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Variable Order Riemann-Liouville Fractional Differential Equation with Anti-Periodic Boundary Conditions

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DEDICATION

I dedicate this work to the pillars of my life my dear parents, Ahmed and Houria, whose unwavering love, sacrifices, and endless patience have shaped me into the person I am today. Your strength has always been my refuge, and your belief in me has been the light that guided me through every dark moment.

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Introduction

The field of fractional calculus, which generalizes classical calculus by allowing derivatives and integrals of non-integer orders, has emerged as a powerful framework for modeling complex systems with memory and hereditary properties. Over recent decades, fractional calculus has found widespread applications in physics, engineering, biology, and control theory, owing to its ability to more accurately describe phenomena where traditional integer-order models fall short.

Among recent advancements, variable-order fractional calculus has attracted considerable attention. Unlike classical fractional derivatives of constant order, variable-order operators allow the order of differentiation or integration to vary with respect to time, space, or other parameters, offering a more flexible and realistic modeling tool for dynamic processes whose behavior evolves over time. Such models have been successfully applied in fields including viscoelastic materials, anomalous diffusion, control systems, and electrochemical processes. However, the theoretical analysis of differential equations involving variable-order derivatives remains relatively less explored compared to the constant-order case, particularly concerning boundary value problems (BVPs).

In many physical and engineering applications, boundary conditions play a crucial role. Among them, anti-periodic boundary conditions naturally arise in systems with alternating behaviors, such as heat conduction with sign-reversing sources, electromagnetic oscillations, and biological rhythms. While extensive studies have been conducted on constant-order fractional BVPs, there exists a noticeable gap in the literature regarding variable-order fractional differential equations with anti-periodic boundary conditions. This motivates the present study.

The main objective of this thesis is to investigate the existence of solutions and the stability analysis (in the sense of Ulam-Hyers-Rassias stability) for a class of Riemann-Liouville variable-order fractional differential equations under anti-periodic boundary conditions. Our approach combines the use of fixed point theorems and Green's function construction in suitable Banach spaces, providing rigorous existence criteria and stability results. Moreover, illustrative examples are provided to demonstrate the applicability and sharpness of the theoretical findings.

The study of stability, particularly Ulam-Hyers and Ulam-Hyers-Rassias stability, has become an important area in the qualitative analysis of differential equations. Initially posed by Ulam in 1940 and first addressed by Hyers in 1941, stability theory has since expanded and now forms a cornerstone in understanding the behavior of solutions under perturbations. In this work, we extend these stability concepts to variable-order fractional differential equations, enriching the theory further.

This thesis is organized as follows:

Chapter 1 introduces essential preliminaries, including fundamental definitions, properties of constant and variable-order fractional calculus, key fixed point theorems, different types of stability, and the role of Green's functions in solving boundary value problems.

Chapter 2 is devoted to establishing existence results for the proposed boundary value problem. Using analytical techniques based on fixed point theory, we derive sufficient conditions guaranteeing the existence of solutions.

Chapter 3 focuses on the Ulam-Hyers-Rassias stability of the solutions obtained. The stability results are proven under appropriate assumptions, and their significance is highlighted through theoretical analysis and presents concrete examples that illustrates the validity of the main results. These examples confirms both the existence and the stability properties of the solutions under study.

Through this work, we aim to fill a gap in the current literature by providing a solid theoretical foundation for the study of variable-order fractional differential equations with anti-periodic boundary conditions. The results presented herein not only generalize existing works on constant-order problems but also open pathways for further research in fractional modeling and its applications across various scientific disciplines.

Chapter 1

Preliminary

This chapter introduces some important fundamental definitions which are used throughout this thesis.

1.1 Notations and definitions

The symbol $C(I, \mathbb{R})$ represents the Banach space of continuous functions $\varkappa : I \to \mathbb{R}$ with the norm

$$\|\varkappa\| = Sup\{|\varkappa(t)| : t \in I\},$$

and

$$L^p[a,b] = \left\{ f : [a,b] \longrightarrow \mathbb{R}; f \text{ is mesurable in } [a,b] \text{ and } \int_a^b |f(t)|^p dt < \infty, 1 < p < +\infty \right\}.$$

1.2 The Gamma function

One of the basic functions used in fractional calculus is the Euler Gamma function. It extends the factorial function to real numbers and even to complex numbers.

Definition 1.1. *([9]) For* $z \in \mathbb{C}$ *such that* $\mathrm{Re}(z) > 0$*, Euler's Gamma function is defined by the following integral:*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt. \tag{1.1}$$

Proposition 1.1. *The Gamma function satisfies the following properties:*

1) **The recurrence relation** : for all $z \in \mathbb{C}$ such that $\text{Re}(z) > 0$ then

$$\Gamma(z+1) = z\Gamma(z)$$

And for every $n \in \mathbb{N}^*$ then :

$$\Gamma(n) = (n-1)!$$

Especially $\Gamma(1) = 1$.

2) **The representation of the Gamma function by the limite:**

$$\Gamma(z) = \lim_{n \to +\infty} \frac{n! n^z}{(z+1)\cdots(z+n)}$$

with $\text{Re}(z) > 0$.

3) **The derivative:** The Gamma function is indefinitely differentiable on $\mathbb{R}_+^*$ its derivative is:

$$\Gamma'(z) = \Gamma(z)\Psi(z)$$

$$\Psi(z) = \frac{d}{dz} \ln[\Gamma(z)]$$

4) **Some special values of the Gamma function:**

For $z = \dfrac{1}{2}$,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

For $z = n + \dfrac{1}{2}, n \in \mathbb{N}^*$,

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi}.$$

1.3 The Beta function

Definition 1.2. *([9]) The Beta function is a type of Euler integral defined for complex numbers z and w by:*

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt, \tag{1.2}$$

with $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$

Proposition 1.2. *For $z, w \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$ the Beta function satisfies the following properties:*

1) The Beta function is linked to the Gamma function by the following relationship:

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

2) The Beta function is symmetric, that is to say:

$$B(z, w) = B(w, z).$$

3) The Beta function can also take integral forms:

$$B(z, w) = 2\int_0^{\frac{\pi}{2}} \sin^{2z-1}(\theta)\cos^{2w-1}(\theta) d\theta$$

$$B(z, w) = \int_0^{+\infty} \frac{t^{z-1}}{(1+t)^{z+w}} dt.$$

4) The derivative of the Beta function is given by:

$$\frac{\partial}{\partial z}B(z,w) = B(z,w)\left(\frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+w)}{\Gamma(z+w)}\right).$$

Definition 1.3. *([9]) The generalized binomial formula $\binom{\alpha}{n}$ for $\alpha \in \mathbb{C}$ where $n \in \mathbb{N}$ is given by :*

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, (n \in \mathbb{N}) \tag{1.3}$$

In particular, for $\alpha = m \in \mathbb{N}$, we have:

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}, (m, n \in \mathbb{N}; m \geq n), \tag{1.4}$$

and

$$\binom{m}{n} = 0, (m, n \in \mathbb{N}; 0 \leq m < n) \tag{1.5}$$

Definition 1.4. *([9]) This formula can be expressed in terms of the Gamma function for $\alpha \notin \mathbb{Z}_-^*$ as follows:*

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)}, \left(\alpha \in \mathbb{C}, \alpha \notin \mathbb{Z}_-^*, n \in \mathbb{N}\right) \tag{1.6}$$

Definition 1.5. *The Beta function is defined by:*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, x, y > 0. \tag{1.7}$$

The change of variable

$$u = 1 - t$$

Allows us to show that the Beta function is symmetric, then:

$$B(x,y) = B(y,x)$$

It can also take the following integral forms:

$$B(x,y) = \frac{1}{a^{x+y-1}}\int_0^a t^{x-1}(a-t)^{y-1}dt.$$
$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt.$$
$$B(x,y) = \int_0^{\frac{\pi}{2}} \sin^{2x-1}(\theta)\cos^{2y-1}(\theta)d\theta.$$

The Beta function and the Gamma function are linked by the following formula:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{1.8}$$

1.4 Fractional calculus.

1.4.1 Fractional calculus of constant-order

Definition 1.6. *([13, 15]). The left Riemann-Liouville fractional integral of the function $f \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\alpha)$ is the gamma function.

