$$

1.1.2 The Beta Function

Beta function has been studied by Euler and is related to the gamma function.

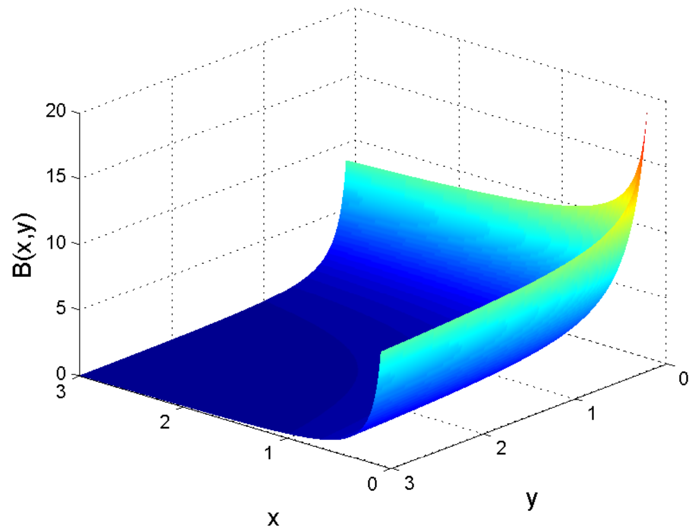

Figure 1.2: Beta Function B(x,y) for x,y in [0.3],[0.3]

Definition 1.1.2 *[3] The beta function is the Euler integral equation of the first kind defined for complex numbers z and v by:*

$$B(z,v) = \int_0^1 t^{z-1}(1-t)^{v-1}dt; \tag{1.18}$$

with $\Re(z) > 0, \Re(v) > 0.$

Properties 1.1.2 *[4] For all $x, y \in \mathbb{C}$ with $\Re(x) > 0, \Re(y) > 0$:*

- *The Beta function is related to the Gamma function by the following relationship:*

$$B(z,v) = \frac{\Gamma(z)\Gamma(v)}{\Gamma(z+v)}. \tag{1.19}$$

- $B(z,v) = B(v,z)$ *(symmetric).*
- $B(z,1) = \frac{1}{z}$.
- $B(z,v) = B(z+1,v) = B(z,v+1)$.

1.1.3 The Error Function and Complementary Error Function

Definition 1.1.3 *[2] We call the error function, the function* erf : $\mathbb{R} \to \mathbb{R}$*, defined by*

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \tag{1.20}$$

As the exponential function is continuous on $\mathbb{R}$*, this function is differentiable, and its derivative is:*

$$\operatorname{erf}'(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}. \tag{1.21}$$

Properties 1.1.3 *[4] The Error Function has four main properties:*

- $\operatorname{erf}(-\infty) = -1.$
- $\operatorname{erf}(+\infty) = 1.$
- $\operatorname{erf}(-z) = -\operatorname{erf}(z).$
- $\operatorname{erf}(z^*) = [\operatorname{erf}(z)]^*.$

The entire series

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)n!} z^{2n+1}$$

converges for all real x *(the radius of convergence is infinite). Moreover, we have*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}.$$

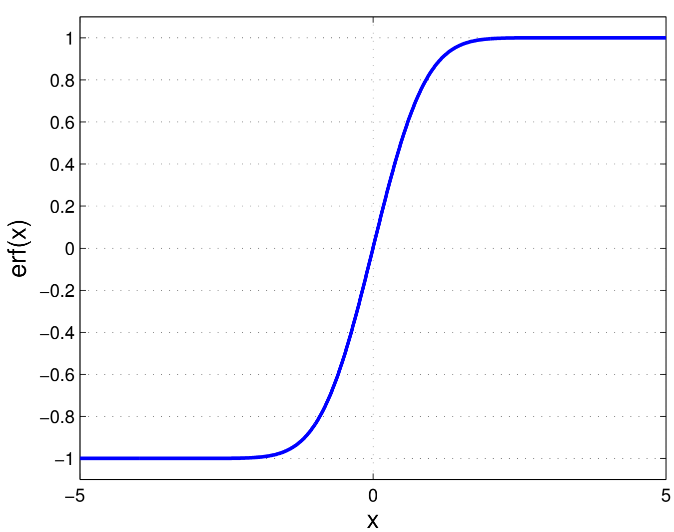

Figure 1.3: Error Function erf(x) on [-5, 5]

Definition 1.1.4 *[1]*
Erfc *is the complementary error function, commonly denoted* erfc(z)*, and is an entire function defined by:*

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt.$$

The derivative is given by

$$\frac{d}{dz} \operatorname{erfc}(z) = -\frac{2e^{-z^2}}{\sqrt{\pi}},$$

and the indefinite integral by

$$\int \operatorname{erfc}(z) dz = z \operatorname{erfc}(z) - \frac{e^{-z^2}}{\sqrt{\pi}} + C,$$

Properties 1.1.4 *[1] The complementary error function, denoted by erfc(x), has the following special values:*

- $erfc(-\infty) = 2$
- $erfc(0) = 1$
- $erfc(+\infty) = 0$
- $erfc(-z) = 2 - erfc(z)$
- $\int_0^{\infty} erfc(t)\, dt = \frac{1}{\sqrt{\pi}}$
- $\int_0^{\infty} erfc^2(t)\, dt = \frac{2-\sqrt{2}}{\sqrt{\pi}}$

1.1.4 The Mittag-Leffler Function

The MLF arises from the solution of fractional-order differential or integral equations. It extends exponential functions and can be represented as a power series.

Definition 1.1.5 *[2]*
(Generalized Mittag-Leffler Function). *The generalized MLF can be defined as*

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\beta + k\alpha)} \frac{z^k}{k!} \quad \text{for } R(\alpha), R(\beta), R(\gamma) > 0 \text{ and } z, \alpha, \beta, \gamma \in \mathbb{C}. \tag{1.7}$$

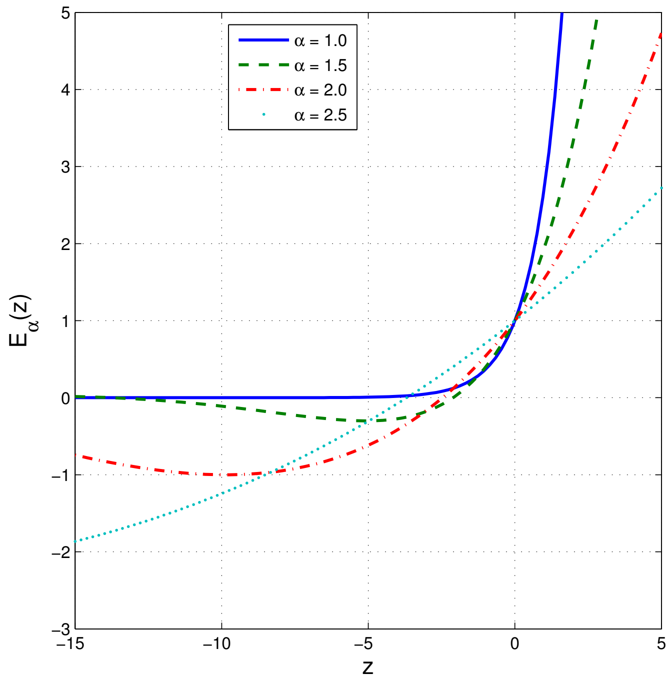

Figure 1.4: Mittag-Leffler Function $E_\alpha(z)$ for Different α Values

Definition 1.1.6 *[1]*
One-parameter MLF is defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\alpha)} \quad \textit{for } z \in \mathbb{C} \textit{ and } \alpha > 0. \tag{1.22}$$

If we put $\alpha = 1$ in 1.1.6, we obtain

$$E_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n)} \quad \textit{for } z \in \mathbb{C}. \tag{1.23}$$

which is the summation form of the exponential function e^z. So, MLF is an extension of the exponential function in one parameter.

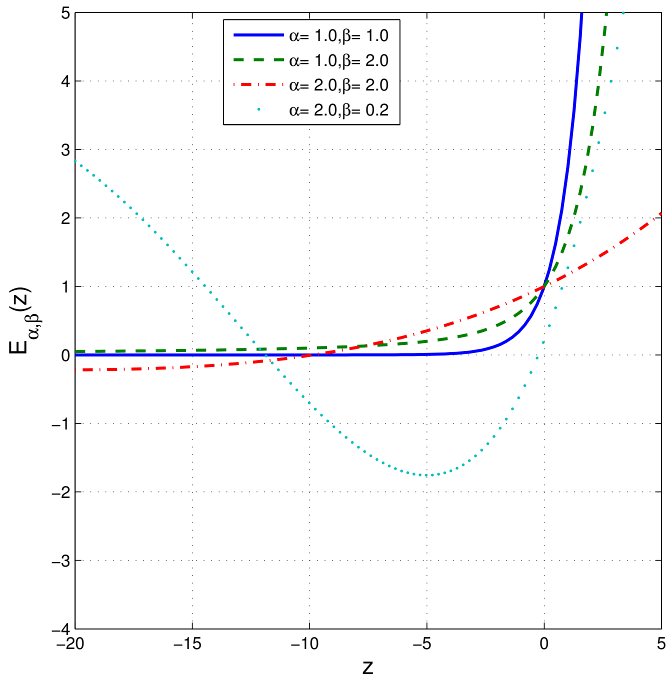

Figure 1.5: Two-Parameter Mittag-Leffler Function $E_{\alpha,\beta}(z)$

Definition 1.1.7 *[1][4]*
Two-parameter representation of the MLF may be written as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta + n\alpha)} \quad \textit{for } z \in \mathbb{C} \textit{ and } \alpha, \beta > 0. \tag{1.6}$$

