République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique
Université Ibn Khaldoun de Tiaret
Faculté des Mathématiques et de l'Informatique

Département mathématique

THÈSE

Présentée par

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EN VUE DE L'OBTENTION DU DIPLOME DE DOCTORAT LMD

Filière : Mathématique
Option : Analyse fonctionnelle et équation différentielle.

Thème

L'application des méthodes numériques aux équations différentielles d'ordre fractionnaire

Soutenue le : 21-05-2025 Devant le Jury composé de :

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Année Universitaire : 2024/2025

Acknowledgment

• **M**y first thanks go to Allah soubhanaho wa tahala for almighty for giving us the health and the will to begin and finish this thesis.

• **M**y sincere thanks are particularly addressed to my thesis director, Mrs *Dr. Sabit Souhila*. I sincerely thank her for the exceptional quality of her supervision, her patience, her rigor, as well as for the trust she has shown in me. I hope you will find here the expression of my deep gratitude. His skills, his valuable advice, his availability and his kindness towards me greatly contributed to the smooth running of this research work. I found her to be a thesis director who was always available.

• **M**y deep gratitude goes to *Pr. Mohammed Said Souid* (Professor at University of Tiaret) who agreed to be my co-supervisor. I sincerely thank him for his valuable help and for guided me on the right path to complete this thesis. His trust, kindness, support and help were invaluable in preparing this work. I found in him a supervisor who was always available.

• **I** warmly thank *Pr. Senouci Abdelkader* (Professor at Tiaret University) for giving me the honor of chairing the jury.

• **A** big thank you to the members of the jury, *Pr. S.B.Baghli* (Professor at Djillali Liabes SBA University), *Dr. Mokhtari Mokhtar* (Lecturer A, at the University of Tiaret) and *Dr. Benkerrouche Amar* (Lecturer A, at the University of Djelfa) for having done me the honor of examining this work.

• **T**his work is the fruit of long years of study, to which the love and unconditional support of my family and friends have largely contributed.

• **I** would also like to thank everyone who helped me directly or indirectly in the realization of this work.

• **T**o all these speakers, I offer my thanks, my respect and my gratitude.

Dedicate

With enormous pleasure, an open heart and immense joy, I dedicate this thesis.

To my dear father ⋆ *Rouai* ⋆ and my adorable mother ⋆ *Mamia* ⋆.
For the love they always gave me, their encouragement and all the help and advice they gave me during the most difficult moments of this long journey.
As a thank you, I hope that they will find here the expression of my deep gratitude for all the efforts they have made to see me succeed in my studies.

To my dear brothers ⋆ *Radouane* ⋆ and ⋆ *Hicham* ⋆.
For their presence and dedication, I thank you for all the moments we spent together and for your diligence by my side throughout my studies, in gratitude for the help you gave me.

To my friends and my future husband ⋆ *Mohammed Amine* ⋆.
Who never stopped advising me, encouraging me to continue my path and supporting me throughout my studies.

To all my friends, especially ⋆ *Bouchra* ⋆ and ⋆ *Amira* ⋆.

To my grandmothers, may God give them a long and happy life.

So to all the ⋆ *Benaouda* ⋆ and ⋆ *Mouffok* ⋆ family.

And to my ⋆ *Bakkar* ⋆ inlaws, especially my mother inlaw.

To all my teachers.

♣ *Benaouda Marwa* ♣

Abstract

In this thesis, we use a novel approach to study the existence, uniqueness, and stability of solutions to a Cauchy-type problem of nonlinear fractional differential equations of variable order with finite and infinite delay. Contrary to the techniques taken in the literature, which were centered on the usage of the concept of generalized intervals and the idea of piecewise constant functions, our approach is straightforward and based on a novel fractional operator that is more appropriate and demonstrates the solvability and stability of the main problem under less restrictive presumptions. All results are achieved by using fixed point theory. In all chapters of this work, we have illustrated our theoretical study with numerical applications to approximate the solution to all our proposed problems, and we have used different methods. One of these methods is the finite difference method. The finite difference method is a numerical technique used to solve differential equations by approximating derivatives with finite differences. It is widely used in physics, engineering, and other fields where differential equations need to be solved numerically. Essentially, it discretizes space and time into small intervals and approximates derivatives with finite differences to obtain a numerical solution.

The second method is the Euler discretization method, also known as the Euler method, which is one of the simplest numerical methods for solving ordinary differential equations (ODEs). It belongs to the family of finite difference methods and applies to first-order equations. The basic idea of this method is to approximate the solution of a differential equation step by step, using a time interval and starting from an initial condition. In this work, comparisons were made between these two methods to confirm the theoretical results founded in this thesis.

Keywords: Derivatives and integrals of variable-order , Fixed point theorem , Cauchy-type problem, functional differential equations with infinite delay , functional differential equations with finite delay , Ulam-Hyers stability, Numerical methods.

AMS (MOS) Subject Classifications: 26A33, 34A08, 34A37, 34A60.

Résumé

Dans cette thèse, nous utilisons une nouvelle approche pour étudier l'existence, l'unicité et la stabilité des solutions d'un problème de type Cauchy pour des équations différentielles fractionnaires non linéaires d'ordre variable avec des retards finis et infinis. Contrairement aux techniques adoptées dans la littérature, qui étaient centrées sur l'utilisation du concept des intervalles généralisés et l'idée des fonctions constantes par morceaux, notre approche est directe et repose sur un nouvel opérateur fractionnaire qui est plus approprié et démontre la solvabilité et la stabilité du problème principal sous des hypothèses moins restrictives. Tous les résultats sont obtenus en utilisant la théorie du point fixe. Dans tous les chapitres de ce travail, nous avons illustré notre étude théorique par des applications numériques pour approximer la solution à tous nos problèmes proposés, et nous avons utilisé différentes méthodes. L'une de ces méthodes est la méthode des différences finies. La méthode des différences finies est une technique numérique utilisée pour résoudre des équations différentielles en approchant les dérivées par des différences finies. Elle est largement utilisée en physique, en ingénierie, et dans d'autres domaines où des équations différentielles doivent etre résolues numériquement. Essentiellement, elle discrétise l'espace et le temps en petits intervalles et approxime les dérivées par des différences finies pour obtenir une solution numérique.

La deuxième méthode est la méthode de discrétisation d'Euler, également appelée méthode d'Euler, qui est l'une des méthodes numériques les plus simples pour résoudre des équations différentielles ordinaires (EDO). Elle appartient à la famille des méthodes de différences finies et s'applique aux équations du premier ordre. L'idée de base de cette méthode est d'approximer la solution d'une équation différentielle pas à pas, en utilisant un intervalle de temps et en partant d'une condition initiale. Dans ce travail, des comparaisons ont été effectuées entre ces deux méthodes pour confirmer les résultats théoriques trouvés dans cette thèse.

Motes clés: Dérivées et intégrales d'ordre variable, Théorème du point fixe, Problème de type Cauchy, équations différentielles fonctionnelles avec retard infini, équations différentielles fonctionnelles avec retard fini, Stabilité d'Ulam-Hyers, méthodes numériques.

AMS (MOS) Subject Classifications: 26A33, 34A08, 34A37, 34A60.