Definition 1.7. *([13, 15]). The left Riemann-Liouville fractional derivative of order $\alpha > 0$ of function $f \in L^1([a,b],\mathbb{R}_+)$, is given by*

$$(D_{a+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)}\Big(\frac{d}{dt}\Big)^n \int_a^t (t-s)^{n-\alpha-1} f(s)ds,$$

here $n = [\alpha] + 1$ *and* $[\alpha]$ *denotes the integer part of* α. *If* $\alpha \in (0, 1]$, *then*

$$(D^{\alpha}_{a+}f)(t) = \frac{d}{dt} I^{1-\alpha}_{a+} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_a^t (t-s)^{-\alpha} f(s) ds.$$

The following properties and lemmas are some of the main ones of the fractional derivatives and integrals.

Proposition 1.3. *We have*

$$\begin{cases} D^{\alpha} f = I^{-\alpha} f, \quad if \ \alpha < 0, \\ D^{\beta} I^{\alpha} f = I^{\alpha-\beta} f, \quad if \ \beta \in [0, \alpha). \end{cases}$$

Example:

$$\begin{cases} D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha},, \quad if \ \beta > -1, \beta \neq \alpha - 1, \alpha - 2, ..., \alpha - n, \\ D^{\alpha} t^{\alpha-i} = 0, i = 1, 2, ..., n. \end{cases}$$

For $f(t) = 1$, *we get*

$$D^{\alpha} 1 = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}, \alpha \notin \mathbb{N}.$$

For $\alpha \in \mathbb{N}$, *we get, of course,* $D^{\alpha} 1 = 0$ *due to the poles of the gamma function at the points* $0, -1, -2, ...$

Lemma 1.1. *([13]). Let* $\alpha > 0$, $a \geq 0$, $f \in L^1(a, b)$, $D^{\alpha}_{a+} f \in L^1(a, b)$. *Then, the differential equation*

$$D^{\alpha}_{a^+} f = 0$$

has unique solution

$$f(t) = \omega_1 (t-a)^{\alpha-1} + \omega_2 (t-a)^{\alpha-2} + ... + \omega_i (t-a)^{\alpha-i} + ... + \omega_n (t-a)^{\alpha-n},$$

where $n = [\alpha] + 1$, $\omega_i \in \mathbb{R}$, $i = 1, 2, ..., n$.

Lemma 1.2. *([13]). Let* $\alpha > 0$, $a \geq 0$, $f \in L^1(a,b)$, $D^{\alpha}_{a^+} f \in L^1(a,b)$. *Then,*

$$I^{\alpha}_{a^+} D^{\alpha}_{a^+} f(t) = f(t) + \omega_1 (t-a)^{\alpha-1} + \omega_2 (t-a)^{\alpha-2} + ... + \omega_i (t-a)^{\alpha-i} + ... + \omega_n (t-a)^{\alpha-n},$$

where $n = [\alpha] + 1$, $\omega_i \in \mathbb{R}$, $i = 1, 2, ..., n$.

Lemma 1.3. *([13]). Let* $\alpha > 0$, $a \geq 0$, $f \in L^1(a,b)$, $D^{\alpha}_{a^+} h \in L^1(a,b)$. *Then,*

$$D^{\alpha}_{a^+} I^{\alpha}_{a^+} f(t) = f(t).$$

Lemma 1.4. *([13]). Let* $\alpha,\ \beta > 0$, $a \geq 0$, $f \in L^1(a,b)$. *Then,*

$$I^{\alpha}_{a^+} I^{\beta}_{a^+} f(t) = I^{\beta}_{a^+} I^{\alpha}_{a^+} f(t) = I^{\alpha+\beta}_{a^+} f(t).$$

1.4.2 Fractional calculus of variable-order

Definition 1.8. *([18, 23]). For* $-\infty < a < b < +\infty$, *we consider the mapping* $\alpha(t) : [a,b] \to (0, +\infty)$. *Then, the left Riemann-Liouville fractional integral (RLInVo) of variable-order* $\alpha(t)$ *for function* $f(t)$ *is expressed by*

$$I^{\alpha(t)}_{a^+} f(t) = \int_a^t \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} f(s) ds, \ \ t > a, \tag{1.9}$$

where the gamma function is denoted by $\Gamma(\alpha)$.

Definition 1.9. *([18, 23]). For* $-\infty < a < b < +\infty$, *we consider the mapping* $\alpha(t) : [a,b] \to (n-1, n)$, $n \in \mathbb{N}$. *Then, the left Riemann-Liouville fractional derivative of variable-order* $\alpha(t)$ *for function* $f(t)$ *is expressed by*

$$D^{\alpha(t)}_{a^+} f(t) = \left(\frac{d}{dt}\right)^n I^{n-\alpha(t)}_{a^+} f(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha(s)-1}}{\Gamma(n-\alpha(s))} f(s) ds, \ \ t > a. \tag{1.10}$$

Obviously, if the order $\alpha(t)$ is a constant function α, then the Riemann-Liouville fractional derivative of variable-order (1.10) and Riemann-Liouville fractional integral of variable-order(1.9) are the usual Riemann-Liouville fractional derivative and Riemann-Liouville fractional integral, respectively; see [13, 18, 19].

Remark ([24]). Generally, for functions $\alpha(t)$ and $\beta(t)$, the semigroup property does not hold, i.e.,

$$I_{a^+}^{\alpha(t)} I_{a^+}^{\beta(t)} f(t) \neq I_{a^+}^{\alpha(t)+\beta(t)} f(t).$$

Example: Let

$$\alpha(t) = \begin{cases} 2, & t \in [0,2], \\ 1, & t \in]2,6], \end{cases}$$

$$\beta(t) = \begin{cases} 1, & t \in [0,2], \\ 2, & t \in]2,6], \end{cases}$$

and $f(t) = t$, $t \in [0,6]$.

Step 1: Compute $I_{0+}^{\alpha(t)} I_{0+}^{\beta(t)} f(t)$

$$I_{0+}^{\alpha(t)} I_{0+}^{\beta(t)} f(t) = \int_0^t \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} \int_0^s \frac{(s-\tau)^{\beta(\tau)-1}}{\Gamma(\beta(\tau))} f(\tau) d\tau ds$$

Splitting over the two intervals:

$$= \int_0^2 \frac{(t-s)^1}{\Gamma(2)} \int_0^s \frac{(s-\tau)^0}{\Gamma(1)} \tau d\tau ds + \int_2^t \frac{(t-s)^0}{\Gamma(1)} \left[\int_0^2 \frac{(s-\tau)^0}{\Gamma(1)} \tau d\tau + \int_2^s \frac{(s-\tau)^1}{\Gamma(2)} \tau d\tau \right] ds$$

Evaluating the inner integrals:

$$= \int_0^2 \frac{(t-s)s^2}{2\Gamma(2)} ds + \int_2^t \left(\frac{s^3}{6} - \frac{s^2}{2} + \frac{5}{6} s \right) ds.$$

Step 2: Compute $I_{0+}^{\alpha(t)+\beta(t)}f(t)$

$$I_{0+}^{\alpha(t)+\beta(t)}f(t) = \int_0^t \frac{(t-s)^{\alpha(s)+\beta(s)-1}}{\Gamma(\alpha(s)+\beta(s))}f(s)ds.$$

Evaluating at $t=4$:

$$I_{0+}^{\alpha(t)}I_{0+}^{\beta(t)}f(t)|_{t=4}$$

$$= \int_0^2 \frac{(4-s)s^2}{2\Gamma(2)}ds + \int_2^4 \left(\frac{s^3}{6} - \frac{s^2}{2} + \frac{5}{6}s\right)ds$$

$$= \frac{5}{24} + \frac{17}{24} = \frac{22}{24}.$$

$$I_{0+}^{\alpha(t)+\beta(t)}f(t)|_{t=4}$$

$$= \int_0^2 \frac{(4-s)^{2+1-1}}{\Gamma(2+1)}sds + \int_2^4 \frac{(4-s)^{1+2-1}}{\Gamma(1+2)}sds$$

$$= \frac{11}{24} + \frac{5}{24} = \frac{16}{24}.$$

Conclusion

$$I_{0+}^{\alpha(t)}I_{0+}^{\beta(t)}f(t)|_{t=4} \neq I_{0+}^{\alpha(t)+\beta(t)}f(t)|_{t=4}.$$

Thus, the semigroup property does not hold for the variable order fractional integral with this choice of $\alpha(t)$ and $\beta(t)$.