Especially,

$$E_{1,2}(z) = \frac{e^z - 1}{z}, \quad E_{2,2}(z^2) = \frac{\sinh(z)}{z}, \quad E_{1/2,1}(z) = e^{z^2} \operatorname{erfc}(-z).$$

where $(\gamma)_n$ is the Pochhammer symbol and is defined as:

$$(\gamma)_n = \begin{cases} 1, & n = 0, \gamma \neq 0, \\ \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = (\gamma + n - 1) \cdots (\gamma + 2)(\gamma + 1), & n \in \mathbb{N}, \gamma \in \mathbb{C}. \end{cases} \tag{1.8}$$

Remark 1.1.1
The derivative of the two-parametric MLF can be expressed in the form of generalized MLF as:

$$\frac{d^n}{dz^n} E_{\alpha,\beta}(z) = n! E^{\gamma+n}_{\alpha,\beta+n}(z), \quad n \in \mathbb{N}, z \in \mathbb{C}. \tag{1.9}$$

Properties 1.1.5 *[2] Some properties of the MLF are given as follows:*

1. $E_{\alpha,\beta}(z) = \frac{1}{z}[E_{\alpha,\alpha+\beta-1}(z) + \beta E_{\alpha,\beta+1}(z)]$,
2. $\frac{d}{dz}E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z E_{\alpha,\beta+1}(z)$,
3. $\left(\frac{d}{dz}\right)^m [z^{\beta-1}E_{\alpha,\beta}(z^\alpha)] = z^{\beta-m-1}E_{\alpha,\beta-m}(z^\alpha), \quad R(\beta-m) > 0, m = 0, 1, 2, \ldots$

1.2 Riemann-Liouville derivative

In this section ; We are intersted in Riemann-Liouville fractional derivative firstly its definition ; then present some examples.

Definition 1.2.1 ***Riemann-Liouville fractional derivative***
The Riemann-Liouville fractional derivative can be defined using the definition of the Riemann-Liouville fractional derivative,its used in the stady of fractional calculs and defined for a function $f(x)$ as:[16]

$$\begin{aligned} D[D^{1-\alpha}f(x)] =& D[D^{-1}D^{-(1-\alpha-1)}f(x) && (1.24)\\ =& D[D^{-1}D^{\alpha}f(x) && (1.25)\\ =& D^{\alpha}f(x) && (1.26) \end{aligned}$$

Hence:

$$D[D^{-1-\alpha}f(x) = D^{\alpha}f(x) \tag{1.27}$$

Now, using the definition of the fractional integral we get ;

$$\begin{aligned} D^{\alpha}f(x) =& \frac{d}{dx}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha}f(t)dt\right] && (1.28)\\ =& \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}f(t)dt && (1.29) \end{aligned}$$

if we differentiate the fractional integral n-times so we have

$$\begin{aligned} D^{\alpha}f(x) =& \frac{d}{dx}\frac{d}{dx}\ldots\frac{d}{dx}D^{-(n-\alpha)}f(x) && (1.30)\\ D^{\alpha}f(x) =& D^n\left[D^{-(n-\alpha)}f(x)\right] \quad ; \quad n-1 \leq \alpha \leq n && (1.31) \end{aligned}$$

Now we can give the definition of The Riemann-Liouville fractional derivative .

Definition 1.2.2
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a coutinous f function . Then Riemann-Liouville fractional derivative of order α of a functions $f(x)$ is giving by :

$$D^{\alpha}f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\dfrac{d^n}{dx^n}\displaystyle\int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt & n-1 \leqslant \alpha \leqslant n \\ \dfrac{d^n}{dx^n}f(x) & \alpha = n \in \mathbb{N} \end{cases} \tag{1.32}$$

Where $\Gamma(\alpha)$ denotes Gamma function.

1.2.1 Examples of Riemann-liouivill fractional derivative :

1. **Constant function:**
   if:$f(x) = k$ Were k is a constant ; then $D^{\alpha} f(x) = \dfrac{k}{\Gamma(1-\alpha)} x^{-\alpha}$
   solution:by the definition of Riemann-Liouville fractional derivative we have :

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt \tag{1.33}$$

$$D^{\alpha} k = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{k}{(x-t)^{\alpha-n+1}} dt \tag{1.34}$$

   tet $t = xu$; $0 \leqslant u \leqslant 1$ and $dt = u$ then

$$D^{\alpha} k = \frac{k}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^1 (x-xu)^{n-\alpha-1} du \tag{1.35}$$

$$= \frac{k}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{n-\alpha} \int_0^1 (1-u)^{n-\alpha-1} du \tag{1.36}$$

$$= \frac{k}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{n-\alpha} \beta(1; n-\alpha) \tag{1.37}$$

$$= \frac{k}{\Gamma(n-\alpha+1)} \frac{d^n}{dx^n} x^{n-\alpha} \tag{1.38}$$

$$= \frac{k}{\Gamma(n-\alpha+1)} \frac{\Gamma(n-\alpha+1)}{\Gamma(1-\alpha)} x^{-\alpha} \tag{1.39}$$

$$= \frac{k}{\Gamma(1-\alpha)} x^{-\alpha} \tag{1.40}$$

$$\tag{1.41}$$

   Tus we have establish that

$$D^{\alpha} k = \frac{k}{\Gamma(1-\alpha)} x^{-\alpha} \tag{1.42}$$

   From this example we can say that the fractional derivative of a constant is not zero by Riemann-Liouiville definition ; not that it is inconsistant result ; since the result is a function of x .

2. **Power Function** If we take $f(x) = x^m ; m \geq 0$ the fractional derivative becomes:
   solution:by the definition of Riemann-Liouiville fractional derivative we obtain :

$$D^{\alpha} x^m = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{t^m}{(x-t)^{\alpha-n+1} dt} \tag{1.43}$$

set $t = ux$ for $0 \leq u \leq 1; dt = xdu$ we got

$$D^\alpha x^m = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^1 (xu)^m (x(1-u))^{n-\alpha-1} x du \tag{1.44}$$

$$D^\alpha x^m = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{m+n-\alpha} \int_0^1 u^m (1-u)^{n-\alpha-1} du \tag{1.45}$$

$$D^\alpha x^m = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{m+n-\alpha} \beta(m+1; n-\alpha) \tag{1.46}$$

$$D^\alpha x^m = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{m+n-\alpha} \frac{\Gamma(m+1)\Gamma(n-\alpha)}{\Gamma(m+n-\alpha+1)} \tag{1.47}$$

$$D^\alpha x^m = \frac{\Gamma(m+1)}{\Gamma(m+n-\alpha+1)} \frac{d^n}{dx^n} x^{m+n-\alpha} \tag{1.48}$$

$$D^\alpha x^m = \frac{\Gamma(m+1)}{\Gamma(m+n-\alpha+1)} \frac{\Gamma(m+1-\alpha+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha} \tag{1.49}$$

$$D^\alpha x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha} \tag{1.50}$$

In the above example which is known as the power rule we obtain .

$$D^\alpha x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha};\ m \geq 0 \tag{1.51}$$

1.3 Caputo derivative

the fractional-order derivative $D_c^\alpha f(x)$ in the Caputo sense is defined as follows :[12]

$$D_c^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt;\ n-1 < \alpha \leq n;\quad n \in \mathbb{N} \tag{1.52}$$

Where $\alpha \in \mathbb{R}_+^*$ is the order of the derivative .
we also have D_c^α is a linear where :

$$D_c^\alpha(\beta f(x) + \gamma g(x)) = \beta D_c^\alpha(f(x) + \gamma D_c^\alpha g(x)) \tag{1.53}$$

where D_c^n is the classical differential operator of order n. For the Caputo derivative we have

$$D_c^\alpha x^\beta = \begin{cases} 0, & \text{for } \beta < \alpha, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \geq \alpha. \end{cases} \tag{1.54}$$

Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of an integer order. Similar to the integer-order differentiation, the Caputo's fractional differentiation is a linear operation; i.e.

$$D_c^\alpha(\lambda f(x) + \mu g(x)) = \lambda D_c^\alpha f(x) + \mu D_c^\alpha g(x), \tag{1.55}$$

where λ and μ are constants.[15]

Definition 1.3.1 *[10][15]*
the left and the right-sided Caputo derivatives of order $\alpha > 0$ are defined by :

$$cD^{\alpha}_{\alpha;x}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{\alpha}^{x}\frac{f^{m}dt}{(x-t)^{\alpha-n+1}}\ ;\ x > \alpha \tag{1.56}$$

and

$$cD^{\alpha}_{x;b}f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b}\frac{f^{m}dt}{(t-x)^{\alpha-n+1}}\ ;\ x < b \tag{1.57}$$

resectivly where n is a positive integer satisfying $m-1 < \alpha \leq m$
it follows from equation 1.56 and 1.57 that the n th order diffferentiation is required for the Caputo derivtive precise results on the existence of Caputo derivative are presented as follows

Remark 1.3.1
if k is a constant then ${}^{c}D^{\alpha}_{t}k = 0$ from this formal we cn find the fractional derivative of any polynominal ; by taking fractional derivatives of each term separately .