الملخص

في هذه ألأطروحة نستخدم نهجًا جديدًا لدراسة وجود الحلول أحاديتها واستقرارها لمسألة من نوع كوشي تتعلق بالمعادلات التفاضلية الكسرية غير الخطية ذات المجهول المتغيرة مع تأخير نهائي ولانهائي. على عكس التقنيات المتبعة في المألوف التي كانت تركز على استخدام مفهوم الفواصل المعممة وفكرة الدوال الثابتة المقطعية، فإن نهجنا مباشر ويستند إلى عامل كسري جديد يتناسب أكثر ويثبت قابلية الحل واستقرار المسألة الرئيسية تحت فرضيات أقل تقييدًا. تم تحقيق جميع النتائج باستخدام نظرية النقطة الثابتة .

في جميع فصول هذا العمل، قمنا بتوضيح دراستنا النظرية بتطبيقات عددية لتقريب الحل لجميع المسائل التي اقترحناها، واستخدمنا طرقًا مختلفة. إحدى هذه الطرق هي طريقة الفرواق المنتهية. تُعد طريقة الفرواق المنتهية تقنية عددية تُستخدم لحل المعادلات التفاضلية عن طريق تقريب المشتقات باستخدام الفروق المنتهية. تُستخدم على نطاق واسع في الفيزياء والهندسة وغيرها من المجالات التي تحتاج فيها المعادلات التفاضلية إلى حل عددي.

بشكل أساسي، تقوم بتقسيم المجال والزمن إلى فواصل صغيرة وتقوم بتقريب المشتقات باستخدام الفروق المنتهية للحصول على حل عددي.

الطريقة الثانية هي طريقة تباعد أولر، والمعروفة أيضًا باسم طريقة أولر، وهي واحدة من أبسط الطرق العددية لحل المعادلات التفاضلية العادية. تنتمي إلى عائلة طرق الفروق المنتهية وتطبق على المعادلات من الدرجة الأولى. الفكرة الأساسية لهذه الطريقة هي تقريب حل المعادلة التفاضلية خطوة بخطوة، باستخدام فترة زمنية والبدء من شرط أولي.

في هذا العمل، تم إجراء مقارنات بين هاتين الطريقتين لتأكيد النتائج النظرية التي تم التوصل إليها في هذه الأطروحة.

الكلمات المفتاحية: المشتقات والتكاملات ذات المجهول المتغيرة، نظرية النقطة الثابتة، مسألة من نوع كوشي، المعادلات التفاضلية الدالية ذات التأخير اللامتناهي، المعادلات التفاضلية الدالية ذات التأخير المحدود، استقرار أولام-هيبرس، الحل العددي.

التصنيفات : 26أ33 ، 34أ08 ، 34أ37 ، 34أ60

Publications and Communications of the thesis

International Publications

1 **M. Benaouda**, S. Sabit, H. Gunerhan, M. S. Souid, Modern Technique to Study Cauchy-Type Problem of Fractional Variable Order, *Journal of Modern Physics Letters B*, **2024**,(2024).

2 **S. Sabit**, M. S. Souid, M. Benaouda, A Novel Fractional Operator Approach to Cauchy-type Problem Existence, Uniqueness, and Stability, *Journal of Studies in Science of Science* , **ISSN: 1003-2053**,(2025).

3 **S. Sabit**, M. S. Souid, M. Benaouda, K. Sudar Mozhi and J. Kiddy K. Asamoah. Fixed Point Analysis for Cauchy-Type Variable-Order Fractional Differential Equations with Finite Delay. (**submitted**).

4 **M. S. Souid**, S. Sabit, M. Benaouda, M. Bensaid and Z. Hammouch, Application of Kuratowski Measure of Noncompactness to Cauchy Problems in Variable Order Fractional Differential Equations,(**submitted**).

International Communications

1 **M. Benaouda**, Boundary value problem for implicit nonlinear fractional differential equations of varaible involving hadamard derivative , *The first International Workshop on Applied Mathematics 1st- IWAM'2022,6-8 December, 2022 Constantine, Algeria*, (Poster).

2 **M. Benaouda**, Existence results for fractional varaible order problem with finite delay

via U.Hyers stability, *The first International confrence on Knowledge data management, ICKDM'2023 11-13 December 2023, Tiaret, Algeria.*

National Communications

1 **M. Benaouda**, Stability of solutions to the value problem for nonlinear implicit fractional differential equations of variable order via Hadamard derivative , Journée Pi Day ,14 mars 2023, Sidi-Belabbas, Algeria, (Poster).

Contents

General introduction

The development of fractional calculus which is a branch of classical calculus, can be traced back to early attempts to extend the concepts of differentiation and integration to non-integer orders in the 17th century. Leibniz and L'Hôpital were among the mathematicians who first explored these concepts, but it wasn't until the 19th century that Augustin-Louis Cauchy and Liouville made significant adsances that the theory of fractional derivatives and integrals was formalized [22, 44, 56]. Fractional calculus has been used in a variety of fields over time. It is used in engineering to simulate intricate systems incorporating electrical circuits, control theory, and viscoelasticity. It is essential to the description of processes in physics like diffusion, wave propagation, and fractional quantum mechanics. Fractional calculus also provides useful tools for deciphering non-Markovian processes and irregular data patterns in biology, finance, and signal processing. Fractional calculus is an essential component of contemporary mathematics and applied sciences due to its versatility [1, 4, 40, 43]. Recent years have seen a huge increase in the number of research publications that examine various qualitative aspects of differential equations while also involving various fractional operators; see the papers [3, 39, 41, 42] for more details. In all these contributions, the fractional operators of constant order were taken into consideration and the conclusions were reached using the appropriate hypotheses.

Variable-order fractional operators have just recently been conceptualized and formally formalized. Variable-order differentiations and integrations are a logical progression from their counterpart in real order. In this situation, the order can continuously change depending on dependent or independent variables of differentiation or integration. The mentioned extension in order is more flexible than the conventional fractional order and is a natural progression [21, 25, 73]. These operators have been successfully used to represent complicated real-world problems in a variety of fields, including biology, mechanics, control theory, and transport systems. This is due to the capability of developing evolutionary governing equations. Due to this widespread area of applications, the scientific community has been actively researching variable order fractional applications to the modeling of engineering and physical systems; see for instance these two old papers [33, 65, 68].

The essential concept behind the extension of constant order fractional calculus to variable-order (sOR) fractional calculus is the substitution of the constant τ from the constant order fractional calculus with the function $\tau(\cdot)$. Though this distinction might seem inconsequential, the (sOR) operator can offer a better understanding of a variety of physical and natural events. The topic of discussion is the concept of variable order. Fractional differential equations are a flexible expansion of the traditional fractional calculus, wherein the degree of differentiation or integration is variable in relation to the independent variable or other parameters. The deviation from a constant fractional order in representing complicated dynamic systems with non-uniform behaviors gives a greater level of flexibility, hence enabling a more precise representation. The notion of variable-order fractional derivatives has its origins in the early 20th century and has since garnered considerable interest in contemporary times owing to its wide-ranging applicability across several fields. The equations presented in this context consider the fractional order as a function or as depending on other system characteristics, allowing for a more accurate depiction of real-world occurrences. Researchers from a variety of fields have used this strategy, including those in physics, biology, economics, engineering, and control theory [7, 8, 9].