Lemma 1.5. *([26]). Let $\alpha : I := [0,T] \to (1,2]$ be a continuous function, then for $f \in C_\delta(I,\mathbb{R}) = \{f(t) \in C(I,\mathbb{R}),\ t^\delta f(t) \in C(I,\mathbb{R})\},\ (0 \leq \delta \leq \min_{t\in I}|\alpha(t)|)$, the variable order fractional integral $I_{0+}^{\alpha(t)}f(t)$ exists for any points on I.*

Lemma 1.6. *([26]). Let $\alpha : I := [0,T] \to (1,2]$ be a continuous function, then $I_{0^+}^{\alpha(t)} f(t) \in C(I, \mathrm{I\!R})$ for $f \in C(I, \mathrm{I\!R})$.*

Definition 1.10. *([1, 25]). A generalized interval is a subset I of $\mathbb{R}$ which is either an interval (i.e. a set of the form $[a,b], (a,b), [a,b)$ or $(a,b]$), a point $\{a\}$, or the empty set $\emptyset$.*

Definition 1.11. *([1, 25]). If I is a generalized interval. A partition of I is a finite set $\mathcal{P}$ of generalized intervals contained in I, such that every x in I lies in exactly one of the generalized intervals I in $\mathcal{P}$.*

Example: The set $\mathcal{P} = \{\{1\}, (1,6), [6,7), \{7\}, (7,8]\}$ of generalized intervals is a partition of $[1,8]$.

Definition 1.12. *([1, 25]). Let I be a generalized interval, let $f : I \to \mathbb{R}$ be a function, and let $\mathcal{P}$ a partition of I. f is said to be piecewise constant with respect to $\mathcal{P}$ if for every $I \in \mathcal{P}$, f is constant on I.*

Example: The function $f : [1,6] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3, & 1 \leq x < 3 \\ 4, & x = 3 \\ 5, & 3 < x < 6 \\ 2, & x = 6 \end{cases}$$

is piecewise constant with respect to the partition $\Big\{[1,3), \{3\}, (3,6), \{6\}\Big\}$ of $[1,6]$.

Definition 1.13. *([1, 25]). Let I be a generalized interval. The function $f : I \to \mathbb{R}$ is called piecewise constant on I, if there exists a partition $\mathcal{P}$ of I such that f is piecewise constant with respect to $\mathcal{P}$.*

Theorem 1.1. *[11](theorem of Ascoli-Arzela). Let $A \subset C(J, \mathrm{I\!R})$, A is relatively compact (i.e $\overline{A}$ is compact) if:*

1. *A is uniformly bounded i.e, there exists* $M > 0$ *such that*

$$|f(x)| < M \text{ for every } f \in A \text{ and } x \in J.$$

2. *A is equicontinuous i.e, for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for each* $x, y \in J$, $|x - y| \leq \delta$ *implies* $|f(x) - f(y)| \leq \epsilon$, *for every* $f \in A$.

1.5 Green's Function

Green's functions are named after the mathematician, **George Green**, who first developed the concept in the 1830s. In the modern study of linear partial differential equations, Green's functions are studied largely from the point of view of fundamental solutions instead. It is an important mathematical tool that has application in many areas of theoretical physics including mechanics, electromagnetism, solid-state physics, thermal physics, and the theory of elementary particles. For the solution of Boundary value problems associated with either ordinary or partial differential equations, one requires a brief knowledge about **Green's function**. Unfortunately, it took many years to emerge from the realms of more formal and abstract mathematical analysis as a potential everyday tool for the practical study of Boundary value problems.

Fundamental Concept

Initially we solve, by fairly elementary methods, a typical one-dimensional boundary value problem for the understanding of Green's function.

Consider the differential equation,

$$Lu(x) = f(x) \tag{1.11}$$

where L is an ordinary linear differential operator, $f(x)$ is a known function while $u(x)$ is an unknown function. To solve the above equation, one method is to find the inverse operator L^{-1} in the form of an integral operator with a kernel $G(x, \xi)$ such that,

$$u(x) = L^{-1} f(x) = \int G(x, \xi) f(\xi)\, d\xi \tag{1.12}$$

The kernel of this integral operator is called **Green's function** for the differential operator. Thus the solution to the non-homogeneous differential equation (1) can be written down, once the Green's function for the problem is known. For this reason, the Green's function is also sometimes called the fundamental solution associated to the operator.

1.6 Some fixed point theorems

Definition 1.14. *Let $T : M \subset X \longrightarrow X$ be a bounded operator from a Banach space X into itself. The operator T is called a k-set contraction if there is a number k $(0 \leq k < 1)$ such that*

$$\mu(T(A)) \leq k\mu(A)$$

for all bounded sets A in M. The bounded operator T is called condensing if
$\mu(T(A)) < \mu(A)$ for all bounded sets A in M with $\mu(M) > 0$.

Obviously, every k-set contraction for $0 \leq k < 1$ is condensing. Every compact map T is a k-set contraction with $k = 0$.

Theorem 1.2. *(Banach's fixed point theorem [13]). Let C be a non-empty closed subset of a Banach space X, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 1.3. *(Schauder fixed point theorem [13]). Let X a Banach space and Q be a convex subset of X and $T : Q \longrightarrow Q$ is compact, and continuous map. Then T has at least one fixed point in Q.*

Theorem 1.4 (Leray–Schauder Alternative [8])**.** *Let X be a Banach space and $T : X \to X$ be a completely continuous operator. Assume that the set*

$$\{x \in X : x = \lambda T(x),\ 0 < \lambda < 1\}$$

is bounded. Then T has at least one fixed point in X, that is, there exists $x^* \in X$ such that $T(x^*) = x^*$.

Remark: This theorem is useful in situations where the operator T is not a contraction but still compact and continuous, and where an a priori bound on the approximate solutions can be established.

1.7 Types of stability

Theorem 1.5. *([2, 17]) Let $\kappa \in C(I, \mathbb{R})$, the boundary value problem is Ulam-Hyers-Rassias stable with respect to κ if there exists $c_f > 0$, such that for any $\epsilon > 0$ and for every solution $z \in C(I, \mathbb{R})$ of the following inequality*

$$|D_{0^+}^{\alpha(t)} z(t) + f(t, z(t))| \leq \epsilon\kappa(t),\ t \in I, \tag{1.13}$$

there exists a solution $x \in C(I, \mathbb{R})$ *of boundary value problem with*

$$|z(t) - x(t)| \leq c_f \epsilon \kappa(t), \ t \in I.$$

Theorem 1.6. *([2, 17]). The boundary value problem is Ulam-Hyers stable if there exists* $c_f > 0$*, such that for any* $\epsilon > 0$ *and for every solution* $z \in C(I, \mathbb{R})$ *of the following inequality*

$$|D_{0^+}^{\alpha(t)} z(t) + f(t, z(t))| \leq \epsilon, \ t \in I, \tag{1.14}$$

there exists a solution $x \in C(I, \mathbb{R})$ *of boundary value problem with*

$$|z(t) - x(t)| \leq c_f \epsilon, \ t \in I.$$

CHAPTER 2

STUDY EXISTENCE AND UNIQUENESS OF SOLUTIONS

2.1 Introduction

In this chapter we deal with the existence of solutions for Riemann-Liouville fractional differential equations with anti-periodic boundary conditions of variable order in the format

$$\begin{cases} D^{\alpha(t)}y(t) = f(t, y(t)),\ t \in I, \\ D^{\alpha(t)-2}y(0) = -D^{\alpha(t)-2}y(T),, \\ D^{\alpha(t)-1}y(0) = -D^{\alpha(t)-1}y(T), \end{cases} \tag{2.1}$$

where $I = [0, T]$, $0 < T < +\infty$, $1 < \alpha(t) \leq 2$, respectively, $f : I \times \mathbb{R} \to \mathbb{R}$, and $D^{\alpha(t)}$ denotes the Riemann-Liouville fractional differential of variable order $\alpha(t)$.

This chapter is divided into the following sections : two important results are as follows : one is relied on the Banach contraction principle, and the other one is relied on Leray–Schauder Alternative fixed-point theorem.

2.2 Existence of solutions

All our original main results in this chapter are discussed in this section. Some assumptions are presented as follows.

