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إهداء

بسم الله الرحمن الرحيم
"وقل ربّ زدني علمًا"

إلى من كانت دعواتهم لي زادًا، وصبرهم حافزًا، ورضاهم غايتي...
إلى **والديَّ الكريمين**، تاج رأسي ونور طريقي،
لكما كل المحبة والامتنان، فما كنت لأبلغ هذا المقام لولا توفيق الله ثم عطاؤكما اللامحدود.

إلى من كانت لي وطنًا وملاذًا، إلى **زوجتي الحبيبة،**
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إلى **عائلتي** التي لم تبخل يومًا بالمحبة والدعم،
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Notations

1. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively denote the sets of natural integers, relative integers, rationals, reals and complexes numbers.

2. $A[x]$ denotes the ring of polynomials whose coefficients in ring A.

3. $\deg(P(x))$: polynomial degree $P(x)$.

4. $\binom{n}{k}$ binomial coefficient, such that n and k two integer where $0 \leq k \leq n$.

5. $\binom{z}{k}$ generalized binomial coefficient, such that z is a complex number and k an integer.

6. H_n the harmonic number, defined as
$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

7. $H_{n,m}$ the harmonic number of order m, defined as
$$H_{n,m} = 1 + \frac{1}{2^m} + \frac{1}{3^m} + \cdots + \frac{1}{n^m}.$$

8. $[x^n](P(x))$ designates the cofficient of x^n in the polynomial $P(x)$.

9. B_n the Bernoulli number defined by the recurrence relation
$$B_0 = 1 \text{ and } B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k,\ n \geqslant 1.$$

10. ζ the Riemann zeta function, defined as :
$$\zeta(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots,\quad \text{where } \operatorname{Re}(s) > 1.$$

11. $\mathbf{Li}_2(x)$ the dilogarithm function defined by the power series
$$\mathbf{Li}_2(x) = \sum_{n=1}^{+\infty}\frac{1}{n^2}x^n \quad \text{for } |x| < 1.$$

12. $s(n,k)$ the Stirling numbers of the first kind, defined as
$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)\,x^k.$$

13. LHS is informal shorthand for the left-hand side of an equation.

14. RHS is informal shorthand for the right-hand side of an equation.

Abstract

This thesis studies integrals using simple tools from combinatorics and analysis. It starts with basic ideas like binomial coefficients, harmonic numbers, and the Riemann zeta function. These help in solving special sums and series. It also uses tools like the Cauchy product and logarithmic series.

Then, it explains how to compute definite integrals using series, harmonic number rules, and known results from books. A special part is about a method called "differentiation under the integral sign," based on the work of K. N. Boyadzhiev. His method uses binomial expansions and changing parameters to solve hard integrals.

This work shows how combinatorics, series, and classical analysis can work together to solve problems in integration.

Keywords: Harmonic numbers, Binomial coefficients, Logarithm integral.

Résumé

Ce mémoire propose une étude approfondie des intégrales en combinant des approches combinatoires et analytiques. Il s'ouvre sur les concepts fondamentaux tels que les coefficients binomiaux, les nombres harmoniques et la fonction zêta de Riemann, en mettant en lumière leur utilité dans l'évaluation de certaines sommes et séries spéciales. Des outils comme le produit de Cauchy et les développements en séries logarithmiques sont également examinés. L'étude se poursuit avec des techniques d'évaluation d'intégrales définies à l'aide de séries, de nombres harmoniques et d'identités issues de la littérature mathématique. Une attention particulière est portée à la méthode de la dérivation sous le signe intégral, inspirée des travaux de K. N. Boyadzhiev, qui repose sur des développements binomiaux et la différentiation par rapport à un paramètre. L'ensemble du travail met en lumière la richesse des liens entre combinatoire, séries et analyse classique, tout en illustrant des approches modernes pour résoudre des problèmes d'intégration.

Mots-clés : Nombres harmoniques, Coefficients binomiaux, Logarithme integrale.

Contents

Introduction

Introduction

In mathematics, an integral represents the continuous counterpart of a sum. It is used to calculate areas, volumes, and other accumulated quantities. Integration—the process of computing an integral—is one of the two fundamental operations in calculus, alongside differentiation. Initially developed to solve problems in mathematics and physics, integration has since become a core tool across numerous scientific and engineering disciplines.

This thesis is dedicated to an in-depth study of key problems presented in selected mathematical journals. Our work is organized into three chapters.

The first chapter is devoted to generalities, we start by defining the binomial coefficients and the generalized binomail coefficients, we study the important and general properties of these binomial coefficients, then we define in a natural way the sequences of harmonic numbers H_n and their generalization harmonic numbers of order $H_{n,m}$. Constitutes the essential part of this thesis, it is devoted to an in -depth study on the study of certain properties and theorems concerning more specifically the sums involving binomial coefficients and sums involving harmonic numbers such

$$\sum_{k=1}^{n} \frac{H_k}{k}.$$

At the end of this chapter due to the important role played the riemann zeta fonction of the calculation, we given an simple proof of Basel problem and some intersing values of $\zeta(n)$, and we have also given some properties of the dilogarithm function $\mathbf{Li}_2(x)$.

At the end of this chapter, we recall the Cauchy product of tow series and we present some various series such the development of

$$f(x) = \frac{\ln(1-x)}{1-x},$$

to use for solving important integrals in chapter two and three.

The second chapter of the thesis presents a selection of problems taken from recent papers. Most of these problems appeared in mathematical journals such as those published by the MAA.

we present various important integrals evaluated using series, combinatorial identities and harmonic numbers.

The main results of this chapter represent an interesting contribution in integral logarithms. They are obtained by using technical operations on binomial coefficients and harmonic numbers.

Firstly, we solve some problems concerning logarithmic integrals related to the harmonic series like

$$\int_0^1 \frac{\ln^m x}{1+x}dx \quad \text{and} \quad \int_0^1 \frac{\ln^m (1-x)}{x^m}dx.$$

In finaly chapter we present an efficient technique for evaluating challenging definite integrals-differentiation with respect to a parameter inside the integral (or appearing in the limits of integration). The integrals can be proper or improper. We provide numerous examples, many of which are listed in some intersing paper.

Finally, in this chapter, many of these integrals invite the use of combinatorial mathematical techniques that involve elegant connections between integrals and infinite series.

The final chapter of this thesis presents a selection of problems that explore key ideas from two insightful papers by K. N. Boyadzhiev on the evaluation of definite integrals. The first paper introduces a method that employs binomial series to compute integrals in elegant and often unexpected ways. The second paper demonstrates how differentiating with respect to a parameter can simplify the evaluation of otherwise integrals.