The utilization of variable-order fractional differential equations encompasses a wide range of applications. The equations utilized in the field of physics are employed to elucidate the characteristics of materials that exhibit dynamic features, such as porous media or viscoelastic materials. In the field of biology, computational models are employed to simulate and analyze many biological phenomena, such as the distribution of drugs through tissues or the activity of neurons. Fractional Diffrential Equations of Variable-order are employed by economists and finance experts for the purpose of modeling intricate market dynamics and asset pricing. Furthermore, control engineers employ these methodologies to analyze and regulate systems exhibiting diverse dynamics, thereby enhancing the precision and efficiency of control procedures. In general, variable-order fractional differential equations provide a robust foundation for improving modeling and analysis in various fields, rendering them a subject of ongoing research and practical implementation in modern scientific and technological progress [14, 16, 17].

Recent research in this area has been particularly done by many researchers who focused on the study of the existence, uniqueness, and stability of solutions (Exi, uniq and stab of sol) to many different problems of Fractional Diffrential Equations of Variable-order under different conditions see Souid et al [6, 10, 11, 12, 13, 15, 18, 37, 53, 55, 64, 66, 67, 69]. The measure of non-compactness technique, the upper-lower solutions method, the continuation theory, and the techniques based on (Fix pt) are the foundations upon which all of the above-mentioned results are proved. Further, the stability of the proposed problems in the sense of Ulam-Hyers (Ula Hyer) or Ulam-Hyers-Rassias (Ula Hyer Ras) was under observation [19, 36, 54, 61]. It is important to note that the investigation relies heavily on the concept of piece-wise constant function (piece-wise constant funct) which plays a crucial role. The majority of the aforementioned results are obtained using this approach, which first divides the existence interval into subintervals and then defines the differential and integral operators with respect to those subintervals. Using this technique, researchers were able to convert the fractional problems of constant order into their equivalent conventional fractional problems of (constant order).

In this thesis we introduce a novel approach to replace the use of the piece-wise constant funct and existence interval splitting. The creation of a new operator that is more adaptable and doesn't need any additional phases is the keystone of our strategy.

In the following we give an outline of our thesis organization, consisting of **4 chapters**.

The **first chapter** gives some notations, definitions, lemmas, fixed point theorems and coincidence degree theory which are used throughout this thesis.

In Chapter 2, we study the existence of solutions to the proposed multiterm Cauchy-type problem (IVP) for the nonlinear fractional differential equation of variable order in the format

$$\begin{cases} D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(\varphi, w(\varphi)), \ \ \varphi \in D = [0, \sigma] \\ w(0) = 0, \end{cases} \tag{1}$$

where $0 < \sigma < +\infty$, $0 < \wp(\varphi) \le 1$, $\Upsilon : D \times \mathbb{R} \to \mathbb{R}$ is a continuous functions **Cont funct** and $D_{0^+}^{\wp(\varphi)}$, is the Riemann-Liouville fractional derivative of variable-order $\wp(\varphi)$. Using the new technique, we examine the stability and solvability of solutions for (1).

In Chapter 3, we study existence, uniqueness and stability of solutions**(Exi, uniq and stab of sol)** to the cauchy-type with finite delay problem of nonlinear fractional differential equations of variable order (CFDPNFDEVO(2))

$$\begin{cases} D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(\varphi, w(\varphi)), \ \ \varphi \in D := [0, \sigma] \\ w(\varphi) = \eta(\varphi), \quad \varphi \in [-r, 0] \end{cases} \tag{CFDPNFDEVO(2)}$$

where $0 < \sigma < +\infty$, $0 < \wp(\varphi) \le 1$, $\Upsilon : D \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ is a continuous functions **Cont funct** and $D_{0^+}^{\wp(\varphi)}$, is the Riemann Liouville fractional derivative of variable- order

(R-LFDVO) $\wp(\varphi)$ and $\eta(\varphi) \in C([-r,0],\mathbb{R})$ with $\eta(0)=0$.

For any function w defined on $[-r,\sigma]$ and any $\varphi \in D$, we denote by w_φ the element of $C([-r,0],\mathbb{R})$ defined by

$$w_\varphi(\jmath) = w(\varphi + \jmath), \quad \jmath \in [-r,0]$$

In Chapter 4, we investigate the existence of solutions for to the cauchy-type with infinite delay problem of nonlinear fractional differential equations of variable order (CPNFDEVOID(3)) as follows:

$$\begin{cases} D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(\varphi, w(\varphi)) \;\; \varphi \in D = [0,\sigma] \\ \\ w(\varphi) = \eta(\varphi), \quad \varphi \in]-\infty, 0] \end{cases} \tag{CPNFDEVOID(3)}$$

where $0 < \sigma < +\infty$, $0 < \wp(\varphi) \leq 1$, $\Upsilon : D \times \mathfrak{B} \to \mathbb{R}$ is a Continuous Functions **Cont funct** and $D_{0^+}^{\wp(\varphi)}$, is the Riemann Liouville Fractional Derivative of Variable-Order $\wp(\varphi)$, $0 < \wp(\varphi) \leq \wp^* \leq 1$ and $\eta(\varphi) \in \mathfrak{B}$ with $\eta(0) = 0$ and $\mathfrak{B}$ is **Phase sp**.

For each function w defined on $]-\infty,\sigma]$ and each $\varphi \in D$, we note by w_φ the element of $\mathfrak{B}$ defined by

$$w_\varphi(\jmath) = w(\varphi + \jmath), \quad \jmath \in]-\infty, 0]$$

Here $w_\varphi(.)$ represents the history of the state from time ∞ up to the present time φ.

Chapter 1

Preliminary

This chapter introduces some important fundamental definitions, fixed point theoremes which are used throughout this thesis.

1.1 Notations and definitions

The symbol $\Psi = C(D, \mathbb{R}))$ represents the Banach space of continuous functions from D into $\mathbb{R}$ with the norm

$$\|w\|_{\Psi} = \sup_{\varphi \in D} |w(\varphi)|.$$

1.2 Fractional calculus.

1.2.1 Fractional calculus of constant-order

Definition 1.2.1 *([43, 51]). The left Riemann-Liouville fractional integral of the function $h \in L^1([a,b], \mathbb{R}_+)$ of order $\wp \in \mathbb{R}+$ is defined by*

$$I_a^{\wp} h(t) = \frac{1}{\Gamma(\wp)} \int_a^{\varphi} (\varphi - s)^{\wp - 1} h(s) ds,$$

where $\Gamma(.)$ is the gamma function.

Definition 1.2.2 *([43, 51]). The left Riemann-Liouville fractional derivative of order $\wp > 0$ of function $h \in L^1([a,b], \mathbb{R}_+)$, is given by*

$$(D_{a+}^{\wp} h)(\varphi) = \frac{1}{\Gamma(n - \wp)} \Big(\frac{d}{d\varphi}\Big)^n \int_a^{\varphi} (\varphi - s)^{n - \wp - 1} h(s) ds,$$

here $n=[\wp]+1$ and $[\wp]$ denotes the integer part of $\wp$. If $\wp\in(0,1]$, then

$$(D^{\wp}_{a+}h)(\varphi)=\frac{d}{d\varphi}I^{1-\wp}_{a+}h(\varphi)=\frac{1}{\Gamma(1-\wp)}\frac{d}{ds}\int_a^{\varphi}(\varphi-s)^{-\wp}h(s)ds.$$

The following properties are some of the main ones of the fractional derivatives and integrals.