(H1) Let $n \in \mathbb{N}$ and $\mathcal{P} = \{I_1 := [0, T_1], I_2 := (T_1, T_2], T_3 := (T_2, T_3], ...T_n := (T_{n-1}, T]\}$ represent a partition of the interval I, and let $\alpha(t) : I \to (1, 2]$ be a piecewise constant function respect to $\mathcal{P}$, i.e.,

$$\alpha(t) = \sum_{i=1}^{n} \alpha_\iota I_i(t) = \begin{cases} \alpha_1, & if\ t \in I_1, \\ \alpha_2, & if\ t \in I_2, \\ \quad . & \\ \quad . & \\ \quad . & \\ \alpha_n, & if\ t \in I_n, \end{cases}$$

where $1 < \alpha_i \leq 2$ are constants and I_i indicates the interval for $I_i := (T_{i-1}, T_i], i = 1, 2, ..., n$, (with $T_0 = 0,\ T_n = T$) such that

$$I_i(t) = \begin{cases} 1, & for\ t \in T_i, \\ 0, & for\ elsewhere. \end{cases}$$

For each $i \in \{1, 2, ..., n\}$, the symbol $E_i = C(J_i, \mathbb{R})$, indicated the Banach space of continuous functions $x : I_i \to \mathbb{R}$ equipped with the norm

$$\|y\|_{E_i} = \sup_{t \in I_i} |y(t)|,$$

then, for any $t \in I_i,\ i = 1, 2, \ldots, n$, the left R-Liouville fractional derivative of variable order $\alpha(t)$ for function $y(t) \in C(I, \mathbb{R})$ is

$$\begin{aligned} D_{0^+}^{\alpha(t)} y(t) &= \frac{1}{\Gamma(2-\alpha(t))} \frac{d^2}{dt^2} \int_0^t (t-s)^{1-\alpha(t)} y(s) ds \\ &= \frac{1}{\Gamma(2-\alpha(t))} \Big(\sum_{k=1}^{m-1} \frac{d^2}{dt^2} \int_{T_{k-1}}^{T_k} (t-s)^{1-\alpha_k} y(s) ds \\ &\quad + \frac{d^2}{dt^2} \int_{T_{i-1}}^{t} (t-s)^{1-\alpha_i} y(s) ds \Big). \end{aligned} \tag{2.2}$$

Thus, for any $t \in I_i,\ i = 1, 2, \ldots, n$, (2.1) can be written as

$$\begin{aligned} &\frac{1}{\Gamma(2-\alpha(t))} \Big(\sum_{k=1}^{i-1} \frac{d^2}{dt^2} \int_{T_{k-1}}^{T_k} (t-s)^{1-\alpha_k} y(s) ds + \frac{d^2}{dt^2} \int_{T_{i-1}}^{t} (t-s)^{1-\alpha_i} y(s) ds \Big) \\ = \;& f(t, y(t)). \end{aligned} \tag{2.3}$$

In the case, when $\tilde{y}(t) \equiv 0$ on $t \in [0, T_{i-1}]$, the equation (2.3) is reduced to

$$D_{T_{i-1}^+}^{\alpha_i} \tilde{y}(t) = f(t, \tilde{y}(t)), \quad t \in I_i.$$

So, we consider the following Riemann-Liouville fractional differential equations with anti-periodic boundary conditions of constant order

$$\begin{cases} D^{\alpha_i}_{T^+_{i-1}} y(t) = f(t, y(t)), \ t \in I_i, \\ D^{\alpha_i - 2}_{T^+_{i-1}} y(T_{i-1}) = -D^{\alpha_i - 2}_{T^+_{i-1}} y(T_i), \\ D^{\alpha_i - 1}_{T^+_{i-1}} y(T_{i-1}) = -D^{\alpha_{i-1}}_{T^+_{i-1}} y(T_i). \end{cases} \tag{2.4}$$

Lemma 2.1. *Let $i \in \{1, 2 \ldots, n\}$, $f : I_i \times \mathbb{R} \to \mathbb{R}$ be continuous and there exists $\delta \in (0, 1)$ such that $t^\delta f \in C(I_i \times \mathbb{R}, \mathbb{R})$.*

The solution of (2.4) is given by

$$\begin{aligned} y(t) \; = \; & \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} (t-s)^{\alpha_i - 1} f(s, y(s)) ds - \frac{t^{\alpha_i - 1}}{2\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} f(s, y(s)) ds \\ - \; & \frac{t^{\alpha_i - 2}}{4\Gamma(\alpha_i - 1)} \int_{\alpha_{i-1}}^{T_i} (T_i - T_{i-1} - 2s) f(s, y(s)) ds. \end{aligned} \tag{2.5}$$

Proof Let $y \in C(I_i, \mathbb{R})$ be a solution of (2.4). Employing the operator $I^{\alpha_i}_{T^+_{i-1}}$ to both sides of (2.4), we find (see Lemma1.2)

$$y(t) = w_1 t^{\alpha_{i-1}} + w_0 t^{\alpha_{i-2}} + I^{\alpha_i}_{T^+_{i-1}} f(t, y(t)), \tag{2.6}$$

where w_0, w_1 are constants.

Using (2.6), we have

$$D^{\alpha_i-1}y(t) = w_1\Gamma(\alpha_i) + I^1 f(t, y(t)).$$

From $D^{\alpha_i-1}_{T^+_{i-1}}y(T_{i-1}) = -D^{\alpha_{i-1}}_{T^+_{i-1}}y(T_i)$, we conclude that

$$w_1 = -\frac{1}{2\Gamma(\alpha_i)}\int_{T_{i-1}}^{T_i} f(s, y(s))ds,$$

since $I^{2-\alpha_i}(t^{\alpha_i-1}) = \Gamma(\alpha_i)t$ and $I^{2-\alpha_i}(t^{\alpha_i-2}) = \Gamma(\alpha_i-1)$, from the boundary condition $D^{\alpha_i-2}_{T^+_{i-1}}y(T_{i-1}) = -D^{\alpha_i-2}_{T^+_{i-1}}y(T_i)$ we get

$$\begin{aligned} w_0 &= \frac{(T_i + T_{i-1})}{4\Gamma(\alpha_i - 1)}\int_{T_{i-1}}^{T_i} f(s, y(s))ds \\ &- \frac{1}{2\Gamma(\alpha_i - 1)}\int_{T_{i-1}}^{T_i} (T_i - s)f(s, y(s))ds. \end{aligned}$$

Thus

$$y(t) = \int_{T_{i-1}}^{T_i} G_i(t, s)f(s, y(s))ds,$$

where $G_i(t, s)$ is Green's function defined by:

$$G_i(t,s) = \begin{cases} -\frac{t^{\alpha_i-1}}{2\Gamma(\alpha_i)} - \frac{(T_i-T_{i-1}-2s)t^{\alpha_i-2}}{4\Gamma(\alpha_i-1)} + \frac{1}{\Gamma(\alpha_i)}(t-s)^{\alpha_i-1}, \\ \qquad \alpha_{i-1} \le s \le t \le T_i, \\ -\frac{t^{\alpha_i-1}}{2\Gamma(\alpha_i)} - \frac{(T_i-T_{i-1}-2s)t^{\alpha_i-2}}{4\Gamma(\alpha_i-1)}, \\ \qquad T_{i-1} \le t \le s \le T_i, \end{cases}$$

where $i = 1, 2, ..., n$.

Then, y solves integral equation (2.5).

Conversely, let $y \in C(I_i, \mathbb{R})$ be a solution of integral equation (2.5). Regarding the continuity of function $t^\delta f$, we deduce that y is the solution of (2.4).

Introduce the following assumption:

(H2) Let the function $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $K > 0$, such that,

$$t^\delta |f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|,$$

for any $y_1,\ y_2 \in \mathbb{R}$, $t \in I$ and $\delta = 2 - \alpha$.

Theorem 2.1. *Assume that condition (H2) hold. Then the problem (2.4) has a unique solution in $C_\delta[T_{i-1}, T_i]$ if*

$$\frac{K(\alpha_i^{1-2\delta} - T_{i-1}^{1-2\delta})}{(1-2\delta)\Gamma(\alpha_i)}\Big[T_i^\delta (T_i - T_{i-1})^{\alpha_i - 1} + \frac{T_i}{2} + \Big|\frac{(T_i - T_{i-1})(\alpha_i - 1)}{4}\Big|$$

$$+\Big|\frac{(\alpha_i - 1)}{2}\Big|\Big(\frac{(1-2\delta)(T_i^{2-2\delta} - T_{i-1}^{2-2\delta})}{(2-2\delta)(T_i^{1-2\delta} - T_{i-1}^{1-2\delta})}\Big)\Big] < 1. \tag{2.7}$$

Proof. For any function $y \in C_\delta[T_{i-1}, T_i]$, The operator defined as follows

$$Fy(t) = \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} (t-s)^{\alpha_i - 1} f(s, y(s)) ds - \frac{t^{\alpha_i - 1}}{2\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} f(s, y(s)) ds$$

$$- \frac{t^{\alpha_i - 2}}{4\Gamma(\alpha_i - 1)} \int_{T_{i-1}}^{T_i} (T_i - T_{i-1} - 2s) f(s, y(s)) ds. \qquad (2.8)$$

It follows from the properties of fractional integrals and from the continuity of function $t^\delta f$ that the operator $F : C_\delta[T_{i-1}, T_i] \to C_\delta[T_{i-1}, T_i]$ defined by (2.8) is well defined.