Together, these techniques provide powerful tools grounded in series expansions and algebraic manipulation. A central theme of this chapter is a general rule that connects certain definite integrals to infinite series involving binomial coefficients. This approach yields surprising and beautiful results in the realm of classical analysis.

Boyadzhiev's work illustrates how combinatorial thinking can deepen our understanding of integration, bridging discrete and continuous mathematics in insightful and meaningful ways. His methods are not only practically effective but also conceptually enriching, inspiring a greater appreciation for the interplay between series and integrals.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we revisit several foundational concepts in mathematical analysis and number theory. We begin by recalling the definitions and key properties of the binomial coefficients and harmonic numbers, which play crucial roles in combinatorics and series analysis. Next, we explore the Riemann zeta function, including specific evaluations such as $\zeta(2)$, $\zeta(3)$, and $\zeta(4)$, which are of particular interest in analytic number theory. We then discuss the Cauchy product of two infinite series, a fundamental operation in series manipulation. The chapter also presents sums involving harmonic numbers and binomial coefficients, revealing rich interconnections between these sequences. Finally, we introduce the dilogarithm function, highlighting its properties and notable values.

1.2 Binomial coefficients

Binomial coefficients play a central role in various branches of mathematics, including algebra (through binomial expansions), combinatorics (in counting problems), series expansions, and probability theory.

A binomial coefficient is a numerical value that represents the number of ways to choose a subset of elements from a larger set, without regard to the order of selection. It is typically denoted as $\binom{n}{k}$

This is read as "n choose k" and is defined mathematically as:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Properties

For all natural numbers n and k, the binomial coefficients satisfy

1. $$\binom{n}{k+1} = \binom{n-1}{k+1} + \binom{n-1}{k} \tag{1.1}$$

2. $$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \tag{1.2}$$

3. $$\binom{n}{k} = \binom{n}{n-k} \tag{1.3}$$

4. $$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k} \tag{1.4}$$

5. $$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k} \tag{1.5}$$

Historically, Al-Karaji was the first known mathematician to study what is now called Pascal's Triangle. He provided an early explanation of this numerical arrangement in one of his works, which, unfortunately, has not survived. His contributions laid the foundation for later developments in combinatorial mathematics.

In the 11th century, the Persian mathematician and poet Omar Khayyam expanded on Al-Karaji's ideas. He used the already known binomial theorem to systematically construct and analyze the triangular array of coefficients. His work played a crucial role in the mathematical advancements of the Islamic Golden Age. For a detailed analysis of this aspect, refer to [28].

Around the same period, independent developments were occurring in China. The mathematician Jia Xian (1010–1070) introduced a similar triangular arrangement to represent the coefficients of binomial expansions. His work provided a systematic method for generating these coefficients, making it a valuable tool for algebraic calculations. In the 13th century, Yang Hui (1238–1298) built upon Jia Xian's findings, further exploring the properties of the triangle and documenting additional patterns within it. His contributions helped preserve and expand the mathematical knowledge of the time.

From the 13th century onward, European mathematicians also began to investigate the properties of this triangular structure. Several scholars contributed to its study, uncovering various relationships between its elements. The most significant contribution came in 1665 when the French mathematician Blaise Pascal published Traité du Triangle Arithmétique (Treatise on the Arithmetical Triangle). In this work, Pascal formalized many of the known properties of the triangle and applied them to solve problems in probability theory. He also discovered new patterns and provided rigorous proofs to support his findings.

Although the triangular arrangement had been studied centuries earlier in different parts of the world, it was eventually named after Pascal. The designation was popularized by Pierre Raymond de Montmort and Abraham de Moivre [18], who recognized Pascal's significant contributions to its mathematical treatment and applications. Today, Pascal's Triangle remains a fundamental object of study in combinatorics, number theory, and algebra, with applications extending to probability, polynomial expansions, and even modern computer science.

Remark 1.1 *Andreas von Ettingshausen introduced the notation* $\binom{n}{k}$ *in 1826, although the numbers were known centuries earlier as* $C(n,k)$, C_n^k, *and* $C_{n,k}$.

1.2.1 Newton's binomial

Let a and b are two complex numbers, and n is a non-zero natural number, then

$$(a+b)^n = \sum_{k=0}^{n} a^k b^{n-k} \binom{n}{k}.$$

Although this formula is named after Newton, it was known to Arab mathematicians six centuries earlier. Thus, we know that the Persian mathematician Muhammed al-Karaji established it around the year 1000. Isaac Newton generalized this formula around 1676 to fractional exponents: we then find a series expansion, and not a finite sum. However, it would take until the 19th century for a rigorous proof of Newton's formulas.

1.3 Harmonic numbers

Harmonic numbers form a fundamental sequence in mathematics, with deep connections to number theory, mathematical analysis, and combinatorics. They frequently appear in various fields such as approximation theory, asymptotic analysis, and the study of algorithms, particularly in complexity analysis.

The n-th harmonic number represents the partial sum of the harmonic series and is defined as the sum of the reciprocals of the first n natural numbers. Formally, it is given by:

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}, \text{ for } n \in \mathbb{N}^*.$$

Recall that harmonic numbers of order m are given by

$$H_{0,m} = 0, \quad H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}, \text{ for } n \in \mathbb{N}^*.$$

with $H_{n,1} = H_n$ are the classical harmonic numbers.

These numbers are named after the harmonic series, which has been studied since ancient times due to its intriguing properties, particularly its divergence as . Despite the divergence of the harmonic series, harmonic numbers themselves remain finite and exhibit fascinating asymptotic behavior.

1.4 Some values of Riemann zeta function

The Riemann zeta function or Euler–Riemann zeta function, denoted by the Greek letter ζ (zeta), is an analytic function defined, for any complex number s such that $\operatorname{Re}(s) > 1$, by the Riemann series

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

Euler calculated (as part of his solution to the Basel problem) the value of the function ζ for even strictly positive integers; he deduced the formula

$$\zeta(2m) = \sum_{n=1}^{+\infty} \frac{1}{n^{2m}} = \frac{|B_{2m}| (2\pi)^{2m}}{2(2m)!}$$

where Bernoulli numbers $(B_n)_{n\in\mathbb{N}}$ are defined by the recurrence relation:

$$B_0 = 1 \text{ and } B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k, \ n \geqslant 1.$$

B_n is called the n-th Bernoulli number.
These values of $\zeta(2k)$ are therefore expressed using the even powers of π, for example

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^2}{90},\ \zeta(6) = \frac{\pi^2}{945}, \cdots$$

1.4.1 Basel problem $\zeta(2)$

The Basel Problem is a famous and historically significant problem in number theory. It asks for the exact value of the infinite series:

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

First posed by Pietro Mengoli in 1650, the problem remained unsolved for decades until Leonhard Euler provided a remarkable solution in 1734. He showed that the sum of the series equals $\frac{\pi^1}{6}$.