1.3.1 Example of caputo derivative

Calculation of the Caputo Fractional Derivative of the Function $f(t) = t$
The derivation of the function $f(t) = t$ using the Caputo fractional derivative of order α (where $0 < \alpha \leq 1$) can be summarized as follows:
1. Definition of the Caputo Fractional Derivative
The Caputo fractional derivative of a function $f(t)$ of order α is defined as:

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau$$

where $n-1 < \alpha \leq n$. Given that $0 < \alpha \leq 1$ in our case, we take $n = 1$. Consequently, the definition simplifies to:

$$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{f'(\tau)}{(t-\tau)^{\alpha}}d\tau$$

2. Application to the Function $f(t) = t$
First, we calculate the first ordinary derivative of the function $f(t) = t$:

$$f'(t) = \frac{d}{dt}(t) = 1$$

Substituting this into the Caputo derivative definition:

$$D^{\alpha}(t) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{1}{(t-\tau)^{\alpha}}d\tau$$

3. Solving the Integral
We then evaluate the definite integral:

$$\int_{0}^{t}(t-\tau)^{-\alpha}d\tau = \left[-\frac{(t-\tau)^{1-\alpha}}{1-\alpha}\right]_{0}^{t}$$

Applying the limits of integration, where the upper limit ($\tau = t$) yields zero (since $1-\alpha > 0$) and the lower limit ($\tau = 0$) yields $-t^{1-\alpha}/(1-\alpha)$:

$$= 0 - \left(-\frac{t^{1-\alpha}}{1-\alpha}\right) = \frac{t^{1-\alpha}}{1-\alpha}$$

4. Final Result

Substituting the result of the integral back into the Caputo derivative equation:

$$D^{\alpha}(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{t^{1-\alpha}}{1-\alpha}\right)$$

By utilizing the Gamma function property $\Gamma(z+1) = z\Gamma(z)$, which allows us to simplify $(1-\alpha)\Gamma(1-\alpha)$ to $\Gamma(2-\alpha)$, we arrive at the final result:

$$D^{\alpha}(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$$

Proposition 1.3.1 *We have the following properties :*

1.
$$^{R}D_a^{-\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau\right), \quad f(a) \tag{1.58}$$
$$^{C}D_a^{\alpha}{}^{R}D_a^{-\alpha}f(t) = \frac{f(a)}{\Gamma(1-\alpha)(t-a)^{1-\alpha}} + {}^{C}D_a^{\alpha}f(t). \tag{1.59}$$

2.
$$^{C}D_a^{\alpha}f(t) = {}^{R}D_a^{\alpha}(f(t) - f(a)). \quad (2.35) \tag{1.60}$$

3. *if $f$is continuous on $[a, b]$, then :*
$$^{C}D_a^{\alpha}I_a^{\alpha}f(t) = f(t). \quad (2.36) \tag{1.61}$$

4. *if $f \in C^m[a, b]$,then :*
$$I_a^{\alpha C}D_a^{\alpha}f(t) = f(t) - \sum_{i=0}^{m-1}\frac{f^{(i)}(a)}{i!}(t-a)^i. \tag{1.62}$$

Thus the Caputo fractional differentiation operator is a left inverse of the Riemann-Liouville fractional integration operator of the same order ; but it is not a right inverse.

1.4 Relationship between fractional derivatives of Riemann-Liouville and Caputo

Let $Re(\alpha) > 0$ with $n - 1 < Re(\alpha) < n$, $(n \in \mathbb{N}^*)$ assume that f is a function such that ${}^{C}D^{\alpha}f(x)$ and ${}^{RL}D^{\alpha}f(x)$ exist, then:

$$
{}^{C}D^{\alpha}f(x) = {}^{RL}D^{\alpha}f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} x^{k-\alpha}. \tag{1.63}
$$

We deduce that if $f^{(k)}(0) = 0$ for $k = 0, 1, 2, \ldots, m-1$, then:

$$
{}^{C}D^{\alpha}f(x) = {}^{RL}D^{\alpha}f(x). \tag{1.64}
$$

Chapter 2

Classical orthogonal polynomials

This chapter is dedicated to introducing some fundamental notions; we will also give a generality about orthogonal polynomials, notions of orthogonality with some definitions that we will need in our study.

2.1 Orthogonal Polynomials

The simplest scalar product of functions is the integral of the product of these functions, over a bounded interval :[5]

$$(f, g) = \int_a^b f(y)g(y)dy$$

More generally, we can introduce a weight function $w(y)$ in the integral (over the integration interval $[a, b]$), $w(y)$ must take finite and strictly positive values, and the integral of the product of the weight function by a polynomial must be finite, the bounds a, b can be infinite):

$$(f, g) = \int_a^b f(y)g(y)w(y)dy$$

With this definition of the scalar product, two functions are orthogonal to each other if their scalar product is zero (in the same way that two vectors are orthogonal if their scalar product is zero). We also introduce the norm $\|f\| = \sqrt{(f, f)}$, the scalar product makes the set of all functions of finite norm a Hilbert space, the integration interval is called the orthogonality interval.

2.1.1 Rodrigues' Formula

In this section, we assume that we have a family of orthogonal polynomials given by a Rodrigues' formula of the form:

$$p_n(y) = \frac{1}{K_n w(y)} \frac{d^n}{dy^n} [w(y)y^n]$$

Where:

- $p_n(y)$: is a polynomial of degree n.
- K_n: is a number depending on the normalization.
- y: is a polynomial in y of degree k.
- $w(y)$: is the weight function of the orthogonal polynomials.

For $n = 1$:

$$k_1 p_1(y) = y' + y\frac{w'(y)}{w(y)}$$

2.1.2 Differential Equation and Form

Let $(p_n(y))_{n\in\mathbb{N}}$ be a sequence of orthogonal polynomials defined using a Rodrigues' formula. Then $p_n(y)$ satisfies, for $n \geq 0$, a differential equation of the form:

$$A(y)Y'' + B(y)Y' + \lambda_n Y = 0$$

Where $A(y)$ and $B(y)$ do not depend on n, and λ_n depends on n.

Remark 2.1.1
The associated formally self-adjoint operator to this differential equation is:

$$\frac{d}{dy}\left[yw\frac{dY}{dy}\right] + \lambda_n wY = 0$$

2.1.3 Classical Orthogonal Polynomials [6]

2.1.3.1 Jacobi Polynomials [6]

The study interval: $[-1, 1]$ Weight function: where $a > -1$ and $b > -1$

$$w(y) = (1-y)^a(1+y)^b$$

Explicit formulat:

$$J_n^{(a,b)}(y) = \frac{1}{2^n}\sum_{m=0}^{n}\binom{n+a}{m}\binom{n+b}{n-m}(y-1)^{n-m}(y+1)^m$$

Where we have:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Differential equation:

$$(1-y^2)Y'' + (b-a-(a+b+2)y)Y' + n(n+a+b+1)Y = 0$$

$$Y(y) = J_n^{(a,b)}(y)$$

Rodrigues' formula:

$$J_n^{(a,b)}(y) = \frac{(-1)^n}{2^n n!(1-y)^a(1+y)^b}\frac{d^n}{dy^n}\left[(1-y)^{n+a}(1+y)^{n+b}\right]$$

The orthogonality relation for Jacobi polynomials $J_n^{(a,b)}(x)$ is given by:

$$\int_{-1}^{1}(1-y)^a(1+y)^b J_n^{(a,b}(y)J_m^{(a,b)}(y)dy = \begin{cases} 0, & \text{if } n \neq m \\ \frac{2^{a+b+1}}{2n+a+b+1}\frac{\Gamma(n+a+1)\Gamma(n+b+1)}{\Gamma(n+1)\Gamma(n+a+b+1)}, & \text{if } n = m \end{cases}$$

where the parameters satisfy $a > -1$ and $b > -1$
The well-known Jacobi polynomials are defined on the interval $[-1, 1]$ and can be generated with the aid of the following recurrence formula:

$$J_i^{(a,b)}(y) = \frac{(a+b+2i-1)((a^2-b^2+y(a+b+2i)(a+b+2i-2))}{2i(a+b+i)(a+b+2i-2)}J_{i-1}^{(a,b)}(y) \quad (2.1)$$

$$-\frac{(a+i-1)(b+i-1)(a+b+2i)}{i(a+b+i)(a+b+2i-2)}J_{i-2}^{(a,b)}(y), \quad (2.2)$$

$i = 2, 3, \ldots$.

where

$$J_0^{(a,b)}(y) = 1 \quad \text{and} \quad J_1^{(a,b)}(y) = \frac{a+b+2}{2}y + \frac{a-b}{2}.$$

The three polynomials of Jacobi are:

$$J_0^{(a,b)}(y) = 1 \quad (2.3)$$

$$J_1^{(a,b)}(y) = \frac{1}{2}[2(a+1)+(a+b+2)(y-1)] \quad (2.4)$$

$$J_2^{(a,b)}(y) = \frac{1}{8}[4(a+1)(a+2)+4(a+2)(a+b+3)(y-1) \quad (2.5)$$

$$+(a+b+3)(a+b+4)(y-1)^2] \quad (2.6)$$

2.1.3.2 Special Jacobi Polynomials

(a) Legendre

$$P_n(y) = J^{(0,0)}(y)$$

(b) Chebyshev 1^{st} kind: $T_n(y) = \frac{1}{\binom{\frac{1}{2}+k}{k}}J_k^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$

2^{nd} kind : $U_n(y) = \frac{k+1}{\binom{\frac{1}{2}+k}{k}}J_k^{\left(\frac{1}{2},\frac{1}{2}\right)}$

(c) Gegenbauer (ultraspherical)

$$G_n(y) = \frac{\binom{2Y+k+1}{k}}{\binom{y-\frac{1}{2}+k}{k}}J_k^{\left(Y-\frac{1}{2},Y-\frac{1}{2}\right)}, \quad \text{for } 0 \neq Y > -\frac{1}{2}$$