Lemma 1.2.1 *([43]). Let $\wp>0$, $a\geq 0$, $h\in L^1(a,b)$, $D^{\wp}_{a+}h\in L^1(a,b)$. Then,*

$$D^{\wp}_{a^+}I^{\wp}_{a^+}h(\varphi)=h(\varphi).$$

Lemma 1.2.2 *([43]). Let $\wp,\ \varpi>0$, $a\geq 0$, $h\in L^1(a,b)$. Then,*

$$I^{\wp}_{a^+}I^{\varpi}_{a^+}h(\varphi)=I^{\varpi}_{a^+}I^{\wp}_{a^+}h(\varphi)=I^{\wp+\varpi}_{a^+}h(\varphi).$$

1.2.2 Fractional calculus of variable-order

Definition 1.2.3 *([58], [59], [68]) Let $-\infty<\nu_1<\nu_2<+\infty$, and $\wp(\varphi):[\nu_1,\nu_2]\to(0,+\infty)$, the left **RLFIVO** for function $F(\varphi)$ is defined by*

$$I^{\wp(\varphi)}_{\nu_1^+}F(\varphi)=\int_{\nu_1}^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}F(s)ds,\ \ \varphi>\nu_1, \tag{1.1}$$

Definition 1.2.4 *([58], [59], [68]) Let $-\infty<\nu_1<\nu_2<+\infty$, $n\in\mathbb{N}$ and $\wp(\varphi):[\nu_1,\nu_2]\to(n-1,n)$, the left Riem-Liov-frac-deriv-var-oder for function $F(\varphi)$ is defined by*

$$D^{\wp(\varphi)}_{\nu_1^+}F(\varphi)=\left(\frac{d}{d\varphi}\right)^n I^{n-\wp(\varphi)}_{\nu_1^+}F(\varphi)=\left(\frac{d}{d\varphi}\right)^n\int_{\nu_1}^{\varphi}\frac{(\varphi-s)^{n-\wp(\varphi)-1}}{\Gamma(n-\wp(\varphi))}F(s)ds,\ \ \varphi>\nu_1. \tag{1.2}$$

We notice that, if the order $\wp(\varphi)$ is a constant function $\wp$, then the Riemann-Liouville fractional derivative of varaible order (1.2) and Riemann-Liouville fractional integral of varaible order (1.1) are the usual Riemann-Liouville fractional derivative and integral, respectively(see [43, 58, 59]).

Remark 1.2.1 *For arbitrary functions $\wp(\varphi)$, $\varrho(\varphi)$, we notice that the semigroup property doesn't hold, i.e:*

$$I^{\wp(\varphi)}_{\nu_1^+}I^{\varrho(\varphi)}_{\nu_1^+}F(\varphi)\neq I^{\wp(\varphi)+\varrho(\varphi)}_{\nu_1^+}F(\varphi).$$

The above identity was very well proved and justified in the literature see ([10], [11], [12]).
Example: Let

$$u(\varphi)=\begin{cases}2, & \varphi\in[0,1],\\ 1, & \varphi\in]1,3],\end{cases}\qquad v(\varphi)=\begin{cases}1, & \varphi\in[0,1],\\ 2, & \varphi\in]1,3],\end{cases}$$

and $\quad h(\varphi)=\varphi, \quad \varphi\in[0,3].$

$$\begin{aligned}
I_{0^+}^{u(\varphi)}I_{0^+}^{v(\varphi)}h(\varphi) &= \int_0^1 \frac{(\varphi-s)^{u(s)-1}}{\Gamma(u(s))}\int_0^s \frac{(s-\tau)^{v(\tau)-1}}{\Gamma(v(\tau))}h(\tau)d\tau ds \\
&+ \int_1^{\varphi} \frac{(\varphi-s)^{u(s)-1}}{\Gamma(u(s))}\int_0^s \frac{(s-\tau)^{v(\tau)-1}}{\Gamma(v(\tau))}h(\tau)d\tau ds, \\
&= \int_0^1 \frac{(\varphi-s)^{1}}{\Gamma(2)}\int_0^s \frac{(s-\tau)^{0}}{\Gamma(1)}\tau d\tau ds \\
&+ \int_1^{\varphi} \frac{(\varphi-s)^{0}}{\Gamma(1)}[\int_0^1 \frac{(s-\tau)^{0}}{\Gamma(1)}\tau d\tau + \int_1^s \frac{(s-\tau)^{1}}{\Gamma(2)}\tau d\tau] ds, \\
&= \int_0^1 \frac{(\varphi-s)s^2}{2\Gamma(2)}ds + \int_1^{\varphi}\left(\frac{s^3}{6}-\frac{s}{2}+\frac{5}{6}\right)ds,
\end{aligned}$$

$$I_{0^+}^{u(\varphi)+v(\varphi)}h(\varphi) = \int_0^{\varphi}\frac{(\varphi-s)^{u(s)+v(s)-1}}{\Gamma(u(s)+v(s))}h(s)ds,$$

we see that

$$\begin{aligned}
I_{0^+}^{u(\varphi)}I_{a^+}^{v(\varphi)}h(\varphi)|_{\varphi=2} &= \int_0^1 \frac{(2-s)s^2}{2\Gamma(2)}ds + \int_1^2\left(\frac{s^3}{6}-\frac{s}{2}+\frac{5}{6}\right)ds, \\
&= \frac{5}{24}+\frac{17}{24}=\frac{22}{24},
\end{aligned}$$

$$\begin{aligned}
I_{0^+}^{u(\varphi)+v(\varphi)}h(\varphi)|_{\varphi=2} &= \int_0^1 \frac{(2-s)^{2+1-1}}{\Gamma(2+1)}sds + \int_1^2 \frac{(2-s)^{1+2-1}}{\Gamma(1+2)}sds \\
&= \frac{11}{24}+\frac{5}{24}=\frac{16}{24}.
\end{aligned}$$

Therefore, we obtain

$$I_{0^+}^{u(\varphi)}I_{0^+}^{v(\varphi)}h(\varphi)|_{\varphi=2} \neq I_{0^+}^{u(\varphi)+v(\varphi)}h(\varphi)|_{\varphi=2}.$$

Lemma 1.2.3 *([74]) Let $\wp : D \to (0,1]$ be a **Cont funct**, then for $w\in C_\iota(D,\mathbb{R}) = w(\varphi)\in\Psi,\ \varphi^\iota w(\varphi)\in\Psi,\ (0\le\iota\le1)$ and for each points on D the $I_{0^+}^{\wp(\varphi)}w(\varphi)$ exists.*

Lemma 1.2.4 *([74]) Let $\wp : D \to (0,1]$ be a Cont funct, then*

$$I_{0^+}^{\wp(\varphi)}w(\varphi)\in\Psi \ \textit{for}\ w\in\Psi.$$

Remark 1.2.2 *[74] : As $\wp(\varphi)$ is **Cont funct** , for $0\le s\le\varphi\le\sigma$ we let $\wp_* = \min_{0\le\varphi\le\sigma}|\wp(\varphi)|$, then we get*

if

$$0\le\sigma\le1, \quad \textit{then}\ \sigma^{\wp(s)-1}\le\sigma^{\wp_*-1},$$

if

$$1 \leq \sigma \leq \infty, \ then \ \sigma^{\wp(s)-1} \leq 1,$$

thus for $-\infty \leq \sigma \leq +\infty$, *we know*

$$\sigma^{\wp(s)-1} \leq max\{1, \sigma^{\wp_*-1}\} = \sigma^*.$$

1.3 Phase Space

The notion of the phase space $\mathfrak{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [34] (see also Kappel and Schappacher [38] and Schumacher [62]). For a detailed discussion on this topic we refer the reader to the book by Hino et al [35].