The problem is named after the Swiss city of Basel, where both Jakob and Johann Bernoulli—renowned mathematicians—served as professors of mathematics at the University of Basel. Coincidentally, Euler himself was born in Basel in 1707, and he would go on to become one of the most influential mathematicians in history. His solution to the Basel Problem marked a turning point in the development of mathematical analysis.

The Basel problem is an important problem in number theory that was first posed by Pietro Mengoli in 1650 and solved by Leonhard Euler in 1734. The Basel problem so is named for the Swiss city in whose university two of the Bernoulli brothers successively served as professor of mathematics (Jakob, 1687 - 1705, and Johann, 1705 - 1748).

The Basel problem resisted solution for some 84 years until the then 26 year old Euler finally solved it. Euler's surprising solution is

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{1.6}$$

1.4.2 Apéry's constant $\zeta(3)$

In mathematics, Apéry's constant is the sum of the reciprocals of the positive cubes. That is, it is defined as the number

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}$$

where ζ is the Riemann zeta function. It has an approximate value

$$\zeta(3) = 1.202056903159594285399738161511449990764986292\ldots$$

It is called Apéry's constant because in 1978, the French mathematician Roger Apéry famously proved that this number is irrational, meaning it cannot be expressed as a fraction of two integers. This was a surprising and important result in number theory at the time.

1.5 Sum involving harmonic numbers and binomial coefficients

The study of harmonic numbers dates back to Ancient Greece and remains fundamentally important across various branches of mathematics. These numbers are closely linked to the Riemann zeta function and frequently appear in expressions involving other special functions.

Sums that combine harmonic numbers and binomial coefficients have attracted considerable interest, particularly during the 20th century, due to their deep connections with combinatorics, number theory, and special functions. In this chapter, we extend certain classes of such sums, which are related to logarithmic integral representations and play a role in analytical proofs.

To support the key examples, problems, and theorems in Chapters Two and Three, we first establish a series of auxiliary results, presented through the following lemmas.

Lemma 1.1 *Let $n \in \mathbb{N}$, the following identity holds*

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx \tag{1.7}$$

Proof. we have

$$\begin{aligned} H_n &= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \\ &= \int_0^1 dx+\int_0^1 xdx+\int_0^1 x^2dx+\cdots+\int_0^1 x^{n-1}dx \\ &= \int_0^1 \left(1+x+x^2+\cdots+x^{n-1}\right)dx, \end{aligned}$$

■

Lemma 1.2 *Let n be positive integers, the following identies hold*

$$\sum_{k=1}^{n}(-1)^k\frac{1}{k}\binom{n}{k}=-H_n \tag{1.8}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_k=-\frac{1}{n}, \tag{1.9}$$

and

$$\sum_{k=1}^{n}(-1)^k\frac{1}{k^2}\binom{n}{k}=\frac{H_n}{n}. \tag{1.10}$$

Proof. Using the integral representation of harmonic number and the the substitution $x=1-u$, we have

$$\begin{aligned} H_n &= \int_0^1 \frac{1-x^n}{1-x}dx=\int_0^1 \frac{1-(1-u)^n}{u}du \\ &= -\int_0^1 \sum_{k=1}^{n}(-1)^k\binom{n}{k}u^{k-1}du=-\sum_{k=1}^{n}(-1)^k\binom{n}{k}\int_0^1 u^{k-1}du \\ &= -\sum_{k=1}^{n}(-1)^k\frac{1}{k}\binom{n}{k}. \end{aligned}$$

■

Corollay 1.1 *The following equality holds*

$$\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}H_{k+1}=-\frac{1}{n+1}\sum_{k=0}^{n}(-1)^{k+1}\binom{n+1}{k+1}H_{k+1}=\frac{1}{(n+1)^2}. \tag{1.11}$$

Corollay 1.2 *Note that, by the Binomial Inversion Formula*

$$a_k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k \Longleftrightarrow b_k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k$$

where $(a)_n$ *and* $(b)_n$ *are two complexe sequences, we have*

$$\sum_{k=1}^{n} (-1)^k \frac{1}{k} \binom{n}{k} = -H_n \Longleftrightarrow \sum_{k=0}^{n} (-1)^k \binom{n}{k} H_k = -\frac{1}{n}.$$

Lemma 1.3 *Let* n *and* q *be positive integers. The following identity holds:*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{k}{q} \frac{(-1)^k}{k+1} = \frac{(-1)^q}{n+1}. \tag{1.12}$$

1.6 Cauchy product of two infinite series

In mathematics, more specifically in mathematical analysis, the Cauchy product is the convolution of two infinite series. It is named after the French mathematician Augustin-Louis Cauchy.

Let $\sum\limits_{n\geq 0} a_n$, $\sum\limits_{n\geq 0} b_n$ be two infinite series with complex terms. The Cauchy product of these two infinite series is defined by

$$\left(\sum_{j\geq 0} a_j\right)\left(\sum_{j\geq 0} b_j\right) = \sum_{n\geq 0} c_n \text{ where } c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

1.6.1 Cauchy product of two power series

Let $\sum\limits_{n\geq 0} a_n x^n$, $\sum\limits_{n\geq 0} b_n x^n$ be two power series, with complex terms.
The Cauchy product formula of these two power series is defined by

$$\left(\sum_{j\geq 0} a_j x^i\right)\left(\sum_{j\geq 0} b_j x^j\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i b_{n-i}\right). \tag{1.13}$$

The general formula is

$$\left(\sum_{j\geq r} a_j x^i\right)\left(\sum_{j\geq s} b_j x^j\right) = \sum_{n\geq r+s}\left(\sum_{i=r}^{n-s} a_i b_{n-i}\right) x^n \text{ or } = \sum_{n\geq r+s}\left(\sum_{j=s}^{n-r} a_{n-j} b_j\right) x^n. \tag{1.14}$$

1.6.2 Some applications

Lemma 1.4 *we have*

$$\frac{\ln(1+x)}{1+x} = \sum_{n=1}^{\infty}(-1)^{n+1}H_nx^n \qquad |x|<1, \tag{1.15}$$

$$-\frac{\ln(1-x)}{1-x} = \sum_{n=1}^{\infty}H_nx^n \qquad |x|<1. \tag{1.16}$$

Proof. Stating from the known expansion of two series

$$\ln(1+x)=\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}x^n \text{ and } \frac{1}{1+x}=\sum_{n\geq 0}(-1)^n x^n, \quad |x|<1 \tag{1.17}$$

from the relation (1.14), we have

$$\begin{aligned}\frac{\ln(1+x)}{1+x} &= \left(\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}x^n\right)\left(\sum_{n\geq 0}(-1)^n x^n\right)=\sum_{n\geq 1}\left(\sum_{i=1}^{n-0}\frac{(-1)^{i+1}}{i}(-1)^{n-i}\right)x^n \\ &= \sum_{n\geq 1}(-1)^{n+1}\left(\sum_{i=1}^{n}\frac{1}{i}\right)x^n \\ &= \sum_{n\geq 1}(-1)^{n+1}H_nx^n.\end{aligned}$$