In view of (H2), for every $t \in [T_{i-1}, T_i]$, we have

$$\begin{aligned}
& t^\delta |(Fu)(t) - (Fv)(t)| \\
& \leq \tfrac{t^\delta}{\Gamma(\alpha_i)} \int_{T_i - 1}^{t} (t-s)^{\alpha_i - 1} \Big| f(s, u(s)) - f(s, v(s)) \Big| ds \\
& + \tfrac{t}{2\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} \Big| f(s, u(s)) - f(s, v(s)) \Big| ds \\
& + \tfrac{1}{4\Gamma(\alpha_i - 1)} \int_{T_{i-1}}^{T_i} |-T_i + T_{i-1} + 2s| \Big| f(s, u(s)) - f(s, v(s)) \Big| ds \\
& \leq K \Big[\tfrac{t^\delta}{\Gamma(\alpha_i)} \int_{T_i - 1}^{t} s^{-\delta} (t-s)^{\alpha_i - 1} |u(s) - v(s)| ds + \tfrac{t}{2\Gamma(\alpha_i)} \int_{T_i - 1}^{T_i} s^{-\delta} |u(s) - v(s)| ds \\
& + \tfrac{1}{4\Gamma(\alpha_i - 1)} \int_{T_{i-1}}^{T_i} s^{-\delta} \Big| T_i - T_{i-1} - 2s \Big| |u(s) - v(s)| ds \Big].
\end{aligned}$$

By the definition of $\|.\|_\delta$, we obtain

$$
\begin{aligned}
&\|(Fu)(t)-(Fv)(t)\|_\delta \\
\leq K\Big[&\tfrac{t^\delta}{\Gamma(\alpha_i)}\int_{T_i-1}^{t} s^{-2\delta}(t-s)^{\alpha_i-1}|u(s)-v(s)|ds+\tfrac{t}{2\Gamma(\alpha_i)}\int_{T_{i-1}}^{T_i} s^{-2\delta}|u(s)-v(s)|ds \\
&+\tfrac{1}{4\Gamma(\alpha_i-1)}\int_{T_{i-1}}^{T_i} s^{-2\delta}\Big|T_i-T_{i-1}-2s\Big||u(s)-v(s)|ds\Big] \\
&\leq \tfrac{K(T_i^{1-2\delta}-T_{i-1}^{1-2\delta})}{(1-2\delta)\Gamma(\alpha_i)}\Big[T_i^\delta(T_i-T_{i-1})^{\alpha_i-1}+\tfrac{T_i}{2}+\Big|\tfrac{(T_i-T_{i-1})(\alpha_i-1)}{4}\Big| \\
&+\Big|\tfrac{(\alpha_i-1)}{2}\Big|\Big(\tfrac{(1-2\delta)(T_i^{2-2\delta}-T_{i-1}^{2-2\delta})}{(2-2\delta)(T_i^{1-2\delta}-T_{i-1}^{1-2\delta})}\Big)\Big]\|u-v\|_\delta.
\end{aligned}
$$

Accordingly, by (2.7), the operator F has a contraction structure. Thus, F involves a fixed point uniquely which is the unique solution of of problem (2.4).

We will prove the existence result for (2.4). This result is based on Leray–Schauder Alternative fixed point theorem.
Assuming that the following assumption is satisfied:

(H3) Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function and there exists a constant $N > 0$ such that
$t^\delta|f(t,y)| \leq N, \quad \forall t \in I_i, y \in \mathbb{R}$, and $\delta = 2-\alpha$.

Theorem 2.2. *Let the conditions of Lemma 2.1 and (H3) be satisfied. Then, the (2.4) possesses at least one solution in* $C_\delta[T_{i-1}, T_i]$.

Proof: Consider the set

$$B_{R_i} = \{y \in C_\delta[T_{i-1}, T_i],\ \|y\|_\delta \leq R_i\}.$$

Let

$$\begin{aligned} R_i \ &\geq \ \frac{N(T_i^{1-\delta} - T_{i-1}^{1-\delta})}{(1-\delta)\Gamma(\alpha_i)}\Big[T_i^\delta (T_i - T_{i-1})^{\alpha_i - 1} + \frac{T_i}{2} + \Big|\frac{(T_i - T_{i-1})(\alpha_i - 1)}{4}\Big| \\ &+ \ \Big|\frac{(\alpha_i - 1)(1-\delta)(T_i^{2-\delta} - T_{i-1})^{2-\delta}}{2(2-\delta)(T_i^{1-\delta} - T_{i-1})^{1-\delta}}\Big|\Big]. \end{aligned}$$

For all $i \in \{1, 2, ..., n\}$ the ball B_{R_i} is a convex, closed bounded non-empty of $C_\delta[T_{i-1}, T_i]$. Now, we shall show that F satisfies the assumption of the Theorem 1.4. The proof will be given in three Steps.

Step 1:

Let B_{R_i} be a bounded set in $C_\delta[T_{i-1}, T_i]$. Hence B_{R_i} is bounded on $C[T_{i-1}, T_i]$ and there exists a constant N such that

$t^{\delta}|f(t,y(t))| \leq N, \forall y \in B_{R_i}, t \in [T_{i-1}, T_i]$. Thus

$$
\begin{aligned}
& t^{\delta}|(Fy)(t)| \\
\leq \quad & \frac{Nt^{\delta}}{\Gamma(y_i)} \int_{T_i - 1}^{t} s^{-\delta}(t-s)^{\alpha_i - 1} ds + \frac{Nt}{2\Gamma(\alpha_i)} \int_{T_{m-1}}^{T_i} s^{-\delta} ds \\
+ \quad & \frac{N}{4\Gamma(\alpha_i - 1)} \int_{T_{i-1}}^{T_i} \Big| T_i - T_{i-1} - 2s \Big| s^{-\delta} ds \\
\leq \quad & \frac{N(T_i^{1-\delta} - T_{i-1}^{1-\delta})}{(1-\delta)\Gamma(\alpha_i)} \Big[T_i^{\delta}(T_i - T_{i-1})^{\alpha_i - 1} + \frac{T_i}{2} + \Big| \frac{(T_i - T_{i-1})(\alpha_i - 1)}{4} \Big| \\
+ \quad & \Big| \frac{(\alpha_i - 1)(1-\delta)(T_i^{2-\delta} - T_{i-1}^{2-\delta})}{2(2-\delta)(T_i^{1-\delta} - T_{i-1}^{1-\delta})} \Big| \Big],
\end{aligned}
$$

which implies that

$$
\|(Fy)\|_{\delta} \leq \frac{N(T_i^{1-\delta} - T_{i-1}^{1-\delta})}{(1-\delta)\Gamma(\alpha_i)} \Big[T_i^{\delta}(T_i - T_{i-1})^{\alpha_i - 1} + \frac{T_i}{2} + \Big| \frac{(T_i - T_{i-1})(\alpha_i - 1)}{4} \Big| + \Big| \frac{(\alpha_i - 1)(1-\delta)(T_i^{2-\delta} - T_{i-1}^{2-\delta})}{2(2-\delta)(T_i^{1-\delta} - T_{i-1}^{1-\delta})} \Big| \Big].
$$

Hence $F(B_{R_i})$ is uniformly bounded.

Step 2:

For $t_1, t_2 \in I_i,\ t_1 < t_2$ and $y \in B_{R_i}$, estimate

$$\begin{aligned}
&\Big|t_1^{\delta}(Fy)(t_1) - t_2^{\delta}(Fy)(t_2)\Big| \\
&= \Big|\tfrac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{t_1}\Big[t_1^{\delta}(t_1-s)^{\alpha_i-1} - t_2^{\delta}(t_2-s)^{\alpha_i-1}\Big]f(s,y(s))ds \\
&-\tfrac{1}{\Gamma(\alpha_i)}\int_{t_1}^{t_2} t_2^{\delta}(t_2-s)^{\alpha_i-1}f(s,y(s))ds - \tfrac{(t_1-t_2)}{2\Gamma(\alpha_i-1)}\int_{T_i-1}^{T_i} f(s,y(s))ds\Big| \\
&\leq N\Big(\Big|\tfrac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{t_1}\Big[t_1^{\delta}(t_1-s)^{\alpha_i-1} - t_2^{\delta}(t_2-s)^{\alpha_i-1}\Big]ds \\
&-\tfrac{1}{\Gamma(\alpha_i)}\int_{t_1}^{t_2} t_2^{\delta}(t_2-s)^{\alpha_{i-1}}ds\Big| + \Big|\tfrac{(t_1-t_2)}{2\Gamma(\alpha_i-1)}\int_{T_i-1}^{T_i} ds\Big|\Big).
\end{aligned}$$

Hence $\Big|t_1^{\delta}(Fy)(t_1) - t_2^{\delta}(Fy)(t_2)\Big| \to 0$ as $|t_1 - t_2| \to 0$. Thus $t^{\delta}F(B_{R_i})$ is equicontinuity. Consequently, the operator F is compact.