Fractional differential equations have been of great interest recently. In cause, in part to both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc. For details, see the monographs of Miller and Ross [47], Podlubny [51] and Samko et al [60], and the papers of Delboso and Rodino [24], Diethelm et al ([27] , [26] , [28]), Gaul et al [30], Glockle and Nonnenmacher [31], Mainardi [45], Metzler et al [46], Momani and Hadid [48], Momani et al [49], Podlubny et al [52], Yu and Gao [71] and the references therein.

Our approach is based on the Banach fixed point theorem and on the nonlinear alternative of Leray-Schauder type [32]. These results can be considered as a contribution to this emerging field.

In this thesis, we take on that the state space $(\mathfrak{B}, \|.\|_\sigma)$ is a **Seminormed line of funct map** $(-\infty, 0]$ into $\mathbb{R}$, and check the fundamental axioms of Hale and kate given in [34].

$(\mathcal{E})$ If $w : (-\infty, \nu_2] \to \mathbb{R}$, and $w_0 \in \mathfrak{B}$, then $\forall \varphi \in D$ the following conditions are satisfied:

- **(i)** w_φ is in $\mathfrak{B}$.
- **(ii)** $\|w_\varphi\|_{\mathfrak{B}} \leq \kappa(\varphi) sup\{|w(s)| : 0 \leq s \leq \varphi\} + L(\varphi)\|w_0\|_{\mathfrak{B}}$,
- **(iii)** $|w(\varphi)| \leq T\|w_\varphi\|_{\mathfrak{B}}$,

  where the constant $T \geq 0$, the **Cont funct** $\kappa : D \to [0, \infty)$, the locally bounded L $[0, \infty) \to [0, \infty)$, the κ, T, L are independent of $w(.)$.

($\mathcal{E}$-1) For the function $w(.)$ in ($\mathcal{E}$), w_φ is a $\mathfrak{B}$-valued **Cont funct** on D.

($\mathcal{E}$-2) The space $\mathfrak{B}$ is complete.

1.3.1 Examples of Phase Spaces

In this section, we present some examples of phase spaces.

Example 1.3.1 *The space C_γ.*
For any real constant γ, we define the functional space C_γ by

$$C_\gamma = \{\eta \in C((-\infty, 0], \Psi)\ , lim_{\jmath \to -\infty}\ \eta(\jmath)\ existe\ in\ \Psi\}$$

endowed with the following norm

$$\|\eta\| = sup\{e^{\gamma\jmath}|\eta(\jmath)|; \jmath < 0\}.$$

Then in the space C_γ (see [35]) the axioms ($\mathcal{E}$) - ($\mathcal{E}$-2) are satisfied.

Example 1.3.2 *The spaces BC, BUC , C^∞ and C^0. Let*
BC the space of bounded continuous functions defined from $(-\infty, 0]$ to Ψ
BUC the space of bounded uniformly continous functions defined from $(-\infty, 0]$ to Ψ

$C^\infty := \{\eta \in BC : lim_{\jmath \to -\infty}\eta(\jmath)\ existe\ in\ \Psi\}$

$C^0 := \{\eta \in BC : lim_{\jmath \to -\infty}\eta(\jmath) = 0\}$, *endowed with the uniform norm*

$$\|\eta\| = sup\{\eta(\jmath) : \jmath \leq 0\}.$$

We have ([35]) that the spaces BUC, C^∞ and C^0 satisfy conditions ($\mathcal{E}$) - ($\mathcal{E}$-2). BC satisfies ($\mathcal{E}$-1) - ($\mathcal{E}$-2) but ($\mathcal{E}$) is not satisfied.

Example 1.3.3 *The spaces C_g, UC_g, C_g^∞ and C_g^0.*

Let g be a positive continuous function on $(-\infty, 0]$. We define

$C_g := \{\eta \in C((-\infty, 0], \Psi) : \frac{\eta(\jmath)}{g(\jmath)}$ *is bounded on* $(-\infty, 0]\}$

$C_g^0 := \{\eta \in C_g : lim_{\jmath \to -\infty} \frac{\eta(\jmath)}{g(\jmath)} = 0\}$, *endowed with the uniform norm*

$$\|\eta\| = sup\{\frac{\eta(\jmath)}{g(\jmath)} : \jmath \leq 0\}.$$

Consider the following condition on the function g.

(g_1) *For all* $a > 0$, $sup_{0 \leq \varphi \leq a}\ sup\left\{\frac{g(\varphi+\jmath)}{g(\jmath)} : -\infty \leq \jmath \leq -\varphi\right\} < \infty$.

Then we have ([35]) that the spaces C_g *and* C_g^0 *satisfy conditions (*$\mathcal{E}$*-2). They satisfy conditions (*$\mathcal{E}$*) and (*$\mathcal{E}$*-1) if* g_1 *holds.*

Example 1.3.4 *The product space* $C_r \times L_\gamma^p$.

Let γ *be a real number,* $1 \leq p \leq +\infty$ *and* $r > 0$*, we denote* $C_r \times L_\gamma^p$ *the space of measurable functions* $\eta : (-\infty, 0] \to \Psi$ *which are continuous on* $[-r, 0]$ *such that* $e^{\gamma\jmath}|\eta(\jmath)|^p$ *is integrable on* $(-\infty, 0]$*. We endowed* $C_r \times L_\gamma^p$ *with the following norm*

$$\|\eta(\jmath)\| = sup\{|\eta(\jmath)| : -r \leq \jmath \leq 0\} + \int_{-\infty}^{-r} e^{\gamma\jmath}|\eta(\jmath)|^p d\jmath.$$

Then $C_r \times L_\gamma^p, \|.\|)$ *is a normed space satisfying the axioms (*$\mathcal{E}$*) - (*$\mathcal{E}$*-2).*

1.4 Some fixed point theorems

Theorem 1.4.1 *(**Banach contr princip** [32]). Let* C *be a non-empty closed subset of a **Banach sp*** Ψ*, then any contraction mapping* V *of* C *into itself has a **Fix pt**.*

Theorem 1.4.2 *(**Schauder fix pt thm** [23]). Let* Ψ *a **Banach sp** and* Q *be a convex, closed bounded non-empty subset of* Ψ *and* $V : Q \longrightarrow Q$ *is **Compl cont**. Then* V *has at least one **Fix pt** in* Q.