Stating from the known expansion of two series

$$\ln(1-x)=-\sum_{n\geq 1}\frac{1}{n}x^n \text{ and } \frac{1}{1-x}=\sum_{n\geq 0}x^n, \quad |x|<1$$

and from the relation (1.14), we have

$$\begin{aligned}-\frac{\ln(1-x)}{1-x} &= \left(\sum_{n\geq 1}\frac{1}{n}x^n\right)\left(\sum_{n\geq 0}x^n\right)=\sum_{n\geq 1}\left(\sum_{i=1}^{n-0}\frac{1}{i}\right)x^n \\ &= \sum_{n\geq 1}H_nx^n.\end{aligned}$$

■

Corollay 1.3 *We find by integration of relation (1.15)*

$$\frac{1}{2}\ln^2(1+x)=\sum_{n\geq 0}(-1)^{n+1}\frac{H_n}{n+1}x^{n+1}$$

then

$$\frac{\ln^2(1+x)}{x} = 2\sum_{n\geq 0}(-1)^{n+1}\frac{H_n}{n+1}x^n. \tag{1.18}$$

Also we have

$$\frac{1}{2}\ln^2(1-x) = \sum_{n\geq 0}\frac{H_n}{n+1}x^{n+1}$$

then

$$\frac{\ln^2(1-x)}{x} = 2\sum_{n\geq 0}\frac{H_n}{n+1}x^n$$

1.7 Dialogarithm function $\mathbf{Li}_2(x)$

1.7.1 Introduction

The dilogarithm function, commonly denoted as $\mathbf{Li}_2(x)$, is a special function that arises in number theory, mathematical physics, and algebraic geometry. It's part of the broader family of polylogarithm functions, defined for a complex variable and an integer parameter. The dilogarithm corresponds to the second-order polylogarithm

The dilogarithm function, is a function which has been known for more than 250 years, but which for a long time was familiar only to a few enthusiasts. In recent years it has become much better known, due to its appearance in hyperbolic geometry and in algebraic K-theory on the one hand and in mathematical physics (in particular, in conformal field theory) on the other.

The dilogarithm function is the function defined by the power series

$$\mathbf{Li}_2(x) = \sum_{n=1}^{+\infty}\frac{1}{n^2}x^n \quad \text{for } |x| < 1$$

is a classical function of mathematical physics. Introduced by Leibniz in 1696 and thoroughly discussed by Euler some seventy years later [[17], pp. 124–126], it has subsequently been well studied in the literature (for further historical details concerning the function see, for example, [32]). The canonical integral representation for the dilogarithm is

$$\mathbf{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t}dt. \tag{1.19}$$

1.7.2 Some properties and values of Dialogarithm function

The first work on it seems to have been done by Landen and published in 1760. Independently Euler studied it, and they obtained results such as

$$\mathbf{Li}_2(x) + \mathbf{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x \ln(1-x) \quad \text{Euler's reflexion formula} \tag{1.20}$$

$$\mathbf{Li}_2(x) + \mathbf{Li}_2\left(\frac{x}{x-1}\right) = -\frac{1}{2}\ln^2(1-x) \quad \text{Landen's identity} \tag{1.21}$$

The first book on this function was by Spence [39] in 1809. Considering the importance of this fonction, where he adedThe first book on this function was by Spence [39] in 1809. Considering the importance of this fonction, where he aded

$$\mathbf{Li}_2(x) + \mathbf{Li}_2(-x) = \frac{1}{2}\mathbf{Li}_2\left(x^2\right), \quad \text{duplication formula} \tag{1.22}$$

$$\mathbf{Li}_2(-x) + \mathbf{Li}_2\left(\frac{-1}{x}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2 x, \quad \text{inversion formula.} \tag{1.23}$$

Some Known Values of the Dilogarithm Function:

$$\begin{aligned}
\mathbf{Li}_2(0) &= 0 \\
\mathbf{Li}_2(1) &= \frac{\pi^2}{6} \\
\mathbf{Li}_2(-1) &= -\frac{\pi^2}{12} \\
\mathbf{Li}_2\left(\frac{1}{2}\right) &= \frac{\pi^2}{12} - \frac{1}{2}\ln^2\frac{1}{2}.
\end{aligned}$$

Chapter 2

A binomial formula for evaluating some Logarithmic integrals

2.1 Introduction

This chapter constitutes the essential part of this thesis, contains some techniques of integration which are not found in standard calculs and advanced calculus book. it can be considered as a map to explore many classical approaches to evaluate integrales.

A variety of identities involving harmonic numbers and generalized harmonic numbers have been investigated since the distant past and involved in a wide range of diverse fields such as analysis of algorithms in computer science, various branches of number theory. Here we show how one can obtain certain infinite series involving harmonic numbers.

The following integrals and series that we will study appear in an interesting journal such as The College Mathematics Journal is an expository magazine it publishes well-written and captivating articles exploring new mathematics, or old mathematics in a new way. Most of its articles are accessible to upper-level undergraduate students and American Mathematical Monthly

2.2 Values of $\int\limits_0^1 x^m \ln(1-x)\,dx$ and $\int\limits_0^1 x^m \ln^2(1-x)\,dx$

The follwing formula (2.1), which is quite old, is recorded in various tables of definite integrals. It appears as formula 865.5 in [16] , It is worth mentioning that the origin of such integrals dates back to the time of the English mathematician Joseph Wolstenholme

(1829–1891), and the first proposed integral appeared in his book with mathematical problems.

Problem 2.1 *Let $n \geq 1$ be an integer. The following identity holds*

$$\int_0^1 x^{n-1} \ln(1-x)\, dx = -\frac{H_n}{n}. \tag{2.1}$$

Proof. we have

$$\begin{aligned}
\int_0^1 x^{n-1} \ln(1-x)\, dx &= \int_0^1 (1-x)^{n-1} \ln x dx \\
&= \int_0^1 \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} x^k \ln x dx \\
&= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_0^1 x^k \ln x dx \\
&= \sum_{k=0}^{n-1} (-1)^{k+1} \frac{1}{(k+1)^2} \binom{n-1}{k} \\
&= \frac{1}{n} \sum_{k=0}^{n-1} (-1)^{k+1} \frac{1}{k+1} \binom{n}{k+1} \\
&= \frac{1}{n} \sum_{k=1}^{n} (-1)^k \frac{1}{k} \binom{n}{k}
\end{aligned}$$

using the formula (1.9) the proof of formula (2.1) is given. ■

The following problem is proved in 2016 by Vălean [42].