Step3: Consider the set

$$\Omega = \{y \in \mathbb{R} \setminus y = \eta F y, 0 < \eta < 1\},$$

and show that the set Ω is bounded. Let $y \in \Omega$, then $y = \eta F y, 0 < \eta < 1$. For any $t \in [T_i - 1, T_i]$, we have

$$\begin{aligned}
|y(t)| \leq \eta\Big[&\tfrac{1}{\Gamma(\alpha_i)}\int_{T_i-1}^{t}(t-s)^{\alpha_i-1}|f(s,y(s))|ds + \tfrac{t^{\alpha_i-1}}{2\Gamma(\alpha_i)}\int_{T_{i-1}}^{T_i}|f(s,y(s))|ds \\
&+\tfrac{t^{\alpha_i-2}}{4\Gamma(\alpha_i-1}\int_{T_{i-1}}^{T_i}\Big|T_i - T_{i-1} - 2s\Big|ds\Big].
\end{aligned}$$

We have

$$\|(Fy)\|_\delta \leq \frac{N(T_i^{1-\delta}-T_{i-1}^{1-\delta})}{(1-\delta)\Gamma(\alpha_i)}\Big[T_i^\delta(T_i-T_{i-1})^{\alpha_i-1}+\frac{T_i}{2}+\Big|\frac{(T_i-T_{i-1})(\alpha_i-1)}{4}\Big| + \Big|\frac{(\alpha_i-1)(1-\delta)(T_i^{2-\delta}-T_{i-1}^{2-\delta})}{2(2-\delta)(T_i^{1-\delta}-T_{i-1}^{1-\delta})}\Big|\Big].$$

This implies that the set Ω is bounded independently of $\eta \in (0,1)$. Using Theorem 1.4, we obtain that the operator F has at least a fixed point , which implies that the problem (2.4) has at least one solution.

Now, we present our existence result for (2.1).

Theorem 2.3. *Suppose that both (H1), (H2) hold, and inequality (2.7) be satisfied for all $i \in \{1,2,\ldots,n\}$. Then, (2.1) possesses unique solution in $C_\delta[0,T]$.*

Proof. According to Theorem 2.1 the (2.4) possesses unique solution $\widetilde{y}_i \in C_\delta[T_{i-1},T_i]$.
For any $i \in \{1,2,...,n\}$ we define the function

$$y_i = \begin{cases} 0, & t \in [0,T_{i-1}], \\ \widetilde{y}_i, & t \in I_i, \end{cases}$$

Thus, the function $y_i \in C_\delta[T_{i-1},T_i]$ solves the integral equation (2.3) for $t \in I_i$, which means that $y_i(0)=0,\ y_i(T_i)=\widetilde{y}_i(T_i)=0$ and solves (2.3) for $t \in I_i,\ \ i \in \{1,2,\ldots,n\}$.

Then the function,

$$y(t) = \begin{cases} y_1(t), \quad t \in I_1, \\ y_2(t) = \begin{cases} 0, \quad t \in I_1, \\ \widetilde{y}_2, \quad t \in I_2, \end{cases} \\ \quad . \\ \quad . \\ \quad . \\ y_n(t) = \begin{cases} 0, \quad t \in [0, T_{n-1}], \\ \widetilde{y}_n, \quad t \in I_n, \end{cases} \end{cases} .$$

is a solution of (2.1) in $C_\delta[0, T]$.

Chapter 3

Stability and Applications

3.1 Ulam-Hyers-Rassias Stability

We introduce the following assumption:

(H4) The function $\kappa \in C(I, \mathbb{R}_+)$ is increasing and there exists $\beta_\kappa > 0$ such that

$$I^{\alpha_i}_{T_{i-1}{}^+}\kappa(t) \leq \beta_\kappa\ \kappa(t), \quad \text{for} \ \ t \in I_i, \quad i = 1, 2, \ldots, n.$$

Theorem 3.1. *Let the conditions (H1), (H2), (H3) and (H4) be satisfied.*

Then, the (2.1) is Ulam-Hyers-Rassias stable with respect to κ.

Proof Assume that $x(t)$ satisfies the inequality (1.13), then the integral inequality

$$\begin{aligned}\Big| x_i(t) &- \tfrac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i-1}f(s,x_i(s))ds + \tfrac{t^{\alpha_i-1}}{2\Gamma(\alpha_i)}\int_{T_{i-1}}^{T_i} f(s,x_i(s))ds \\ &+ \tfrac{u^{\alpha_i-2}}{4\Gamma(\alpha_i-1)}\int_{T_{i-1}}^{T_i}(T_i-T_{i-1}-2s)f(s,x_i(s))ds\Big| \\ &\leq \epsilon\, I_{T_{i-1}^+}^{\alpha_i}\kappa(t) \leq \epsilon\, \beta_\kappa\, \kappa(t).\end{aligned}$$

holds.

Let y be a the unique solution of (2.1). According to Theorem 2.3, y is given by

$y(t) = y_i(t)$ for $t \in I_i$, where

$$y_i = \begin{cases} 0, & t \in [0, T_{i-1}],\ i = 1, 2, \ldots, n, \\ \widetilde{y}_i, & t \in I_i. \end{cases} \tag{3.1}$$

and $\widetilde{y}_i \in E_i$ is a solution of (2.4). According to Lemma 2.1 the integral equation

$$\begin{aligned}\widetilde{y}_i(t) &= \frac{1}{\Gamma(\alpha_i)}\int_{T_{i-1}}^{t}(t-s)^{\alpha_i-1}f(s,\widetilde{y}(s))ds - \frac{t^{\alpha_i-1}}{2\Gamma(\alpha_i)}\int_{T_{i-1}}^{T_i} f(s,\widetilde{y}(s))ds \qquad (3.2) \\ &- \frac{u^{\alpha_i-2}}{4\Gamma(\alpha_i-1)}\int_{T_{i-1}}^{T_i}(T_i-T_{i-1}-2s)f(s,\widetilde{y}(s))ds\end{aligned}$$

holds.

Let $t \in I_i$ where $i \in \{1, 2, \dots, n\}$. Then we obtain

$$
\begin{aligned}
& |x(t) - y(t)| = |x(t) - y_i(t)| = |x_i(t) - \widetilde{y}_i(t)| \\
\leq & \left| x_i(t) - \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} (t-s)^{\alpha_i - 1} f(s, x_i(s)) ds + \frac{t^{\alpha_i - 1}}{2\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} f(s, x_i(s)) ds \right. \\
+ & \left. \frac{t^{\alpha_i - 2}}{4\Gamma(\alpha_i - 1)} \int_{T_{i-1}}^{T_i} (T_i - T_{i-1} - 2s) f(s, x_i(s)) ds \right| \\
+ & \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} (t-s)^{\alpha_i - 1} \Big| f(s, x_i(s)) - f(s, \widetilde{y}_i(s)) \Big| ds \\
+ & \frac{t^{\alpha_i - 1}}{2\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} \Big| f(s, x_i(s)) - f(s, \widetilde{y}_i(s)) \Big| ds \\
+ & \frac{t^{\alpha_i - 2}}{4\Gamma(\alpha_i - 1)} \int_{T_{i-1}}^{T_i} (T_i - T_{i-1} - 2s) \Big| f(s, x_i(s)) - f(s, \widetilde{y}_i(s)) \Big| ds \\
\leq & \beta_\kappa \, \epsilon \, \kappa(t) + \frac{1}{\Gamma(\alpha_i)} \int_{T_{i-1}}^{t} s^{-\delta} (t-s)^{\alpha_i - 1} (K |x_i(s) - \widetilde{y}_i(s)|) ds \\
+ & \frac{t^{\alpha_i - 1}}{2\Gamma(\alpha_i)} \int_{T_{i-1}}^{T_i} s^{-\delta} (K |x_i(s) - \widetilde{y}_i(s)|) ds \\
+ & \frac{t^{\alpha_i - 2}}{4\Gamma(\alpha_i - 1)} \int_{T_{i-1}}^{T_i} (T_i - T_{i-1} - 2s) s^{-\delta} (K |x_i(s) - \widetilde{y}_i(s)|) ds \\
\leq & \beta_\kappa \, \epsilon \, \kappa(t) + K \Big[\frac{(t - T_{i-1})^{\alpha_i - 1} (t^{1-\delta} - {T_{i-1}}^{1-\delta})}{(1-\delta)\Gamma(\alpha_i)} + \frac{t^{\alpha_i - 1} (T_i^{1-\delta} - {T_{i-1}}^{1-\delta})}{2(1-\delta)\Gamma(\alpha_i)} \\
+ & \frac{t^{\alpha_i - 2}}{4\Gamma(\alpha_i - 1)} \Big(\Big(T_i - T_{i-1} \Big) \Big(\frac{T_i^{1-\delta} - {T_{i-1}}^{1-\delta}}{1-\delta} \Big) - \Big(\frac{T_i^{2-\delta} - {T_{i-1}}^{2-\delta}}{2-\delta} \Big) \Big) \Big] \|x_i - \widetilde{y}_i\|_{E_i} \\
\leq & \beta_\kappa \, \epsilon \, \kappa(t) + \eta \|x - y\|.
\end{aligned}
$$