Theorem 1.4.3 *(**Alt nonlinear L-S thm**) ([43]) Let* Ψ *a **Banach sp** and* Q *be a convex, closed bounded non-empty of* Ψ *and* $V \subset Q$ *an open and such that* $0 \in V$ *. Assume that* $\Phi : V \to Q$ *is **Compl cont**. If* $\Phi(V)$ *is **Relat comp** then, either*

(i) : Φ *has a **Fix pt**, or*

(ii) : there is a point $u \in \partial V$ *and* $\lambda \in (0; 1)$ *with* $u = \lambda\Phi u$.

Lemma 1.4.1 *Let $\phi : [0,\sigma] \to [0,\infty)$ be a* ***Real funct*** *and $\psi(.)$ is a nonnegative,* ***Locally int funct*** *on $[0,\sigma]$ and there are constants $\gamma > 0$ and $0 < \wp(\varphi) \leq \wp^* \leq 1$ such that*

$$\phi(\varphi) \leq \psi(\varphi) + \gamma \int_0^\varphi \frac{\phi(s)}{(\varphi - s)^{\wp(s)-1}} ds,$$

then there exists a constant $\kappa = \kappa(\wp^*)$ such that

$$\phi(\varphi) \leq \psi(\varphi) + \kappa\gamma \int_0^\varphi \frac{\psi(s)}{(\varphi - s)^{\wp(s)-1}} ds,$$

for every $\varphi \in [0,\sigma]$.

1.5 Types of stability

Theorem 1.5.1 *: The (CPNFDEVOID(3)) is* ***Ula Hyer Stab*** *if there exists $c_\Upsilon > 0$, such that for each $\varepsilon > 0$ and for every solution $\chi \in \Psi$ of the following inequality*

$$|D_{0^+}^{\wp(\varphi)} \chi(\varphi) - \Upsilon(\varphi, \chi(\varphi))| < \varepsilon, \quad \varphi \in D$$

there exists a solution $w \in \Psi$ of (CPNFDEVOID(3))

$$|\chi(\varphi) - w(\varphi)| < c_\Upsilon \varepsilon, \quad \varphi \in D$$

1.6 Finite Difference Method

1.6.1 Finite Difference Method Principle

The principle of this method is outlined in the following lines: We assume that the interval $[a,b]$ is subdivided into n subintervals $[\varphi_k, \varphi_{k+1}]$ of length $h = \frac{b-a}{n}$, using equally spaced nodes $\varphi_k = a + kh$ for $k = 0, 1, \ldots, n$ [20, 57]. The composite trapezoidal rule for n subintervals allows us to write:

$$T(f,h) = \frac{h}{2} \sum_{k=1}^{n} \left[f(\varphi_{k-1}) + f(\varphi_k) \right], \tag{1.3}$$

and

$$\int_a^b f(\varphi) d\varphi \approx T(f,h). \tag{1.4}$$

1.6.2 Approximation of the Fractional Derivative in the Riemann-Liouville Sense Using the Finite Difference Method

We have the following Cauchy-type problem of fractional variable order

$$\begin{cases} D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(\varphi, w(\varphi)), \ \ \varphi \in D = [0, \sigma] & (1) \\ w(0) = 0, & (2) \end{cases} \tag{1.5}$$

where $0 < \sigma < +\infty$, $0 < \wp(\varphi) \leq 1$, $\Upsilon : D \times \mathbb{R} \to \mathbb{R}$ is a continuous functions **Cont funct** and $D_{0^+}^{\wp(\varphi)}$, is the Riemann-Liouville fractional derivative of variable-order $\wp(\varphi)$.
The solution of this problem is:

$$w(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s)) ds, \qquad \wp(s) > 0.$$

We apply (1.3) and (1.4), we find[57]:

$$w(\varphi_i) = \frac{h}{2} \Big(\sum_{k=1}^{k=i} \frac{(\varphi_i - s_{k-1})^{\wp(s_{k-1})-1}}{\Gamma(\wp(s_{k-1}))} \Upsilon(s_{k-1}, w(s_{k-1})) + \sum_{k=1}^{k=i} \frac{(\varphi_i - s_k)^{\wp(s_k)-1}}{\Gamma(\wp(s_k))} \Upsilon(s_k, w(s_k)) \Big)$$

This further simplifies as:

$$\begin{aligned} w(\varphi_i) &= h \Big(\sum_{k=1}^{k=i-1} \frac{(\varphi_i - s_k)^{\wp(s_k)-1}}{\Gamma(\wp(s_k))} \Upsilon(s_k, w(s_k)) \Big) + \frac{h(\varphi_i)^{\wp(0)-1}}{2\Gamma(\wp(0))} \Upsilon(0, w(0)) \\ &= h \Big(\sum_{k=1}^{k=i-1} \frac{(\varphi_i - s_k)^{\wp(s_k)-1}}{\Gamma(\wp(s_k))} \Upsilon(s_k, w(s_k)) \Big) + \frac{h(\varphi_i)^{\wp(0)-1}}{2\Gamma(\wp(0))} \Upsilon(0, 0), i = 1..n \end{aligned}$$

We try to write the linear part separately and the nonlinear part, then we write the system in matrix form and use methods to solve it.

1.7 Euler's discretization method

The discretization process is introduced to discretize fractional-order differential equations/systems. It has been observed that as the fractional-order parameter approaches one, Euler's discretization method is recovered. This discretization method has been applied to fractional-order versions of the Riccati differential equation and Chua's system [2, 29, 47] . In this context, we are particularly interested in applying the discretization method to the Cauchy-type problem for fractional variable order. Let $0 < \wp(\varphi) \leq 1$, and consider the fractional-order differential equation given by the system (1.5).
The corresponding equation with a piecewise constant argument is

$$D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(r\frac{\varphi}{r}, w(r\frac{\varphi}{r})), \ \ \varphi \in D = [0, \sigma]$$

Let $\varphi \in [0, r]$, then $\frac{\varphi}{r} \in [0, 1]$. We get $D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(0, w(0)), \varphi \in [0, r]$.
Thus $w_1 = w_0 + \frac{\varphi^{\wp(0)}}{\Gamma(1+\wp(0))} \Upsilon(0, w_0)$.
Let $\varphi \in [r, 2r]$, then $\frac{\varphi}{r} \in [1, 2]$. We get $D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(r, w(r)), \varphi \in [r, 2r]$.
Thus $w_2 = w_1 + \frac{(\varphi - r)^{\wp(r)}}{\Gamma(1+\wp(r))} \Upsilon(r, w_1)$.
Let $\varphi \in [2r, 3r]$, then $\frac{\varphi}{r} \in [2, 3]$. We get $D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(2r, w(2r)), \varphi \in [2r, 3r]$.
Thus $w_3 = w_2 + \frac{(\varphi - 2r)^{\wp(2r)}}{\Gamma(1+\wp(2r))} \Upsilon(2r, w_2)$.
Repeating the process, we get when $\varphi \in [nr, (n+1)r]$, then $\frac{\varphi}{r} \in [n, n+1]$.
So we get $D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(nr, w(nr)), \varphi \in [nr, (n+1)r]$.
thus

$$w_{n+1}(\varphi) = w_n(nr) + \frac{(\varphi - nr)^{\wp(nr)}}{\Gamma(1 + \wp(nr))} \Upsilon(nr, w_n(nr)).$$

We can compute this sequence w_n for larger n to obtain a more accurate approximation.