Problem 2.2 *Let $n \geq 1$ be an integer. The following identity holds*

$$\int_0^1 x^{n-1} \ln^2(1-x)\, dx = \frac{H_n^2 + H_{n,2}}{n}.$$

Proof. Let

$$J_n = \int_0^1 x^{n-1} \ln^2(1-x)\, dx.$$

By applying integration by parts, as done in the evaluation of the integral J_n, we derive the following recurrence relation in terms of k,

$$kJ_k - (k-1) J_{k-1} = 2\frac{H_k}{k}.$$

Giving values to k from $k=2$ to n and using that $\int\limits_0^1 \ln^2(1-x)\,dx = 2$, we obtain that

$$J_n = 2\sum_{k=1}^{n} \frac{H_k}{k},$$

using the above relation, the proof is complete. ■

2.3 Values of $\int\limits_0^1 \frac{\ln^m x}{1+x} dx$

Problem 2.3 *Let m be a positive integer. Then the following equality holds*

$$\int\limits_0^1 \frac{\ln^m t}{1+t} dt = \frac{(-1)^{m+1}\, m!\,(1-2^m)}{2^m} \sum_{n=1}^{+\infty} \frac{1}{n^{m+1}}$$

Proof. Note that

$$\int\limits_0^1 \frac{\ln^m t}{1+t} dt = \int\limits_0^1 \ln^m t \sum_{n=0}^{+\infty} (-1)^n t^n dt = \sum_{n=0}^{+\infty} (-1)^n \int\limits_0^1 t^n \ln^m t dt,$$

using the subtitition $t = e^{-x}$ and using

$$I_{n,k} = \int\limits_0^{+\infty} t^n e^{-(m+1)t} dt$$

we have by part

$$\begin{aligned} u &= t^n \longrightarrow u' = nt^{n-1} \\ v' &= e^{-(m+1)t} \longrightarrow v = -\frac{1}{m+1} e^{-(m+1)t} \end{aligned}$$

hence, we have the identity

$$I_{n,m} = \frac{n}{k+1} I_{n-1,m} = \frac{n!}{(m+1)^{n+1}}$$

and from this identity, we immediately have:

$$\begin{aligned}\int_0^1 t^n \ln^m t dt &= -\int_{+\infty}^0 e^{-nx} (-x)^m e^{-x} dx \\ &= (-1)^m \int_0^{+\infty} x^m e^{-(n+1)x} dx \\ &= \frac{(-1)^m m!}{(n+1)^{m+1}}\end{aligned}$$

then

$$\int_0^1 \frac{\ln^m t}{1+t} dt = (-1)^m m! \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+1)^{m+1}} = (-1)^{m+1} m! \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^{m+1}}$$

and by the remark

$$\sum_{n=1}^{+\infty} \frac{(-1)^n + 1}{n^{\grave{u}+1}} = 2 \sum_{n=1}^{+\infty} \frac{1}{(2n)^{m+1}} = \frac{1}{2^m} \sum_{n=1}^{+\infty} \frac{1}{n^{m+1}} \tag{2.2}$$

we have

$$\begin{aligned}\int_0^1 \frac{\ln^m t}{1+t} dt &= (-1)^{m+1} m! \left(\frac{1}{2^m} \sum_{n=1}^{+\infty} \frac{1}{n^{m+1}} - \sum_{n=1}^{+\infty} \frac{1}{n^{m+1}} \right) \\ &= \frac{(-1)^{m+1} m! (1-2^m)}{2^m} \sum_{n=1}^{+\infty} \frac{1}{n^{m+1}}\end{aligned}$$

■

Corollay 2.1 *As an application for some values of m we have*

$$\int_0^1 \frac{\ln t}{1+t} dt = \frac{-1}{2} \sum_{n=1}^{+\infty} \frac{1}{n^2} = -\frac{1}{12}\pi^2 \tag{2.3}$$

$$\int_0^1 \frac{\ln^2 t}{1+t} dt = \frac{3}{2} \sum_{n=1}^{+\infty} \frac{1}{n^3} = \frac{3}{2}\zeta(3) \tag{2.4}$$

$$\int_0^1 \frac{\ln^3 t}{1+t} dt = \frac{-21}{4} \sum_{n=1}^{+\infty} \frac{1}{n^4} = \frac{-21}{4}\zeta(4) \tag{2.5}$$

2.4 Some values of $\int\limits_0^1 \frac{\ln^n(1-t)}{t^m} dt$

In the following problem, we present a detailed proof of Problem 1117, published in the College Mathematics Journal (2018). The problem was proposed by C. I. Vălean, and the solution was provided by Khristo N. Boyadzhiev [7].

Problem 2.4 *Let n and m be a positive integer. Then the following equality holds*

$$\int\limits_0^1 \frac{\ln^n(1-x)}{x^m} dx = (-1)^n n! \sum_{k=1}^{+\infty} \frac{1}{k^{n+1}} \binom{m+k-2}{k-1}. \tag{2.6}$$

Proof. We have by the substitution $x = 1 - e^{-t}$ we write

$$\begin{aligned}
\int\limits_0^1 \frac{\ln^n(1-x)}{x^m} dx &= (-1)^n \int\limits_0^{+\infty} \frac{t^n e^{-t}}{(1-e^{-t})^m} dt = (-1)^n \int\limits_0^{+\infty} t^n e^{-t} \sum_{k=0}^{\infty} \binom{-m}{k} (-1)^k e^{-kt} dt \\
&= (-1)^n \sum_{k=0}^{\infty} (-1)^k \binom{-m}{k} \int\limits_0^{+\infty} t^n e^{-(k+1)t} dt \\
&= (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{n+1}} \binom{-m}{k} \\
&= (-1)^n n! \sum_{k=0}^{\infty} \frac{1}{(k+1)^{n+1}} \binom{m+k-1}{k}. \\
&= (-1)^n n! \sum_{k=1}^{+\infty} \frac{1}{k^{n+1}} \binom{m+k-2}{k-1}.
\end{aligned}$$

■

Corollay 2.2 *As an application of the above problem , we have the following identies*

$$\int_0^1 \frac{\ln(1-x)}{x} dx = -\xi(2) \tag{2.7}$$

$$\int_0^1 \frac{\ln^2(1-x)}{x} dx = 2\xi(3) \tag{2.8}$$

$$\int_0^1 \left(\frac{\ln(1-x)}{x}\right)^2 dx = 2\xi(2) \tag{2.9}$$

$$\int_0^1 \frac{\ln^3(1-x)}{x} dx = -6\xi(4) \tag{2.10}$$

2.5 Some applications of Logarithmic Integrals

in this section we stady some deficult convergence series and given her values like the following problem

We have detailed the proof of the problem 11682 (American Mathematical Monthly, Vol.119, December 2012) Proposed by Ovidiu Furdui.