Then,

$$
\|x - y\| \Big(1 - \eta \Big) \;\; \leq \;\; \beta_\kappa \, \epsilon \, \kappa(t)
$$

or for any $t \in I$

$$|x(t) - y(t)| \leq \|x - y\| \leq \frac{\beta_\kappa}{1-\eta}\epsilon\kappa(t).$$

Therefore, the problem (2.1) is Ulam-Hyers-Rassias stable with respect to κ.

3.2 Numerical results

Example 1:

Let $I := [0,2]$, $T_0 = 0$, $T_1 = 1$, $T_2 = 2$. Consider the following Riemann-Liouville fractional differential equations with anti-periodic boundary conditions of variable order

$$\begin{cases} D_{0^+}^{\alpha(t)} y(t) = \frac{te^t}{9+e^t}\Big(1 + \sin y(t)\Big), \quad t \in I, \\ D_{0^+}^{\alpha(t)-2} y(0) = -D_{0^+}^{\alpha(t)-2} y(2), \;\; D_{0^+}^{\alpha(t)-1} y(0) = -D_{0^+}^{\alpha(t)-1} y(2), \end{cases} \tag{3.3}$$

where

$$\alpha(t) = \begin{cases} \frac{9}{8}, & t \in I_1 := [0,1], \\ \frac{9}{5}, & t \in I_2 :=]1,2]. \end{cases} \tag{3.4}$$

Denote

$$f(t,y) = \frac{te^t}{9+e^t}\Big(1 + \sin y(t)\Big), \;\; (t,y) \in [0,2] \times \mathbb{R}.$$

By (3.4), we consider two auxiliary Riemann-Liouville fractional differential equations with anti-periodic boundary conditions of constant order

$$\begin{cases} D_{0^+}^{\frac{9}{8}}y(t) = \frac{te^t}{9+e^t}\Big(1+\sin y(t)\Big), \quad t \in I_1, \\ D_{0^+}^{-\frac{7}{8}}y(0) = -D_{0^+}^{-\frac{7}{8}}y(1), \ \ D_{0^+}^{\frac{1}{8}}y(0) = -D_{0^+}^{\frac{1}{8}}y(1), \end{cases} \tag{3.5}$$

and

$$\begin{cases} D_{1^+}^{\frac{9}{5}}y(t) = \frac{te^t}{9+e^t}\Big(1+\sin y(t)\Big), \quad t \in I_2, \\ D_{1^+}^{-\frac{1}{5}}y(1) = -D_{1^+}^{-\frac{1}{5}}y(2), \ \ D_{1^+}^{\frac{4}{5}}y(1) = -D_{1^+}^{\frac{4}{5}}y(2). \end{cases} \tag{3.6}$$

For $i = 1$. Clearly,

$$t^{\delta}\Big|f(t,y_1) - f(t,y_2)\Big| \ \leq \ t^{\frac{7}{8}}\Big|\frac{te^t}{9+e^{\pi}}\Big(1+\sin y_1(t)\Big) - \frac{te^t}{9+e^t}\Big(1+\sin y_2(t)\Big)\Big| \leq |y_1 - y_2|$$

$$t^{\delta}\Big|f(t,y)\Big| \ \leq \ t^{\frac{7}{8}}\Big|\frac{te^t}{9+e^t}\Big(1+\sin y(t)\Big)\Big| \leq 2.$$

Hence conditions (H2), (H3) hold with $K = 1$, $N = 2$.

Let $\kappa(t) = t^{\frac{1}{2}}$. Then, we obtain

$$\begin{aligned} I_{0^+}^{\alpha_1}\kappa(t) &= \tfrac{1}{\Gamma(\frac{9}{8})}\textstyle\int_0^t (t-s)^{\frac{1}{8}} s^{\frac{1}{2}} ds \\ &\leq \tfrac{1}{\Gamma(\frac{9}{8})}\textstyle\int_0^t (t-s)^{\frac{1}{8}} ds \\ &\leq \tfrac{8}{9\Gamma(\frac{9}{8})}\kappa(t) := \beta_{\kappa(t)}\kappa(t), \end{aligned}$$

where $\beta_{\kappa} = \frac{8}{9\Gamma(\frac{9}{8})}$. Hence, assumption (H4) is satisfied.

Then, the problem (3.5) has a solution $\widetilde{y}_1 \in C(I_1, \mathbb{R})$(Theorem 2.2).

For $i = 2$. Clearly,

$$t^{\delta}\Big|f(t,y_1) - f(t,y_2)\Big| \ \leq \ t^{\frac{1}{5}}\Big|\frac{te^t}{9+e^t}\Big(1+\sin y_1(t)\Big) - \frac{te^t}{9+e^t}\Big(1+\sin y_2(t)\Big)\Big| \leq 2^{\frac{6}{5}}|y_1 - y$$

$$t^{\delta}\Big|f(t,y)\Big| \ \leq \ t^{\frac{1}{5}}\Big|\frac{te^t}{9+e^t}\Big(1+\sin y(t)\Big)\Big| \leq 2^{\frac{11}{5}}.$$

Hence conditions (H2), (H3) hold with $K = 2^{\frac{6}{5}}$, $N = 2^{\frac{11}{5}}$.

Let $\kappa(t) = t^{\frac{1}{2}}$. Then, we obtain

$$\begin{aligned} I_{1^+}^{\alpha_2}\kappa(t) &= \tfrac{1}{\Gamma(\frac{9}{5})}\int_1^t (t-s)^{\frac{4}{5}} s^{\frac{1}{2}} ds \\ &\leq \tfrac{\sqrt{2}}{\Gamma(\frac{9}{5})}\int_1^t (t-s)^{\frac{4}{5}} ds \\ &\leq \tfrac{5\sqrt{2}}{9\Gamma(\frac{9}{5})}\kappa(t) := \beta_{\kappa(t)}\kappa(t), \end{aligned}$$

where $\beta_\kappa = \frac{5\sqrt{2}}{9\Gamma(\frac{9}{5})}$. Thus, condition (H4) is satisfied.

By Theorem 2.2, the problem (3.6) has a solution $\widetilde{y}_2 \in C(I_2,\mathbb{R})$. Thus, according to Theorem 2.3 the problem (3.3) possesses a solution

$$y(t) = \begin{cases} \widetilde{y}_1(t), & t \in I_1, \\ y_2(t), & t \in I_2, \end{cases}$$

where

$$y_2(t) = \begin{cases} 0, & t \in I_1 \\ \widetilde{y}_2(t), & t \in I_2. \end{cases}$$

In addition, according to Theorem (3.1) the problem (3.3) is Ulam-Hyers-Rassias stable with respect to κ.