And

$$J_k^{(0)} = \lim_{Y\to 0}\frac{G_k^{(Y)}}{Y}$$

2.1.3.3 Legendre Polynomials [6]

Study interval: $[-1, 1]$

Weight function:

$$w(y) = 1$$

Explicit formula:

$$P_n(y) = \frac{1}{2^n} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \binom{n}{m} \binom{2n-2m}{n} y^{n-2m}$$

Differential equation:

$$(1-y^2)Y'' - 2yY' + n(n+1)Y = 0 \tag{2.7}$$

$$Y = P_n(y) \tag{2.8}$$

Rodrigues' Formula:

$$P_n(y) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dy^n} \left\{ (1-y^2)^n \right\}$$

Orthogonality Relation:

$$\int_{-1}^{+1} P_n(y) P_m(y) dy = \begin{cases} 0, & \text{if } n \neq m \\ \frac{2}{2n+1}, & \text{if } n = m \end{cases}$$

The first three functions of these polynomials are:

$$P_0(y) = 1, \quad P_1(y) = y, \quad P_2(y) = \frac{3}{2}y^2 - \frac{1}{2}$$

$$P_n(1) = 1, \quad P_n(-1) = (-1)^n$$

2.1.3.4 Chebyshev Polynomials (first kind)

Study interval: $[-1, 1]$

Weight function:

$$w(y) = (1-y^2)^{-\frac{1}{2}}$$

Explicit formula:

$$T_n(y) = \frac{n}{2} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \frac{(n-m-1)!}{m!(n-2m)!} (2y)^{n-2m} = \cos(n \arccos y)$$

Differential equation:

$$(1-y^2)Y'' - yY' + n^2 Y = 0 \tag{2.9}$$

$$Y = T_n(y) \tag{2.10}$$

Rodrigues' Formula:

$$T_n(y) = \frac{(-1)^n(1-y^2)^{\frac{1}{2}}}{2^n\sqrt{\pi}\Gamma(n+\frac{1}{2})}\frac{d^n}{dy^n}\left\{(1-y^2)^{n-\frac{1}{2}}\right\}$$

Orthogonality Relation:

$$\int_{-1}^{+1}\frac{T_n(y)T_m(y)}{\sqrt{1-y^2}}dy = \begin{cases}0, & n \neq m\\ \pi, & n = m = 0\\ \frac{\pi}{2}, & n = m > 0\end{cases}$$

The three polynomials of Chebyshev Polynomials (first kind)are:

$$T_0(y) = 1, \quad T_1(y) = x, \quad T_2(y) = 2y^2 - 1$$

2.1.3.5 Chebyshev Polynomials (second kind)

Study interval: $[-1, 1]$

Weight function:

$$w(y) = (1-y^2)^{\frac{1}{2}}$$

Explicit Formula:

$$U_n(y) = \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^m\binom{n-m}{m}(2y)^{n-2m} = \frac{\sin((n+1)\arccos y)}{\sqrt{1-y^2}}$$

Differential equation:

$$(1-y^2)Y'' - 3yY' + n(n+2)Y = 0 \tag{2.11}$$

$$Y = U_n(y) \tag{2.12}$$

Rodrigues' Formula:

$$U_n(y) = \frac{(-1)^n(n+1)\sqrt{\pi}}{(1-y^2)^{\frac{1}{2}}2^{n+1}\Gamma(n+\frac{3}{2})}\frac{d^n}{dy^n}\left\{(1-y^2)^{n+\frac{1}{2}}\right\}$$

Orthogonality Relation:

$$\int_{-1}^{+1}U_n(y)U_m(y)\sqrt{1-y^2}dy = \begin{cases}0, & n \neq m\\ \frac{\pi}{2}, & n = m\end{cases}$$

three polynomials of Chebyshev Polynomials (second kind) are:

$$U_0(y) = 1, \quad U_1(y) = 2y, \quad U_2(y) = 4y^2 - 1$$

2.1.3.6 Gegenbauer Polynomials (or ultraspherical)

Study interval: $[-1, 1]$

Weight function:

$$w(y) = (1-y^2)^{\alpha-\frac{1}{2}}$$

Explicit formula:

$$G_n^{(a)}(y) = \frac{1}{\Gamma(a)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{\Gamma(n-k+a)}{k!(n-2k)!} (2x)^{n-2k}$$

$$G_n^{(a)}(y) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \frac{\Gamma(n-m+a)}{\Gamma(a)m!(n-2m)!} (2y)^{n-2m}$$

Differential equation:

$$(1-y^2)Y'' - (2a+1)yY' + n(n+2a)Y = 0 \tag{2.13}$$

$$Y = G_n^{(a)}(y) \tag{2.14}$$

Rodrigues' Formula:

$$G_n^{(a)}(y) = \frac{(-1)^n (1-y^2)^{\frac{1}{2}-a} \Gamma(n+2a)}{2^n n! \Gamma(a+\frac{1}{2}) \Gamma(n+a+\frac{1}{2})} \frac{d^n}{dy^n} \left\{ (1-y^2)^{n+a-\frac{1}{2}} \right\}$$

Orthogonality Relation:

$$\int_{-1}^{+1} (1-y^2)^{a-\frac{1}{2}} G_n^{(a)}(y) G_m^{(a)}(y) dy = \begin{cases} 0, & \text{if } n \neq m \\ \frac{\pi 2^{1-2a} \Gamma(n+2a)}{n!(n+a)[\Gamma(a)]^2}, & \text{if } n = m \end{cases}$$

where $a > \frac{1}{2}$

The three polynomials of Gegenbauer are :

$$G_0^{(a)}(y) = 1, \quad G_1^{(a)}(y) = 2ay, \quad G_2^{(a)}(y) = -a + 2a(1+a)y^2$$

2.1.3.7 Laguerre Polynomials [14]

Interval of Study $[0, +\infty[$

Weight Function

$$W(y) = e^{-y}$$

Explicit Formula: $L_n(y) = \sum_{k=0}^{n} \frac{(-1)^k n! y^k}{k!^2 (n-k)!}$

Differential Equation

$$Y'' + (1-y)Y' + nY = 0$$

Rodrigues' Formula:

$$L_n(y) = \frac{e^y}{n!} \frac{d^n}{dy^n} (x^n e^{-y})$$

Orthogonality Relation:

$$\int_0^{+\infty} e^{-y} L_n(y) L_m(y) dy = \begin{cases} 0, & \text{if } n \neq m \\ (n!)^2, & \text{if } n = m \end{cases}$$

The Three Polynomials of Laguerre are:

$$L_0(y) = 1, \quad L_1(y) = 1 - y, \quad L_2(y) = y^2 - 4y + 2$$

2.1.3.8 Hermite Polynomials [7]

Study interval: $y \in \mathbb{R} =]-\infty, +\infty[$

Weight function:

$$w(y) = e^{-y^2}$$

Explicit formula:

$$H_n(y) = n! \sum_{k=0}^{E\left(\frac{n}{2}\right)} \frac{(-1)^k (2y)^{n-2k}}{k!(n-2k)!}$$

Differential equation:

$$H_n''(y) - 2y H_n'(y) + 2n H_n(y) = 0$$

Rodrigues' Formula:

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$$

Orthogonality Relation:

$$\int_{-\infty}^{+\infty} H_n(y) H_m(y) e^{-y^2} dy = 0, \quad \text{if } n \neq m$$

$$\int_{-\infty}^{+\infty} H_n(y) H_m(y) w(y) dy = \gamma_n \delta_{mn}, \quad \gamma_n = \sqrt{\pi} 2^n n!, \quad \text{if } n = m$$

The Three Polynomials of Hermite are:

$$H_0(y) = 1, \quad H_1(y) = 2y, \quad H_2(y) = 4y^2 - 2$$

2.2 Numerical Integration

2.2.1 Gaussian Methods [6]

Gauss methods are among the most common and most precise methods; the integration is exact for any polynomial of degree less than or equal to $2n+1$. Let $\{\Psi_i\}$ be a family of orthogonal polynomials for the weight function $w(y)$ over the interval $[a, b]$. We are looking for an approximate integral $\int_a^b f(y) w(y) dy$ using a formula of Lagrange type:

$$f(y) = \sum_{i=0}^{n} L_i(y) f(y_i) + \prod_{j=0}^{n} \frac{f^{(n+1)}(c)}{(n+1)!}, \quad \text{with } c \in [a, b]$$

$$L_i(y) + \prod_{0\leqslant j\leqslant n, j\neq i}^{n} \frac{(y-y_j)}{(y_i-y_j)}$$

If $\{\Psi_n\}$ is a basis of orthogonal polynomials for the weight function $w(y)$ on $[a,b]$, we have:

$$\int_a^b \Psi_n(y)\Psi_m(y)\omega(y)dy = 0, \quad \text{if } n \neq m$$

Let's develop the product based on this property:

$$\prod_{i=0}^{n}(y-y_i) = \sum_{i=0}^{n+1} a_i\Psi_i(y)$$

And if f is a polynomial of degree $(2n+1)$, let's denote:

$$Q_n(y) = \frac{f^{(n+1)}(y)}{(n+1)!} = \sum_{i=0}^{n+1} b_i\Psi_i(y)$$

The remainder is expressed by:

$$R_n(y) = \prod_{i=0}^{n}(y-y_i)\frac{f^{(n+1)}(c)}{(n+1)!} = \sum_{i=0}^{n} a_i b_i \Psi_i(y)\Psi_j(y) + a_{n+1}\sum_{i=0}^{n} b_i\Psi_i(y)\Psi_{i+1}(y)$$

whose integral is:

$$\int_a^b f(y)f(y)\omega(y)dy = \int_a^b \sum_{i=0}^{n} L_i(y)f(y_i)w(y)dy + \int_a^b R_n(y)\omega(y)dy + \varepsilon$$