Problem 2.5 *The following identity holds*

$$\sum_{n=1}^{+\infty} \frac{H_n}{n^2} = 2\xi(3).$$

Proof. We proceed as follow

$$\sum_{n=1}^{+\infty} \frac{H_n}{n^2} = \sum_{n=1}^{+\infty} \frac{1}{n}\frac{H_n}{n} = -\sum_{n=1}^{+\infty} \frac{1}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx$$

by the relations (2.1), (1.17) and (2.8) we have

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{H_n}{n^2} &= \sum_{n=1}^{+\infty}\frac{1}{n}\frac{H_n}{n} = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 x^{n-1}\ln(1-x)\,dx \\ &= -\int_0^1 \ln(1-x)\sum_{n=1}^{+\infty}\frac{1}{n}x^{n-1}dx \\ &= -\int_0^1 \frac{\ln(1-x)}{x}\sum_{n=1}^{+\infty}\frac{x^n}{n}dx \\ &= \int_0^1 \frac{\ln^2(1-x)}{x}dx \\ &= 2\xi(3).\end{aligned}$$

■

The proof of the following problem (2.11) is given in 2022 by K. N. Boyadzhiev (see [10])

Problem 2.6 *The following identity holds*

$$\sum_{n=0}^{+\infty}(-1)^n\left(\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{n+k}\right)^2 = \frac{\pi^2}{24}. \tag{2.11}$$

Proof. Let

$$S = \sum_{n=0}^{+\infty}(-1)^n\left(\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{n+k}\right)^2,$$

And starting again from the representations

$$\frac{1}{n+k} = \int_0^1 x^{n+k-1}dx$$

then

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{n+k} = \int_0^1 x^n\sum_{k=1}^{\infty}(-x)^{k-1}dx = \int_0^1\frac{x^n}{1+x}dx$$

we continue this way

$$\begin{aligned} S &= .\sum_{n=0}^{+\infty}(-1)^n\left(\int_0^1\frac{x^n}{1+x}dx\right)^2=\sum_{n=0}^{+\infty}(-1)^n\left(\int_0^1\frac{x^n}{1+x}dx\right)\left(\int_0^1\frac{y^n}{1+y}dy\right)\\ &= \sum_{n=0}^{+\infty}(-1)^n\left(\int_0^1\int_0^1\frac{(xy)^n}{(1+x)(1+y)}dxdy\right)\\ &= \int_0^1\int_0^1\left\{\sum_{n=0}^{+\infty}(-xy)^n\right\}\frac{dxdy}{(1+x)(1+y)}\\ &= \int_0^1\int_0^1\frac{dxdy}{(1+xy)(1+x)(1+y)} \end{aligned}$$

Here we set $u=\frac{y}{x}$ to get

$$S=\int_0^1\left\{\int_0^x\frac{1}{(1+u)(u+x)}du\right\}\frac{1}{x+1}dx.$$

Using partial fractions

$$\frac{1}{(1+u)(u+x)}=\frac{1}{x-1}\left(\frac{1}{(u+1)}-\frac{1}{(u+x)}\right)$$

we can evaluate the inside integral. The result is

$$S=\int_0^1\frac{1}{x^2-1}\ln\frac{2}{1+x}dx.$$

The substitution ($\frac{2}{1+x}=1+t$, $x=\frac{1-t}{1+t}$, $dx=\frac{-2}{(1+t)^2}dt$) transforms this integral into a more transparent one

$$S=\frac{1}{2}\int_0^1\frac{\ln(1+t)}{t}dt.$$

By the relation (2.2) and the folowing remark

$$S=\frac{1}{2}\int_0^1\left\{\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}t^k\right\}\frac{1}{t}dt=\frac{1}{2}\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k},$$

the proof is complete of this problem. ■

Chapter 3

Binomial Expansions and Parameter Methods in Integral Evaluation: Insights from Boyadzhiev

3.1 introduction

In this chapter, we explore several intriguing ideas and theorems that appear in two insightful papers authored by K. N. Boyadzhiev. The first, titled "A Binomial Formula for Evaluating Integrals," introduces a powerful technique that connects definite integrals with binomial series, offering elegant and often surprising results. The second paper, "Evaluation of Integrals by Differentiation with Respect to a Parameter," presents a complementary method, showing how parameter differentiation can simplify the evaluation of otherwise challenging integrals.

Together, these works provide a rich framework for understanding integrals through series expansions and algebraic manipulation. In particular, we will focus on a general rule derived from Boyadzhiev's ideas that allows for the evaluation of certain definite integrals in terms of infinite series involving binomial expressions. These techniques not only deepen our appreciation for classical analysis but also highlight the creative connections between combinatorics and integration.

3.2 A Binomial Formula for Evaluating Integrals

In this section we present a special formula for transforming integrals to series. The resulting series involves binomial transforms with the Taylor coefficients of the integrand.

Five applications are provided for evaluating challenging integrals.

Theorem 3.1 *[9] Let $f(x)$ be a function defined and integrable on $]-r,\lambda]$ for some $r>0$, $\lambda>0$.*
Let also $f(x)$ be analytic in a neighborhood of the origin with Taylor series

$$f(x)=\sum_{n=0}^{\infty}a_n x^n$$

Then we have

$$\int_0^{\lambda} f(x)dx=\sum_{n=0}^{\infty}\left(\frac{\lambda}{\lambda+1}\right)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{k+1}$$

In particular, for $\lambda=1$, we have

$$\int_0^{1} f(x)dx=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{k+1} \tag{3.1}$$

and for $\lambda\to\infty$

$$\int_0^{\infty} f(x)dx=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{k+1}. \tag{3.2}$$

Proof. With the substitution $x=t/(1-t)$, $t=x/(1-x)$, we get:

$$\begin{aligned}\int_0^{\lambda} f(x)dx &= \int_0^{\frac{\lambda}{\lambda+1}}\frac{1}{(1-t)^2}f\left(\frac{t}{1-t}\right)dt\\ &= \int_0^{\frac{\lambda}{\lambda+1}}\frac{1}{1-t}\left\{\frac{1}{1-t}f\left(\frac{t}{1-t}\right)\right\}dt\\ &= \int_0^{\frac{\lambda}{\lambda+1}}\frac{1}{1-t}\left\{\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}\binom{n}{k}a_k\right\}dt\end{aligned}$$

by using Euler's series transformation formula

$$\frac{1}{1-t}f\left(\frac{t}{1-t}\right)=\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}\binom{n}{k}a_k$$

■

By expanding $(1+t)^{-1}$ as a geometric series and applying Cauchy's rule for power series multiplication, the proof of theorem is complete.