Chapter 2

A Novel Fractional Operator Approach to Cauchy-type Problem Existence, Uniqueness, and Stability

2.1 Introduction

[1]In this chapter , we introduce a novel approach to replace the use of the **piece-wise constant funct** and existence interval splitting. The creation of a new operator that is more adaptable and doesn't need any additional phases is the keystone of our strategy. We apply the new technique on the following Cauchy-type problem of fractional variable order

$$\begin{cases} D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(\varphi, w(\varphi)), \quad \varphi \in D = [0, \sigma] \\ w(0) = 0, \end{cases} \tag{2.1}$$

where $0 < \sigma < +\infty$, $0 < \wp(\varphi) \leq 1$, $\Upsilon : D \times \mathbb{R} \to \mathbb{R}$ is a continuous functions **Cont funct** and $D_{0^+}^{\wp(\varphi)}$, is the Riemann-Liouville fractional derivative of variable-order $\wp(\varphi)$. Using the new technique, we examine the stability and solvability of solutions for (2.1). We assert that the method used is original and hasn't been used in any earlier publications.

2.2 Existence and uniqueness of solutions

Throughout the remainder portion of the study, the following assumptions are made available.

(H1) $\wp : [0, \sigma] \to (0, \wp^*]$ is Cont funct, such that $\frac{1}{2} < \wp(\varphi) \leq \wp^* \leq 1$.

[1]**S. Sabit**, M. S. Souid,M. Benaouda, *Journal of Studies in Science of Science* , **ISSN: 1003-2053**,(2025).

(H2) Let $\varphi^{\iota}\Upsilon : D \times \mathbb{R} \to \mathbb{R}$ is Cont funct $(0 \leq \iota \leq 1)$. Then, there exist a constant, $\ell > 0$, such that

$$\varphi^{\iota}|\Upsilon(\varphi, x_1) - \Upsilon(\varphi, x_2)| \leq \ell|x_1 - x_2| \ \textit{for any}\ x_1, x_2, \in \mathbb{R} \ \textit{and}\ \varphi \in D.$$

Definition 2.2.1 *A function $w \in C(D, \mathbb{R})$ is said to be a solution for (2.1) if and only if it verifies (2.1) simultaneously.*

For the existence of solutions for the (2.1), an auxiliary lemma is needed as follows:

Lemma 2.2.1 *[72] The function $w \in \Psi$ is a solution of (2.1) if and only if it satisfies the integral equation*

$$w(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s))ds, \qquad \wp(s) > 0.$$

The first outcome validates the existence of the solutions found using **Schauder fix pt thm**.

Theorem 2.2.1 *Let conditions $(H1)$ and $(H2)$ hold. If*

$$\frac{\sigma^*\sigma^{1-\wp_*}\Upsilon^*\Gamma(1-\iota)\sigma^{\wp_*-\iota}\varphi^{\iota}}{(\iota+1)\Big(\Gamma(1-\iota+\wp^*) - \sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}\Big)} < 1, \tag{2.2}$$

then the (2.1) has at least one solution on Ψ.

Proof 2.2.1 *Consider the operator $\Im : \Psi \to \Psi$, defined by*

$$\Im(w)(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s))ds. \tag{2.3}$$

We consider the set

$$B_{R_\iota} = \{w \in \Psi, \|w\|_{\Psi} \leq R_\iota\},$$

where

$$R_\iota = \frac{\sigma^*\sigma^{1-\wp_*}\Upsilon^*\Gamma(1-\iota)\sigma^{\wp_*-\iota}\varphi^{\iota}}{(\iota+1)\Big(\Gamma(1-\iota+\wp^*) - \sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}\Big)}.$$

Clearly B_{R_ι} is ***Convex, closed bounded non-empty****.*

The proof will be presented in three steps following this.

Step 1: $\Im$ is ***Cont ope****.*

Let w_n be a sequence such that $w_n \to w$ in Ψ then

$$\|(\Im w_n) - (\Im w)\|_\Psi \to 0.$$

For $\varphi \in D$, we have

$$|\Im(w_n)(\varphi) - \Im(w)(\varphi)| = \Big|\int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s, w_n(s))ds - \int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s, w(s))ds\Big|.$$

It follows that

$$\begin{aligned}
|\Im(w_n)(\varphi) - \Im(w)(\varphi)| &\leq \frac{1}{\Gamma(\wp^*)}\int_0^\varphi (\varphi - s)^{\wp(s)-1}|\Upsilon(s, w_n(s)) - \Upsilon(s, w(s))|ds \\
&\leq \frac{\ell}{\Gamma(\wp^*)}\int_0^\varphi \sigma^{\wp_*-1}\left(\frac{(\varphi - s)}{\sigma}\right)^{\wp_*-1} s^{-\iota}|w_n(s) - w(s)|ds \\
&\leq \frac{\ell\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)}\|w_n - w\|_\Psi \int_0^\varphi (\varphi - s)^{\wp_*-1}s^{-\iota}ds \\
&\leq \frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)\Gamma(\wp^*)}\sigma^{\wp_*-\iota}\|w_n - w\|_\Psi.
\end{aligned}$$

Therefore, we have

$$\|\Im(w_n) - \Im(w)\| \leq \frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|w_n - w\|_\Psi,$$

which implies that

$$\|(\Im w_n) - (\Im w)\|_\Psi \to 0 \quad as \quad n \to \infty.$$

Consequently, $\Im$ is ***Cont ope*** *on Ψ.*

Step 2: $\Im(B_{R_\iota}) \subseteq (B_{R_\iota})$.

For $w \in B_{R_\iota}$, *and by* $(H2)$, *we get*

$$
\begin{aligned}
|\Im(w)(\varphi)| &= \Big|\int_0^\varphi \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,w(s))ds\Big| \\
&\le \frac{1}{\Gamma(\wp^*)}\int_0^\varphi \sigma^{\wp_*-1}\left(\frac{(\varphi-s)}{\sigma}\right)^{\wp_*-1}|\Upsilon(s,w(s))|ds \\
&\le \frac{\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)}\int_0^\varphi (\varphi-s)^{\wp_*-1}|\Upsilon(s,w(s))-\Upsilon(s,0)+\Upsilon(s,0)|ds \\
&\le \frac{\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)}\int_0^\varphi (\varphi-s)^{\wp_*-1}\Big(|\Upsilon(s,w(s))-\Upsilon(s,0)|+|\Upsilon(s,0)|\Big)ds \\
&\le \frac{\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)}\int_0^\varphi (\varphi-s)^{\wp_*-1}|\Upsilon(s,w(s))-\Upsilon(s,0)|ds \\
&+ \frac{\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)}\int_0^\varphi (\varphi-s)^{\wp_*-1}|\Upsilon(s,0)|ds \\
&\le \frac{\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)}\int_0^\varphi (\varphi-s)^{\wp_*-1}s^{-\iota}|w(s)|ds + \frac{\sigma^*\sigma^{1-\wp_*}\Upsilon^*}{\Gamma(\wp^*)}\int_0^\varphi s^{-\iota}(\varphi-s)^{\wp_*-1}s^\iota ds \\
&\le \frac{\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)}\|w\|_\Psi\int_0^\varphi (\varphi-s)^{\wp_*-1}s^{-\iota}ds + \frac{\sigma^*\sigma^{1-\wp_*}\Upsilon^*\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\int_0^\varphi s^\iota ds \\
&\le \frac{\sigma^*\sigma^{1-\wp_*}\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)\Gamma(\wp^*)}\sigma^{\wp_*-\iota}\|w\|_\Psi + \frac{\sigma^*\sigma^{1-\wp_*}\Upsilon^*\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^\iota \\
&\le \frac{\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|w\|_\Psi + \frac{\sigma^*\sigma^{1-\wp_*}\Upsilon^*\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^\iota.
\end{aligned}
$$