Thus, by virtue of the orthogonality of the polynomials:

$$\int_a^b R_n(y)\omega(y)dy = \sum_{i=0}^{n} a_i b_i \int_a^b \Psi^2{}_i(y)\omega(y)$$

By choosing the points $\{y_i\}$ of the subdivision as the $(n+1)$ roots of the polynomial of degree $n+1$, we set $a_i = 0$ for $i = 0, 1, ..., n$ and $a_{n+1} \neq 0$, that is to say:

$$\prod_{i=0}^{n}(y-y_i) = \sum_{i+0}^{n+1} a_i\Psi_i(y) = a_{n+1}\Psi_{n+1}(y)$$

From where:

$$\int_a^b R_n(y)\omega(y)dy = 0.$$

Consequently, the Gaussian method applied to a function f leads to an approximation of the form:

$$\int_a^b f(y)\omega(y)dy = \sum_{i=0}^{n} \omega_i f(y_i) + \epsilon$$

With

$$\omega_i = \int_a^b L_i(y)\omega(y)dy.$$

The error is of the form $\epsilon = \varepsilon_n f^{(2n+2)}(c)$ where $c \in (a,b)$ depends on the choice of the orthogonal polynomials $\{\Psi_n\}$.[8]

2.2.2 Gauss-Legendre Integration [6]

When the family of orthogonal polynomials is the family of Legendre polynomials relative to the weight function $w(y) = 1$ over the interval $[-1, 1]$, the integral is approximated by the formula:

$$\int_{-1}^{1} f(y)dy = \sum_{i=0}^{n} w_i f(y_i) + \epsilon$$

Where the numbers w_i are given by:

$$w_i = \int_{-1}^{1} \prod_{0\leq j\leq n, j\neq i} \frac{(y - y_j)}{(y_i - y_j)} dy$$

And the y_i are the roots of the Legendre polynomial P_{n+1}.
The error is expressed by:
$\epsilon = \frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^3} f^{(2n+2)}(c), \quad$ with $c \in [-1, 1]$

2.2.3 Shifted Jacobi polynominals

In order to use these polynomials on the interval $x \in [0, L]$, we defined the so-called shifted Jacobi polynomials by introducing the change of variable $y = \frac{2x}{L} - 1$. Let the shifted Jacobi polynomials $J_{L,i}^{(a,b)}(x)$ be denoted by $J_{L,i}^{(a,b)}(x)$. Then $J_{L,i}^{(a,b)}(x)$ can be generated from:

$$J_{L,i}^{(a,b)}(x) = \frac{(a+b+2i-1)(a^2-b^2+(\frac{2x}{L}-1))(a+b+2i)(a+b+2i-2)}{2i(a+b+i)(a+b+2i-2)} J_{L,i-1}^{(a,b)}(x) \tag{2.15}$$

$$-\frac{(a+i-1)(b+i-1)(a+b+2i)}{i(a+b+i)(a+b+2i-2)} J_{L,i-2}^{(a,b)}(x) \tag{2.16}$$

i = 2, 3, ... where

$$J_{L,0}^{(a,b)}(x) = 1 \quad \text{and} \quad J_{L,1}^{(a,b)}(x) = \frac{a+b+2}{2}\left(\frac{2x}{L} - 1\right) + \frac{a-b}{2}. \tag{2.17}$$

The analytic form of the shifted Jacobi polynomials $J_{L,i}^{(a,b)}(x)$ of degree i is given by

$$J_{L,i}^{(a,b)}(x) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+b+1)\Gamma(i+k+a+b+1)}{\Gamma(k+b+1)\Gamma(i+a+b+1)(i-k)!k!L^k} x^k \tag{2.18}$$

where

$$J_{L,i}^{(a,b)}(0) = (-1)^i \frac{\Gamma(i+b+1)}{\Gamma(b+1)i!} \tag{2.19}$$

and

$$J_{L,i}^{(a,b)}(L) = \frac{\Gamma(i+a+1)}{\Gamma(a+1)i!} \tag{2.20}$$

The orthogonality condition of shifted Jacobi polynomials is

$$\int_0^L J_{L,i}^{(a,b)}(x) J_{L,j}^{(a,b)}(x) w_L^{(a,b)}(x) dx = h_i \tag{2.21}$$

where $w_L^{(a,b)}(x) = x^b (L - x)^a$ and

$$h_i = \begin{cases} \frac{L^{a+b+1}\Gamma(i+a+1)\Gamma(i+b+1)}{(2i+a+b+1)i!\Gamma(i+a+b+1)}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{2.22}$$

Chapter 3

Numerical solution of FI-DEs using spectral collocation methode

In this chapter our goal is to find an approximate solution of this equation:

$$\begin{cases} D^{\alpha}u(x) + \int_0^L k(x,t)u(x)dx = f(x) \\ u(0) = \rho \qquad\qquad x,t \in [0,L] \end{cases}$$

Let $u_N(x)$ the approximate solution of the exact solution $u(x)$, then $u_N(x)$ can be expressed in terms of shifted Jacobi polynomials as[9]

$$u_N(x) = \sum_{j=0}^{N} c_j J_{L,j}^{(a,b)}(x) = \mathbf{C}^T \boldsymbol{\phi}_L(x), \tag{3.1}$$

where the coefficients c_j are given by

$$c_j = \frac{1}{h_j} \int_0^L w_L^{(a,b)}(x)u(x)J_{L,j}^{(a,b)}(x)dx, \quad j = 0, 1, \dots \tag{3.2}$$

If the shifted Jacobi coefficient vector $\mathbf{C}$ and the shifted Jacobi vector $\boldsymbol{\phi}(x)$ are written as

$$\mathbf{C}^T = [c_0, c_1, \dots, c_N], \tag{3.3}$$

and

$$\boldsymbol{\phi}_L(x) = [J_{L,0}^{(a,b)}(x), J_{L,1}^{(a,b)}(x), \dots, J_{L,N}^{(a,b)}(x)]^T. \tag{3.4}$$

Lemma 3.0.1 *[9]*
Let $J_{L,i}^{(a,b)}(x)$ be a shifted Jacobi polynomial. Then

$$D^{\alpha} J_{L,i}^{(a,b)}(x) = 0, \quad i = 0, 1, 2, \dots, \lfloor \alpha \rfloor - 1, \quad \alpha > 0. \tag{3.5}$$

Proof —
Using Eqs 1.54 and 1.55 in Eq 2.18 the lemma can be proved.

The following theorem is generalizing the operational matrix of derivatives of shifted Jacobi polynomials given in Eq. (2.12)[18] to fractional calculus. ∎

3.1 Matrix of the Fractional derivative part

Theorem 3.1 *[11]*
Let $\boldsymbol{\phi}(x)$ be shifted Jacobi vector defined in Eq. 3.1 and let also $\alpha > 0$. Then

$$D^{\alpha}\boldsymbol{\phi}(x) \simeq \mathbf{D}^{(\alpha)}\boldsymbol{\phi}(x), \tag{3.6}$$

where $\mathbf{D}^{(\alpha)}$ is the $(N+1)\times(N+1)$ operational matrix of fractional derivatives of order α in the Caputo sense and is defined by:

$$\mathbf{D}^{(\alpha)} \equiv \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \Delta_{\alpha}(1,1) & \Delta_{\alpha}(1,2) & \cdots & \Delta_{\alpha}(1,N) \\ 0 & \Delta_{\alpha}(2,1) & \Delta_{\alpha}(2,2) & \cdots & \Delta_{\alpha}(2,N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \Delta_{\alpha}(N,1) & \Delta_{\alpha}(N,2) & \cdots & \Delta_{\alpha}(N,N) \end{bmatrix} \tag{3.7}$$

where

$$\Delta_{\alpha}(i,j) = \sum_{k=\lceil\alpha\rceil}^{i} \delta_{ijk} \tag{3.8}$$

and δ_{jk} is given by

$$\delta_{ijk} = \frac{(-1)^{i-k}L^{a+b-\alpha+1}\Gamma(j+b+1)\Gamma(i+k+a+b+1)}{h_j\Gamma(j+a+b+1)\Gamma(k+b+1)\Gamma(i+a+b+1)\Gamma(k-\alpha+1)(i-k)!}\sum_{l=0}^{j} \times \frac{(-1)^{j-1}\Gamma(j+l+a+b+1)\Gamma(a+1)\Gamma(l+k+b-\alpha+1)}{\Gamma(l+b+1)\Gamma(l+k+a+b-\alpha+2)(j-l)!l!} \tag{3.9}$$

Note that in $D^{(\alpha)}$, the first $\lfloor\alpha\rfloor$ rows are all zeros.[13]