3.3 Some applications of Binomial Formula for Evaluating Integrals

Here we give some applications of our theorem in the form of problems.

Problem 3.1 *The following identity holds*

$$\int_0^\infty \frac{\ln(1+t)}{t(1+t)}dt = \frac{\pi^2}{6}.$$

Proof. We start from the series (1.15)

$$\sum_{n=1}^{\infty} (-1)^n H_n t^n = \frac{\ln(1+t)}{1+t}$$

and dividing both sides by t we get

$$\sum_{n=1}^{\infty} (-1)^n H_n t^{n-1} = \frac{\ln(1+t)}{1+t}$$

then

$$\sum_{n=0}^{\infty} (-1)^{n+1} H_{n+1} t^n = \frac{\ln(1+t)}{t(1+t)}$$

and we take $a_k = (-1)^k H_{k+1}$ in formula(3.2) we get

$$\int_0^\infty \frac{\ln(1+t)}{t(1+t)}dt = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k H_{k+1}}{k+1}\binom{n}{k}$$

using the relation (3.2)

$$\sum_{k=0}^{n} \frac{(-1)^k H_{k+1}}{k+1}\binom{n}{k} = \frac{1}{(n+1)^2}$$

and

$$\sum_{k=0}^{n} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} \qquad \text{see relation (1.6)}$$

this way, we obtened

$$\int_0^\infty \frac{\ln(1+t)}{t(1+t)}dt = \frac{\pi^2}{6}.$$

■

Problem 3.2 *Let q be a positive integer, then*

$$\int_0^1 \frac{x^q}{(1+x)^{q+1}}dx = \ln 2 - \sum_{n=1}^{q} \frac{1}{2^n n}.$$

Proof. Note that

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k x^k = \sum_{k=q}^{\infty} (-1)^k \binom{k}{q} x^k \\ &= \frac{(-1)^q x^q}{(1+x)^{q+1}}. \end{aligned}$$

Taking

$$a_k = (-1)^k \binom{k}{q}.$$

in formula(3.1) we get

$$\begin{aligned} \int_0^1 f(x)\,dx &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{a_k}{k+1} \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \binom{k}{q} \frac{(-1)^k}{k+1} \end{aligned}$$

using the relation (1.12), then

$$\begin{aligned} \int_0^1 f(x)\,dx &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)} - \sum_{n=0}^{q-1} \frac{1}{2^{n+1}(n+1)} \\ &= \ln 2 - \sum_{n=1}^{q} \frac{1}{2^n n}. \end{aligned}$$

■

3.4 Solving Integrals by Differentiation with Respect to a Parameter

In this section we present an efficient technique for evaluating challenging definite integrals- differentiation with respect to a parameter inside the integral (or appearing in the limits of integration). The integrals can be proper or improper. We provide numerous examples, many of which are listed in some intersing paper.

Below we evaluate some very different integrals in order to demonstrate the wide scope of the method.

Theorem 3.2 *[10] Suppose the function $f(x, \lambda)$ is defined and continuous on the rectangle $[a, b] \times [c, d]$ together with its partial derivative $f_\lambda(x, \lambda)$. Then*

$$\frac{d}{d\lambda}\int_c^d f(x,\lambda)dx = \int_c^d f_\lambda(x,\lambda)dx$$

Theorem 3.3 *[10] Suppose the function $f(x, \lambda)$ is defined and continuous on the rectangle $[a, b] \times [0, +\infty[$ together with its partial derivative $f_\lambda(x, \lambda)$. Then*

$$\frac{d}{d\lambda}\int_0^{+\infty} f(x,\lambda)dx = \int_0^{+\infty} f_\lambda(x,\lambda)dx$$

Example 3.1 *We start with a very simple example. It is easy to show that the popular integral*

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

is convergent since

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx.$$

Integrating by parts

$$\begin{aligned} u &= \frac{1}{x} \to u' = -\frac{1}{x^2} \\ v' &= \sin x \to v = -\cos x \end{aligned}$$

we find

$$\begin{aligned} \int_1^\infty \frac{\sin x}{x}\,dx &= -\left[\frac{\sin x}{x}\right]_1^{+\infty} - \int_1^\infty \frac{\cos x}{x^2}\,dx \\ &= \cos 1 - \int_1^\infty \frac{\cos x}{x^2}\,dx \end{aligned}$$

We know that $|\cos x| \leq 1$. So

$$\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2} \quad \forall x \geq 1$$

Since $\int_1^\infty \frac{\cos x}{x^2}\,dx$ is convergent (Riemann's integral first type $\alpha = 2 > 1$), then the integral $\int_1^\infty \frac{\cos x}{x^2}\,dx$ converges by the test of comparaison. So $\int_0^\infty \frac{\sin x}{x}\,dx$ convergs and

$$\frac{\sin x}{x} \overset{0}{\sim} 1$$

then

$$\int_0^1 \frac{\sin x}{x}\, dx$$

converge.

For its evaluation we consider the function

$$F(\lambda) = \int_0^\infty e^{-\lambda x} \frac{\sin x}{x}\, dx, \quad \lambda > 0$$

with derivative

$$F'(\lambda) = \int_0^\infty e^{-\lambda x} \sin x\, dx = -\frac{1}{1+\lambda^2}.$$

Integrating back we find

$$F(\lambda) = -\arctan\lambda + C$$

Setting $\lambda \to +\infty$ leads to

$$C = \frac{\pi}{2}.$$

Therefore,

$$\int_0^\infty e^{-\lambda x} \frac{\sin x}{x}\, dx = -\arctan\lambda + \frac{\pi}{2}.$$

and with $\lambda \to 0$ we find

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

3.4.1 Special integrals with arctangents and logarithms

In the following problem, we present the solution to Problem A5 from the 66th Annual William Lowell Putnam Mathematical Competition (2005), which involves the evaluation of a challenging definite integral. [30]