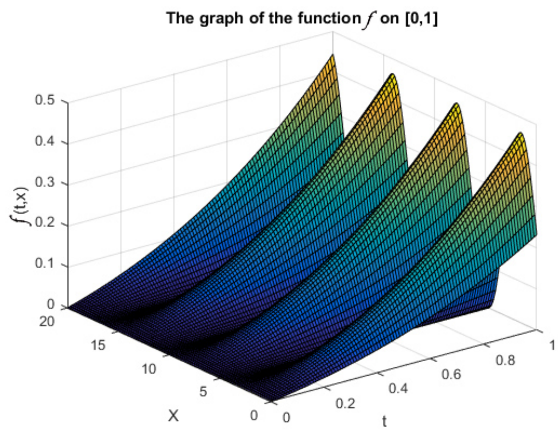

Figure 3.1: The function's f graph for $\alpha(t) = \frac{9}{8}$ on $I_1 = [0, 1]$.

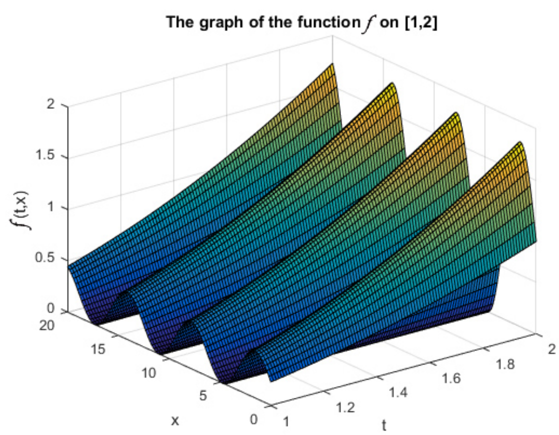

Figure 3.2: The function's f graph for $\alpha(t) = \frac{9}{5}$ on $I_2 =]1, 2]$.

Figures (3.1) and (3.2) translate these analytical results into geometry: monotone 3D surfaces of the nonlinear term $f(t, y)$ verify the Lipschitz bounds required for contraction, while numerical trajectories remain inside the predicted Ulam-Hyers-Rassias stability tube. Example 1 demonstrates robustness when the order switches multiple times.
Many physical and biological media exhibit time dependent memory.
The present framework therefore offers a more faithful modelling paradigm than constant order alternatives—especially for processes that alternate in sign over fixed horizons (e.g., symmetric oscillations in visco elastic beams or cyclic charge–discharge phenomena).

Example 2:

Let $I := [0, 4]$, $T_0 = 0$, $T_1 = 2$, $T_2 = 4$. Consider the following Riemann-Liouville fractional differential equations with anti-periodic boundary conditions of variable order

$$\begin{cases} D_{0^+}^{\alpha(t)} y(t) = \frac{\sin y(t) + 2\cos(t)}{t^2}, \quad t \in I, \\ D_{0^+}^{\alpha(t)-2} y(0) = -D_{0^+}^{\alpha(t)-2} y(4), \;\; D_{0^+}^{\alpha(t)-1} y(0) = -D_{0^+}^{\alpha(t)-1} y(4), \end{cases} \tag{3.7}$$

where

$$\alpha(t) = \begin{cases} \frac{3}{2}, \quad t \in I_1 := [0, 2], \\ \frac{10}{9}, \quad t \in I_2 :=]2, 4]. \end{cases} \tag{3.8}$$

Denote

$$f(t, y) = \frac{\sin y(t) + 2\cos(t)}{t^2}, \;\; (t, y) \in [0, 4] \times \mathbb{R}.$$

By (3.8), we consider two auxiliary Riemann-Liouville fractional differential equations with anti-periodic boundary conditions of constant order

$$\begin{cases} D_{0^+}^{\frac{3}{2}}y(t) = \frac{\sin y(t)+2\cos(t)}{t^2}, \quad t \in I_1, \\ D_{0^+}^{-\frac{1}{2}}y(0) = -D_{0^+}^{-\frac{1}{2}}y(2), \;\; D_{0^+}^{\frac{1}{2}}y(0) = -D_{0^+}^{\frac{1}{2}}y(2), \end{cases} \tag{3.9}$$

and

$$\begin{cases} D_{2^+}^{\frac{10}{9}}y(t) = \frac{\sin y(t)+2\cos(t)}{t^2}, \quad t \in I_2, \\ D_{2^+}^{-\frac{8}{9}}y(2) = -D_{2^+}^{-\frac{8}{9}}y(4), \;\; D_{2^+}^{\frac{1}{9}}y(2) = -D_{2^+}^{\frac{1}{9}}y(4). \end{cases} \tag{3.10}$$

For $i = 1$. Clearly,

$$t^\delta \Big| f(y, y_1) - f(t, y_2) \Big| \;\leq\; t^{\frac{1}{2}} \Big| \frac{\sin y_1(t) + 2\cos(t)}{t^2} - \frac{\sin y_2(t) + 2\cos(t)}{t^2} \Big| \leq 2^{-\frac{3}{2}} |y_1 - y_2|.$$

$$t^\delta \Big| f(t, y) \Big| \leq \; t^{\frac{1}{2}} \Big| \tfrac{\sin y(t)+2\cos(t)}{t^2} \Big| \leq \; \frac{3}{2\sqrt{2}}.$$

Hence conditions (H2), (H3) hold with $K = 2^{-\frac{3}{2}}$, $N = \frac{3}{2\sqrt{2}}$.

Let $\kappa(t) = t^{\frac{1}{2}}$. Then, we acquire

$$\begin{aligned} I_{0^+}^{\alpha_1}\kappa(t) &= \tfrac{1}{\Gamma(\frac{3}{2})} \int_0^t (t-s)^{\frac{1}{2}} s^{\frac{1}{2}} ds \\ &\leq \tfrac{\sqrt{2}}{\Gamma(\frac{3}{2})} \int_0^t (t-s)^{\frac{1}{2}} ds \\ &\leq \tfrac{2\sqrt{2}}{3\Gamma(\frac{3}{2})} \kappa(t) := \beta_{\kappa(t)} \kappa(t), \end{aligned}$$

where $\beta_\kappa = \frac{2\sqrt{2}}{3\Gamma(\frac{3}{2})}$. Hence, assumption (H4) is satisfied.

Then, the problem (3.9) has a solution $\widetilde{y}_1 \in C(I_1, \mathbb{R})$(Theorem 2.2).

For $i = 2$. Clearly,

$$t^{\delta}\Big|y(t,y_1) - f(t,y_2)\Big| \;\leq\; t^{\frac{8}{9}}\Big|\frac{\sin y_1(t) + 2\cos(t)}{t^2} - \frac{\sin y_2(t) + 2\cos(t)}{t^2}\Big| \leq 4^{-\frac{10}{9}}|y_1 - y_2|.$$

$$t^{\delta}\Big|f(t,y)\Big| \leq \; t^{\frac{8}{9}}\Big|\tfrac{\sin y(t)+2\cos(t)}{t^2}\Big| \leq \; 3.4^{\frac{-10}{9}}.$$

Hence conditions (H2), (H3) hold with $K = 4^{-\frac{10}{9}}$, $N = 3.4^{-\frac{10}{9}}$.

Let $\kappa(t) = t^{\frac{1}{2}}$. Then, we acquire

$$\begin{aligned} I_{2^+}^{\alpha_2}\kappa(t) &= \tfrac{1}{\Gamma(\frac{10}{9})}\int_2^t (t-s)^{\frac{1}{9}} s^{\frac{1}{2}} ds \\ &\leq \tfrac{2}{\Gamma(\frac{10}{9})}\int_2^t (t-s)^{\frac{1}{9}} ds \\ &\leq \tfrac{9}{5\Gamma(\frac{10}{9})}\kappa(t) := \beta_{\kappa(t)}\kappa(t), \end{aligned}$$

where $\beta_{\kappa} = \frac{9}{5\Gamma(\frac{10}{9})}$. Thus, condition (H4) is satisfied.

By Theorem 2.2, the problem (3.10) has a solution $\widetilde{y}_2 \in C(I_2,\mathbb{R})$. Thus, according to Theorem 2.3 the problem (3.7) possesses a solution

$$y(t) = \begin{cases} \widetilde{y}_1(t), & t \in I_1, \\ y_2(t), & t \in I_2, \end{cases}$$

where

$$y_2(t) = \begin{cases} 0, & t \in I_1 \\ \widetilde{y}_2(t), & t \in I_2. \end{cases}$$

In addition, according to Theorem (3.1)the problem (3.7) is Ulam-Hyers-Rassias stable with respect to κ.

Conclusion

This study examined variable order Riemann-Liouville fractional differential equations endowed with anti-periodic boundary conditions. Combining fixed-point theory with carefully constructed Green functions, we proved:

1. Existence and uniqueness on every subinterval where
2. Global solvability of the composite variable order problem
3. Ulam–Hyers–Rassias stability with respect to a general weight κ

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