Proof —
The analytic form of the shifted Jacobi polynomials $J_{L,i}^{(a,b)}(x)$ of degree i is given by 2.18 Using Eqs 1.54 and 1.55 in Eq.2.18 we have

$$\begin{aligned} D^{\alpha}P_{L,i}^{(a,b)}(x) &= \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+b+1)\Gamma(i+k+a+b+1)}{\Gamma(k+b+1)\Gamma(i+a+b+1)(i-k)!k!L^k} D^{\alpha}x^k \\ &= \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i-k}\Gamma(i+b+1)\Gamma(i+k+a+b+1)}{\Gamma(k+b+1)\Gamma(i+a+b+1)(i-k)!\Gamma(k-\alpha+1)L^k} x^{k-\alpha}, \end{aligned} \tag{3.10}$$

$$i = \lceil\alpha\rceil, \lceil\alpha\rceil+1, \ldots$$

Now, approximate $x^{k-\alpha}$ by $(N+1)$ terms of shifted Jacobi series, we get

$$x^{k-\alpha} \simeq \sum_{j=0}^{N} b_{k,j}^{(\alpha)} P_{L,j}^{(a,b)}(x), \tag{3.11}$$

where $b_{k,j}$ is given from 3.2 with $u(x) = x^{k-\alpha}$, and this immediately gives

$$b_{k,j} = \frac{L^{a+b+k-\alpha+1}\Gamma(j+b+1)}{h_j\Gamma(j+a+b+1)} \tag{3.12}$$

$$\times \sum_{l=0}^{j} \frac{(-1)^{j-l}\Gamma(j+l+a+b+1)\Gamma(a+1)\Gamma(l+k+b-\alpha+1)}{\Gamma(l+b+1)(j-l)!l!\Gamma(l+k+a+b-\alpha+2)} \tag{3.13}$$

Employing Eqs 3.10 and 3.12 , we get

$$D^\alpha J_{L,i}^{(a,b)}(x) = \sum_{j=0}^{N} \Delta_\alpha(i,j) J_{L,j}^{(a,b)}(x), \quad i = \lceil\alpha\rceil, \lceil\alpha\rceil+1, \ldots, N. \tag{3.14}$$

where $\Delta_\alpha(i,j)$ is given in Eq 3.9. Accordingly, rewriting Eq. 3.14 as a vector form gives

$$D^\alpha J_{L,i}^{(a,b)}(x) \simeq [\Delta_\alpha(i,0), \Delta_\alpha(i,1), \Delta_\alpha(i,2), \ldots, \Delta_\alpha(i,0), \Delta_\alpha(i,N)](x) \tag{3.15}$$

$i = 0, 1, \ldots, \lceil\alpha\rceil - 1$ Also according to 3.0.1 one can write

$$D^\alpha J_{L}^{(a,b)}(x) \simeq [0, 0, \ldots, 0]\boldsymbol{\phi}(x), \quad i = 0, 1, \ldots, \lceil\alpha\rceil - 1. \tag{3.16}$$

A combination of Eqs 3.15 and 3.16 leads to the desired result. ∎

3.2 Matrix of the Integral part

$$I = \int_0^L k(x,t)u(t)dt = \int_0^L \frac{k(x,t)}{W_L^{(a,b)}(t)} W_L^{(a,b)}(t)u(t)dt \tag{3.17}$$

$$= \sum_{j=0}^{N} \frac{k(x,\hat{\sigma}_{L,j})}{W_{L,j}^{(a,b)}(\hat{\sigma}_{L,j})} \hat{\omega}_{L,j} u(\hat{\sigma}_{L,j}) \tag{3.18}$$

$$= \sum_{j=0}^{N} \frac{k(x,\hat{\sigma}_{L,j})}{W_{L,j}^{(a,b)}(\hat{\sigma}_{L,j})} \hat{\omega}_{i,j} \sum_{i=0}^{N} c_i \hat{J}_{L,i}^{(a,b)}(\hat{\sigma}_{L,i}) \tag{3.19}$$

$$= \sum_{i=0}^{N}\sum_{j=0}^{N} c_i \frac{k(x,\hat{\sigma}_{L,j})}{W_L^{(a,b)}(\hat{\sigma}_{L,j})} \hat{\omega}_{L,j} \hat{J}_{L,i}^{(a,b)}(\hat{\sigma}_{L,j}) \tag{3.20}$$

where $\{\hat{\sigma}_{L,j}\}_{j=0}^N$ are the shifted Jacobi Gauss nodes on the interval $[0, L]$ such that

- $\hat{\sigma}_{L,j} = \frac{L}{2}(1 + \sigma_j)$
- $\{\sigma_j\}_{j=0}^N$ are the standard Jacobi Gauss nodes on $[-1, 1]$
- $\{\hat{\omega}_{L,j}\}_{j=0}^N$ are standard shifted Jacobi Gauss weights on $[0, L]$,
  $\hat{\omega}_{L,j} = (\frac{L}{2})^{a+b+1}\omega_j$

- $\{\omega_j\}_{j=0}^{N}$ are standard Jacobi-Gauss weights on $[-1,1]$

The fundamental matrix relation for the fractional derivative part based on collocation points is given by

$$ {}_{c}^{\alpha}D = \mathbf{D}^{(\alpha)}\boldsymbol{\phi}\ \mathbf{C} \tag{3.21}$$

where

$$\boldsymbol{\phi} = \begin{pmatrix} \hat{J}_{L,0}^{(a,b)}(\hat{\sigma}_{L,0}) & \hat{J}_{L,1}^{(a,b)}(\hat{\sigma}_{L,0}) & \dots & \hat{J}_{L,N}^{(a,b)}(\hat{\sigma}_{L,0}) \\ \hat{J}_{L,0}^{(a,b)}(\hat{\sigma}_{L,1}) & \hat{J}_{L,1}^{(a,b)}(\hat{\sigma}_{L,1}) & \dots & \hat{J}_{L,N}^{(a,b)}(\hat{\sigma}_{L,1}) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{J}_{L,0}^{(a,b)}(\hat{\sigma}_{L,N}) & \hat{J}_{L,1}^{(a,b)}(\hat{\sigma}_{L,N}) & \dots & \hat{J}_{L,N}^{(a,b)}(\hat{\sigma}_{L,N}) \end{pmatrix}$$

And

$$C = [c_0, c_1, ..., c_N]^T$$

The fundamental Matrix relation of the Integral part based on collocation points is given by

$$\mathbf{I} = \mathcal{M}W\boldsymbol{\phi}C \tag{3.22}$$

where

$$\mathcal{M} = \left(\frac{k(\hat{\sigma}_{L,i}, \hat{\sigma}_{L,j})}{W_L^{(a,b)}(\sigma_{L,j})} \right)$$

for $0 \leq i, j \leq N$ And

$$W = \text{diag}((\hat{\omega}_{L,j})_{0\leq j\leq N})$$

Using relation 3.21 and 3.22 the equation of the system is reduced to the following system

$$\mathbf{A}C = \mathbf{F} \tag{3.23}$$

where

$$\mathbf{W} = D^{(\alpha)}\boldsymbol{\phi} + \lambda\mathcal{M}W\boldsymbol{\phi}$$

and

$$\mathbf{F} = [f(\hat{\sigma}_{L,0}), f(\hat{\sigma}_{L,1}), ..., f(\hat{\sigma}_{L,N})]^T$$

On the other hand, the fundamental Matrix for The Initial condition can be written as

$$\mathbf{V}C = f \tag{3.24}$$

where

$$\begin{aligned} \mathbf{V} &= [\hat{J}_{L,0}^{(a,b)}(0), \hat{J}_{L,1}^{(a,b)}(0), \hat{J}_{L,2}^{(a,b)}(0), ..., \hat{J}_{L,N}^{(a,b)}(0)] && (3.25) \\ &= [v_0, v_1, ..., v_N] && (3.26) \end{aligned}$$

In order to satisfy the Initial condition in collocation Method we add the equation 3.24to the final system3.23 then we obtain

$$\tilde{\mathbf{A}}C = \tilde{\mathbf{F}}$$

so that the new augmented Matrix is of the form

$$[\tilde{A}|\tilde{F}] = \begin{bmatrix} A_{0,0} & A_{0,1} & \dots & A_{0,N} & f(\hat{\sigma}_{L,0}) \\ A_{1,0} & A_{1,1} & \dots & A_{1,N} & f(\hat{\sigma}_{L,1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{N,0} & A_{N,1} & \dots & A_{N,N} & f(\hat{\sigma}_{L,N}) \\ V_0 & V_1 & \dots & V_N & \rho \end{bmatrix}$$

finally, we have an over determined system with $(N+1)$ linear equations which can be solved by using least square Method by matlab R2009b.

3.3 Examples

3.3.1 Example 1

Let the following Fractional Fredholm Integro-differential equation with initial condition

$$D_c^{0.5}u(x) + \int_0^1 xtu(t)dt = \frac{x}{4} + \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}}$$

with the initial condition $u(0) = 0$, and the exact solution $u(x) = x^2$.
In the following tables 3.1 and 3.2 we show the absolute errors $E_N(x_i) = |u(x_i) - u_N(x_i)|$ using Legendre polynomials and Chebyshev polynomials of first kind respectively .

x_i	$E_4(x_i)$	$E_8(x_i)$	$E_{16}(x_i)$
0	5.042e-17	1.552e-16	1.510e-16
0.1	4.163e-17	2.776e-17	1.457e-16
0.2	6.245e-17	7.633e-17	1.457e-16
0.3	8.327e-17	1.110e-16	1.804e-16
0.4	2.776e-17	8.327e-17	1.943e-16
0.5	2.776e-17	2.776e-17	1.943e-16
0.6	0.000e+00	0.000e+00	2.220e-16
0.7	5.551e-17	0.000e+00	1.665e-16
0.8	0.000e+00	2.220e-16	0.000e+00
0.9	0.000e+00	2.220e-16	2.220e-16
1	1.110e-16	1.110e-16	5.551e-16