Problem 3.3 *We have*

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx = \frac{\pi \ln 2}{8}$$

Proof. Consider the function

$$F(\lambda) = \int_0^1 \frac{\ln(1+\lambda x)}{1+x^2}\, dx.$$

Differentiating this function gives

$$F'(\lambda) = \int_0^1 \frac{dx}{(1+x^2)(1+\lambda x)}.$$

Using partial fractions, we obtain

$$\frac{1}{(1+x^2)(1+\lambda x)} = \frac{A}{1+\lambda x} + \frac{Bx+C}{1+x^2}$$

by identification

$$\begin{cases} A = \frac{-\lambda}{\lambda^2+1} \\ B = \frac{1}{\lambda^2+1} \\ C = \frac{\lambda}{\lambda^2+1} \end{cases}$$

$$\begin{aligned} \int_0^1 \frac{1}{(1+x^2)(1+\lambda x)} dx &= \int_0^1 \frac{A}{1+\lambda x} + \frac{Bx+C}{1+x^2} dx \\ &= \left[\frac{A}{\lambda} \ln|1+ax| + \frac{B}{2} \ln\left(1+x^2\right) + C \arctan x \right]_0^1 \\ &= -\frac{\ln(1+\lambda)}{\lambda^2+1} + \frac{\ln 2}{2\left(\lambda^2+1\right)} + \frac{\pi}{4} \frac{\lambda}{\lambda^2+1} \end{aligned}$$

then

$$F'(x) = -\frac{\ln(1+x)}{x^2+1} + \frac{\ln 2}{2(x^2+1)} + \frac{\pi}{4} \frac{x}{x^2+1}$$

we get

$$\int_0^\lambda F'(x)\, dx = -\int_0^\lambda \frac{\ln(1+x)}{x^2+1} dx + \frac{\ln 2}{2} \int_0^\lambda \frac{1}{x^2+1} dx + \frac{\pi}{4} \int_0^\lambda \frac{x}{x^2+1} dx$$

then

$$F(\lambda) = -\int_0^\lambda \frac{\ln(1+x)}{x^2+1} dx + \frac{\ln 2}{2} \arctan \lambda + \frac{\pi}{8} \ln\left(\lambda^2+1\right)$$

taking $\lambda = 1$, we arrive at the equation

$$F(1) = -F(1) + \frac{\pi \ln 2}{4}.$$

That is,

$$\int_0^1 \frac{\ln(1+x)}{x^2+1} dx = \frac{\pi \ln 2}{8}$$

■

Problem 3.4 *For any* $0 < \alpha < 1$,

$$2 \int_0^{+\infty} \frac{\arctan \alpha x}{1+x^2} dx = \ln \alpha \ln \frac{1-\alpha}{1+\alpha} + \mathbf{Li}_2(\alpha) - \mathbf{Li}_2(-\alpha).$$

Proof. It is easy to show that the generalise integral

$$\int_0^{+\infty} \frac{\arctan \alpha x}{1+x^2} dx$$

is convergent since

$$\lim_{x \to +\infty} x^{\frac{3}{2}} \frac{\arctan \alpha x}{1+x^2} = 0.$$

Let

$$J(\alpha) = 2\int_0^{\infty} \frac{\arctan \alpha x}{1+x^2} dx.$$

By differentiation with respet of α

$$\begin{aligned} J'(\alpha) &= 2\int_0^{\infty} \frac{x}{(1+\alpha^2 x^2)(1+x^2)} dx \\ &= \frac{1}{(1-\alpha^2)} \int_0^{\infty} \frac{2x}{1+x^2} - \frac{2\alpha^2 x}{1+\alpha^2 x^2} dx \\ &= \frac{1}{1-\alpha^2} \left[\ln \frac{1+x^2}{1+\alpha^2 x^2} \right]_0^{\infty} \\ &= \frac{-2\ln \alpha}{1-\alpha^2}. \end{aligned}$$

Thus, since $J(\alpha)$ defined for $\alpha = 0$, $J(0) = 0$,

$$\begin{aligned} \int_0^{\alpha} J'(t)\, dt &= \int_0^{\alpha} \frac{-2\ln t}{1-t^2} dt \\ &= \int_0^{\alpha} \frac{\ln t}{t-1} - \frac{\ln t}{t+1} dt \end{aligned}$$

Integrating by parts we find

$$\begin{aligned} u &= \ln t \to u' = \frac{1}{t} \\ v' &= -\frac{1}{1-t} - \frac{1}{t+1} \to v = \ln(t-1) - \ln(t+1) \end{aligned}$$

thus

$$\begin{aligned} J(\alpha) &= \ln \alpha \ln \frac{1-\alpha}{1+\alpha} + \int_0^{\alpha} \frac{\ln(1+t)}{t} - \frac{\ln(1-t)}{t} dt \\ &= \ln \alpha \ln \frac{1-\alpha}{1+\alpha} + +\mathbf{Li}_2(\alpha) - \mathbf{Li}_2(-\alpha). \end{aligned}$$

■

Problem 3.5 *For every* $0 < \lambda \leq 1$,

$$\int_0^{+\infty} \frac{\ln(1+\lambda x)}{x(1+x)} dx = \ln\lambda \ln(1-\lambda) + \mathbf{Li}_2(\lambda).$$

Proof. It is easy to show that the generalise integral

$$\int_0^{+\infty} \frac{\ln(1+\lambda x)}{x(1+x)} dx = \int_0^1 \frac{\ln(1+\lambda x)}{x(1+x)} dx + \int_1^{+\infty} \frac{\ln(1+\lambda x)}{x(1+x)} d$$

is convergent since

$$\lim_{x\to+\infty} x^{\frac{3}{2}} \frac{\ln(1+\lambda x)}{x(1+x)} = 0 \text{ and } \lim_{x\to 0} x^{\frac{1}{2}} \frac{\ln(1+\lambda x)}{x(1+x)} = 0$$

Let

$$J(\lambda) = \int_0^{\infty} \frac{\ln(1+\lambda x)}{x(1+x)} dx.$$

By differentiation with respet of λ

$$J'(\lambda) = \int_0^{\infty} \frac{1}{(1+x)(1+\lambda x)} dx$$

Using partial fractions, we obtain

$$\frac{1}{(1+x)(1+\lambda x)} = \frac{A}{1+x} + \frac{B}{1+\lambda x}$$

by identification

$$\begin{cases} A = \frac{-1}{\lambda-1} \\ B = \frac{\lambda}{\lambda-1} \end{cases}$$

then

$$\begin{aligned} J'(\lambda) &= \left[\frac{1}{1-\lambda} \ln\frac{1+x}{1+\lambda x}\right]_0^{+\infty} \\ &= -\frac{1}{1-\lambda} \ln\lambda \end{aligned}$$

From here, integrating by parts,

$$\begin{aligned} J(\lambda) &= -\int_0^{\lambda} \frac{1}{1-t} \ln t dt \\ &= \ln\lambda \ln(1-\lambda) - \int_0^{\lambda} \frac{\ln(1-t)}{t} dt \\ &= \ln\lambda \ln(1-\lambda) + \mathbf{Li}_2(\lambda). \end{aligned}$$

■

Remark 3.1 *The William Lowell Putnam Mathematical Competition is a prestigious annual mathematics competition for undergraduate students in North America. It has been organized by the Mathematical Association of America (MAA) since 1938 and is known for its high level of difficulty, focusing on creative thinking and deep mathematical reasoning. It is considered one of the hardest undergraduate math competitions.*

Winners often become prominent mathematicians or gain prestigious academic opportunities.

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