Table 3.1: Absolute errors $E_n(x_i)$ for different values of N at various points x_i using Legendre polynomials $(\alpha, \beta) = (0, 0)$

x_i	$E_4(x_i)$	$E_8(x_i)$	$E_{16}(x_i)$
0	2.261e-07	3.019e-09	3.315e-11
0.1	1.717e-04	5.034e-05	1.399e-05
0.2	4.735e-04	1.436e-04	3.960e-05
0.3	8.791e-04	2.628e-04	7.276e-05
0.4	1.367e-03	4.047e-04	1.120e-04
0.5	1.920e-03	5.661e-04	1.565e-04
0.6	2.526e-03	7.442e-04	2.058e-04
0.7	3.179e-03	9.373e-04	2.593e-04
0.8	3.875e-03	1.145e-03	3.168e-04
0.9	4.619e-03	1.367e-03	3.780e-04
1	5.416e-03	1.601e-03	4.428e-04

Table 3.2: Absolute errors $E_n(x_i)$ for different values of N at various points x_i using Chebyshev polynomials of first kind $(\alpha, \beta) = (-0.5, -0.5)$

3.3.2 Example 2

Let the following Fractional Fredholm Integro-differential equation with initial condition

$$D_c^{0.3}u(x) + \int_0^1 (x-t)u(t)dt = \frac{3418}{2493}x^{\frac{11}{5}} - \frac{11}{14}x - \frac{602}{547}x^{\frac{7}{10}} + \frac{7}{18}$$

with the initial condition $u(0) = 1$, and the exact solution $u(x) = x^{\frac{5}{2}} - x + 1$.
In the following tables 3.3 and 3.4 we show the absolute errors $E_N(x_i) = |u(x_i) - u_N(x_i)|$ using Legendre polynomials and Chebyshev polynomials of first kind respectively .

x_i	$E_4(x_i)$	$E_8(x_i)$	$E_{12}(x_i)$	$E_{16}(x_i)$	$E_{20}(x_i)$
0	1.796e-04	5.619e-06	7.140e-07	1.607e-07	4.773e-08
0.1	5.858e-05	1.250e-05	7.424e-07	4.220e-07	2.480e-08
0.2	3.479e-04	5.025e-06	1.505e-06	5.145e-08	2.018e-08
0.3	4.685e-04	8.001e-06	3.809e-07	4.929e-08	1.121e-07
0.4	2.912e-04	1.200e-05	1.596e-06	2.700e-07	4.149e-08
0.5	3.436e-05	3.896e-07	3.956e-08	1.045e-08	1.015e-08
0.6	8.373e-05	1.311e-06	8.719e-07	3.087e-07	1.020e-07
0.7	4.106e-05	1.244e-05	1.058e-06	1.770e-08	6.261e-08
0.8	3.319e-04	7.289e-06	4.355e-07	3.224e-07	1.723e-08
0.9	4.878e-04	3.115e-06	1.009e-06	2.512e-07	5.004e-09
1	4.610e-05	8.399e-08	5.169e-08	1.423e-08	2.269e-07

Table 3.3: Absolute errors $E_n(x_i)$ for different values of N at various points x_i using Legendre polynomials $(\alpha, \beta) = (0, 0)$

x_i	$E_4(x_i)$	$E_8(x_i)$	$E_{12}(x_i)$	$E_{16}(x_i)$	$E_{20}(x_i)$
0	8.293e-04	2.430e-04	1.036e-04	5.544e-05	3.352e-05
0.1	5.097e-03	1.496e-03	6.127e-04	4.104e-04	2.372e-04
0.2	5.757e-03	1.313e-03	7.130e-04	3.903e-04	2.520e-04
0.3	5.020e-03	1.425e-03	5.935e-04	3.539e-04	2.428e-04
0.4	3.872e-03	1.246e-03	5.765e-04	3.199e-04	1.966e-04
0.5	2.760e-03	8.433e-04	3.959e-04	2.283e-04	1.477e-04
0.6	1.828e-03	5.725e-04	2.861e-04	1.690e-04	1.071e-04
0.7	1.025e-03	3.802e-04	1.381e-04	5.453e-05	3.565e-05
0.8	1.759e-04	3.406e-05	6.330e-05	2.319e-05	3.021e-05
0.9	9.784e-04	4.943e-04	2.198e-04	1.411e-04	9.341e-05
1	2.748e-03	8.684e-04	4.323e-04	2.593e-04	1.838e-04

Table 3.4: Absolute errors $E_n(x_i)$ for different values of N at various points x_i using Chebyshev polynomials of first kind $(\alpha, \beta) = (-0.5, -0.5)$

3.3.3 Example 3

Let the following Fractional Fredholm Integro-differential equation with initial condition

$$D_c^{0.9}u(x) + \int_0^1 e^{-x-t}u(t)dt = e^{2x}$$

with the initial condition $u(0) = 1$, and the exact solution is unknown.
In the following table 3.5 and 3.6 we show the approximate solutions $u_N(x)$ at different values of x_i using Legendre polynomials and Chebyshev polynomials of first kind respectively .

x_i	$u_4(x_i)$	$u_8(x_i)$	$u_{12}(x_i)$	$u_{16}(x_i)$	$u_{20}(x_i)$
0.0	1.0001040678	0.9999982417	0.9999997047	0.9999999133	0.9999999632
0.1	1.0186428919	1.0186159352	1.0186430837	1.0186300718	1.0186308190
0.2	1.0788771264	1.0783976561	1.0783542867	1.0783576700	1.0783550657
0.3	1.1826145020	1.1815209580	1.1815235071	1.1815154181	1.1815131132
0.4	1.3343957761	1.3333257250	1.3333380652	1.3333392091	1.3333387149
0.5	1.5414947332	1.5411615969	1.5411481522	1.5411444397	1.5411428634
0.6	1.8139181849	1.8144716606	1.8144482086	1.8144441437	1.8144430912
0.7	2.1644059692	2.1652924376	2.1652937596	2.1652949619	2.1652934582
0.8	2.6084309516	2.6087449970	2.6087524477	2.6087467958	2.6087470318
0.9	3.1641990241	3.1634868393	3.1634687103	3.1634687585	3.1634679841
1.0	3.8526491058	3.8524889989	3.8524840076	3.8524820624	3.8524827448

Table 3.5: Approximate solutions $u_N(x_i)$ at various points x_i using Legendre polynomials $(\alpha, \beta) = (0, 0)$

x_i	$u_4(x_i)$	$u_8(x_i)$	$u_{12}(x_i)$	$u_{16}(x_i)$	$u_{20}(x_i)$
0.0	1.0002118097	0.9999915015	0.9999980121	0.9999992430	0.9999974319
0.1	1.0164934977	1.0179260357	1.0183335354	1.0184423664	1.0185091253
0.2	1.0751317251	1.0772666688	1.0778010868	1.0780397195	1.0781416305
0.3	1.1776432085	1.1799352171	1.1807866495	1.1810836185	1.1812272203
0.4	1.3283598619	1.3313786967	1.3324189385	1.3328062639	1.3329921861
0.5	1.5344287965	1.5389411883	1.5400915158	1.5405281376	1.5407335053
0.6	1.8058123210	1.8119903757	1.8132615681	1.8137504644	1.8139905267
0.7	2.1552879414	2.1625486714	2.1639892601	2.1645378302	2.1647955330
0.8	2.5984483607	2.6057899624	2.6073570874	2.6079295354	2.6082088775
0.9	3.1537014797	3.1603741384	3.1619803000	3.1626031335	3.1628982588
1.0	3.8422703961	3.8492026857	3.8509217627	3.8515711486	3.8518833501

Table 3.6: Approximate solutions $u_N(x_i)$ at various points x_i using Chebyshev polynomials of first kind $(\alpha, \beta) = (-0.5, -0.5)$

In the following tables 3.7 and 3.8 we show the absolute errors between different approximate solutions $u_N(x)$ at different values of x_i using Legendre polynomials and Chebyshev polynomials of first kind respectively .

x_i	$\lvert u_8 - u_4\rvert$	$\lvert u_{12} - u_8\rvert$	$\lvert u_{16} - u_{12}\rvert$	$\lvert u_{20} - u_{16}\rvert$
0.0	1.058e-04	1.463e-06	2.086e-07	4.989e-08
0.1	2.696e-05	2.715e-05	1.301e-05	7.472e-07
0.2	4.795e-04	4.337e-05	3.383e-06	2.604e-06
0.3	1.094e-03	2.549e-06	8.089e-06	2.305e-06
0.4	1.070e-03	1.234e-05	1.144e-06	4.942e-07
0.5	3.331e-04	1.344e-05	3.712e-06	1.576e-06
0.6	5.535e-04	2.345e-05	4.065e-06	1.052e-06
0.7	8.865e-04	1.322e-06	1.202e-06	1.504e-06
0.8	3.140e-04	7.451e-06	5.652e-06	2.360e-07
0.9	7.122e-04	1.813e-05	4.821e-08	7.744e-07
1.0	1.601e-04	4.991e-06	1.945e-06	6.824e-07

Table 3.7: Absolute errors $E_n(x_i)$ for different values of N at various points x_i using Legendre polynomials $(\alpha, \beta) = (0, 0)$

x_i	$\lvert u_8 - u_4\rvert$	$\lvert u_{12} - u_8\rvert$	$\lvert u_{16} - u_{12}\rvert$	$\lvert u_{20} - u_{16}\rvert$
0.0	2.203e-04	6.511e-06	1.231e-06	1.811e-06
0.1	1.433e-03	4.075e-04	1.088e-04	6.676e-05
0.2	2.135e-03	5.344e-04	2.386e-04	1.019e-04
0.3	2.292e-03	8.514e-04	2.970e-04	1.436e-04
0.4	3.019e-03	1.040e-03	3.873e-04	1.859e-04
0.5	4.512e-03	1.150e-03	4.366e-04	2.054e-04
0.6	6.178e-03	1.271e-03	4.889e-04	2.401e-04
0.7	7.261e-03	1.441e-03	5.486e-04	2.577e-04
0.8	7.342e-03	1.567e-03	5.724e-04	2.793e-04
0.9	6.673e-03	1.606e-03	6.228e-04	2.951e-04
1.0	6.932e-03	1.719e-03	6.494e-04	3.122e-04

Table 3.8: Absolute errors $E_n(x_i)$ for different values of N at various points x_i using Chebyshev polynomials of first kind $(\alpha, \beta) = (-0.5, -0.5)$

3.3.4 Example 4

Let the following Fractional Fredholm Integro-differential equation with initial condition

$$D_c^{0.5}u(x) + \int_0^1 u(t)dt = 2 - erf(\sqrt{x})e^x - e$$

with the initial condition $u(0) = 0$, and the exact solution $u(x) = 1 - e^x$.
In the figures 3.1 and 3.2 we show the comparison between approximate solutions of Legendre polynomials and Chebyshev polynomials of first kind with the exact solution, and comparison between $\log_{10} |(u(x) - u_4(x))|$ of Legendre polynomials and Chebyshev polynomials of first kind respectively .
In table 3.9 we show the comparison of absolute errors at $N = 10$ at various points x_i between Spectral collocation method using Legendre polynomials and method in [17] using Gegenbauer polynomials .

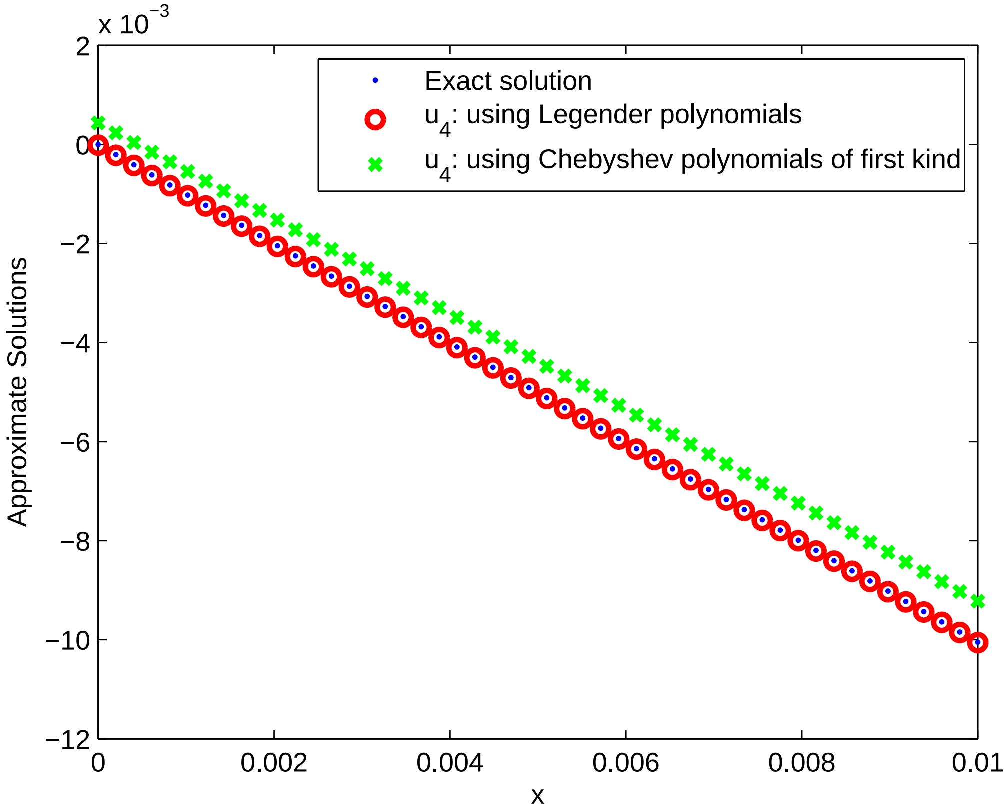

Figure 3.1: Comparison between Approximate solutions of Legendre polynomials and Chebyshev polynomials of first kind with the exact solution

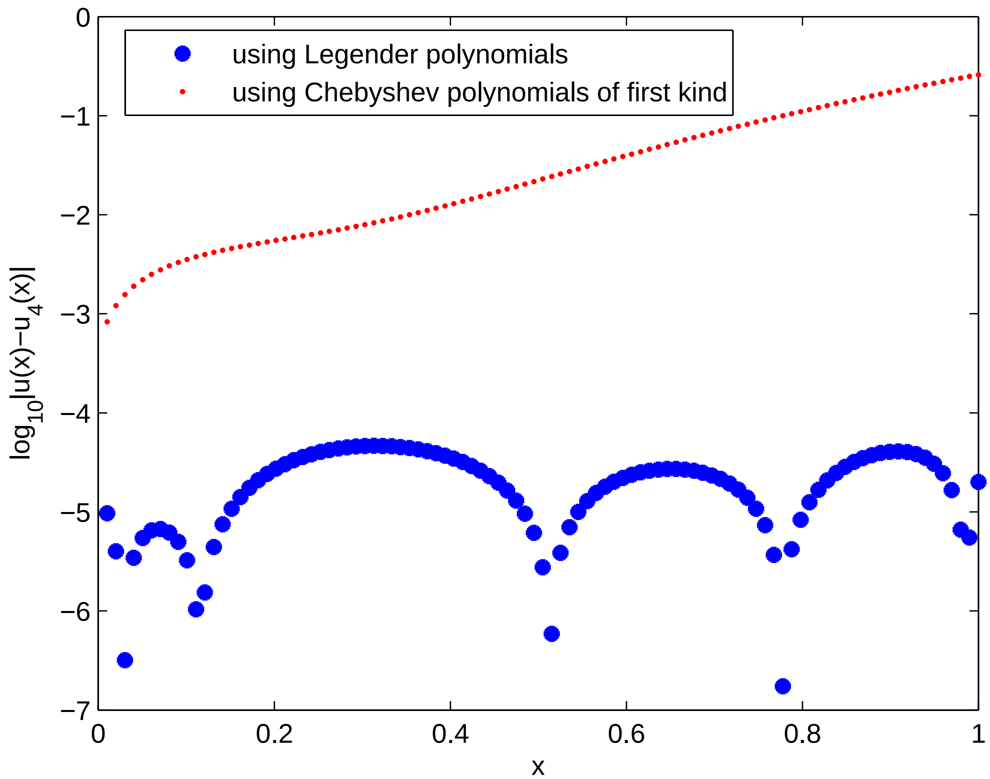

Figure 3.2: Comparison between $\log_{10}|(u(x) - u_4(x))|$ of Legendre polynomials and Chebyshev polynomials of first kind

x_i	$E_{10}(x_i)$ In Method [17] using Gegenbauer polynomials	$E_{10}(x_i)$ using Legendre polynomials
0	3.3225e-12	1.280e-014
0.1	1.1718e-11	1.296e-014
0.2	4.6097e-11	3.580e-015
0.3	1.8265e-10	2.959e-014
0.4	4.0956e-10	1.527e-014
0.5	7.3089e-10	1.354e-014
0.6	1.1368e-09	2.687e-014
0.7	1.6012e-09	1.932e-014
0.8	2.0906e-09	2.398e-014
0.9	2.5908e-09	9.326e-015
1	3.1642e-09	1.443e-014

Table 3.9: Comparison of absolute errors at $N = 10$ at various points x_i between Spectral collocation method using Legendre polynomials and method in [17]

Conclusion

In this thesis, we tackled the challenging task of numerically solving linear fractional integro-differential equations (FI-DEs), with a particular focus on those involving the Caputo fractional derivative. This class of equations serves as a vital modeling tool for systems exhibiting memory effects and non-local properties, which often lie beyond the scope of classical integer-order models. Given the inherent complexity in finding analytical solutions for these equations, developing efficient numerical methods becomes essential.

This thesis successfully presented and applied the spectral collocation method as a robust and effective approach to solve these equations. We began by establishing a solid theoretical foundation.

in Chapter 1, where we reviewed fundamental special functions such as Gamma, Beta, and Mittag-Leffler functions. We then delved into the definitions and properties of both Riemann-Liouville and Caputo fractional derivatives, emphasizing the latter's importance in practical applications due to its compatibility with traditional initial conditions.

In Chapter 2, we highlighted orthogonal polynomials, particularly Jacobi and Legendre polynomials, discussing their optimal properties that make them excellent choices as basis functions in spectral methods. We also included a discussion on Chebyshev polynomials, which are commonly used in this context.

The main contribution of this thesis lay **in Chapter 3**, where the spectral collocation methodology was detailed. We demonstrated how a linear FI-DEs can be transformed into a system of linear algebraic equations by approximating the solution with orthogonal polynomials and enforcing the equation at carefully selected collocation points. Detailed derivations for the fractional derivative of the basis functions and the handling of integral terms were presented. When applying the method to numerical examples using both Legendre and Chebyshev polynomials as basis functions, the results clearly showed that Legendre polynomials yielded superior performance in terms of solution accuracy and convergence rate for the problems studied. These findings confirm the high accuracy and spectral (exponential) convergence rates of the proposed method, underscoring its effectiveness and efficiency in solving this complex class of equations.

This work demonstrates that the spectral collocation method, when applied correctly, provides a powerful and reliable tool for engineers and scientists dealing with phenomena described by linear fractional integro-differential equations. It also offers a valuable insight into the choice of basis functions, suggesting that Legendre polynomials might be the optimal choice for certain classes of these equations.

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