In light of this, we obtain

$$
\begin{aligned}
|\Im(w)(\varphi)| &\le \frac{\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|w\|_\Psi + \frac{\sigma^*\sigma^{1-\wp_*}\Upsilon^*\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^\iota \\
&\le \frac{\sigma^*\sigma^{1-\wp_*}\Upsilon^*\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^\iota \times \frac{\Gamma(1-\iota+\wp^*)}{\Gamma(1-\iota+\wp^*)-\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}} \\
&\le \frac{\sigma^*\sigma^{1-\wp_*}\Upsilon^*\Gamma(1-\iota)\sigma^{\wp_*-\iota}\varphi^\iota}{(\iota+1)\Big(\Gamma(1-\iota+\wp^*)-\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}\Big)} = R_\iota,
\end{aligned}
$$

where

$$\Upsilon^* = sup_{\varphi\in D}|\Upsilon(\varphi,0)|.$$

Therefore, it follows that $\Im(B_{R_\iota}) \subseteq (B_{R_\iota})$.

`Step 3` *:*$\Im$ *is* ***Compact ope****.*

Now, we will show that $\Im(B_{R_\iota})$ *is Relat comp, meaning that* $\Im$ *is Compact ope. Clearly* $\Im(B_{R_\iota})$ *is UB because by Step 2, we have* $\Im(B_{R_\iota}) = \{\Im(w) : w \in B_{R_\iota}\} \subset B_{R_\iota}$. *Thus for each* $w \in B_{R_\iota}$ *we have* $\|\Im(w)\|_\Psi \leq R_\iota$ *which implies that* $\Im(B_{R_\iota})$ *is* ***Unif bnd****. It remains to be demonstrated that* $\Im(B_{R_\iota})$ *is* ***Equi-cont****.*

Firstly, we can remark that the function $\nu(\varphi) = \lambda^\varphi - \mu^\varphi$ *is decreasing for* $\varphi \in (1, 0)$ *and* $0 < \lambda < \mu < 1$. *Indeed, since* $ln\lambda < ln\mu < 0, \lambda^\varphi > \mu^\varphi > 0$, *we have that*

$$\nu'(\varphi) = \lambda^\varphi ln\lambda - \mu^\varphi ln\mu < \mu^\varphi ln\lambda - \mu^\varphi ln\mu = \mu^\varphi(ln\lambda - ln\mu) < 0,$$

which implies that $\nu(\varphi)$ *is decreasing function. Thus, for*

$$\nu_\wp(s) = \left(\frac{\varphi_1 - s}{\sigma}\right)^{\wp(s)-1} - \left(\frac{\varphi_2 - s}{\sigma}\right)^{\wp(s)-1}$$

where $0 < \frac{\varphi_1 - s}{\sigma} < \frac{\varphi_2 - s}{\sigma} < 1$, *we may look* $\nu_\wp(s)$ *as the same type as* $\nu(s)$, *then* $w_{\wp(s)}$ *is decreasing with respect to its exponent* $\wp(s) - 1$.

For $\varphi_1, \varphi_2 \in D$ *where* $\varphi_1 < \varphi_2$ *and* $w \in B_{R_\iota}$, *we have*

$$\begin{aligned}
&|\Im(w)(\varphi_2) - \Im(w)(\varphi_1)| \\
&= \Big| \int_0^{\varphi_2} \frac{(\varphi_2 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s))ds - \int_0^{\varphi_1} \frac{(\varphi_1 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s))ds \Big| \\
&= \Big| \int_0^{\varphi_1} \frac{(\varphi_2 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s))ds + \int_{\varphi_1}^{\varphi_2} \frac{(\varphi_2 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s))ds \\
&- \int_0^{\varphi_1} \frac{(\varphi_1 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s))ds \Big| \\
&\leq \Big| \int_0^{\varphi_1} \Big[(\varphi_2 - s)^{\wp(s)-1} - (\varphi_1 - s)^{\wp(s)-1} \Big] \Big| \Big| \frac{\Upsilon(s, w(s))}{\Gamma(\wp(s))} \Big| ds \\
&+ \int_{\varphi_1}^{\varphi_2} (\varphi_2 - s)^{\wp(s)-1} \Big| \frac{\Upsilon(s, w(s))}{\Gamma(\wp(s))} \Big| ds \\
&\leq \frac{1}{\Gamma(\wp^*)} \int_0^{\varphi_1} \Big[\sigma^{\wp(s)-1} \left(\frac{\varphi_2 - s}{\sigma}\right)^{\wp(s)-1} - \sigma^{\wp(s)-1} \left(\frac{\varphi_1 - s}{\sigma}\right)^{\wp(s)-1} \Big] |\Upsilon(s, w(s)) - \Upsilon(s, 0) + \Upsilon(s, 0)| ds \\
&+ \frac{1}{\Gamma(\wp^*)} \int_{\varphi_1}^{\varphi_2} \sigma^{\wp(s)-1} \left(\frac{\varphi_2 - s}{\sigma}\right)^{\wp(s)-1} |\Upsilon(s, w(s)) - \Upsilon(s, 0) + \Upsilon(s, 0)| ds \\
&\leq \frac{\sigma^*}{\Gamma(\wp^*)} \int_0^{\varphi_1} \Big[\left(\frac{\varphi_2 - s}{\sigma}\right)^{\wp_* - 1} - \left(\frac{\varphi_1 - s}{\sigma}\right)^{\wp_* - 1} \Big] \Big[|\Upsilon(s, w(s)) - \Upsilon(s, 0)| + |\Upsilon(s, 0)| \Big] ds \\
&+ \frac{\sigma^*}{\Gamma(\wp^*)} \int_{\varphi_1}^{\varphi_2} \left(\frac{\varphi_2 - s}{\sigma}\right)^{\wp_* - 1} \Big[|\Upsilon(s, w(s)) - \Upsilon(s, 0)| + |\Upsilon(s, 0)| \Big] ds
\end{aligned}$$

As a result, we get

$$
\begin{aligned}
&|\Im(w)(\varphi_2)-\Im(w)(\varphi_1)|\\
&\leq \frac{\sigma^*}{\Gamma(\wp^*)}\int_0^{\varphi_1}\left[\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]|\Upsilon(s,w(s))-\Upsilon(s,0)|ds\\
&+\frac{1}{\Gamma(\wp^*)}\int_0^{\varphi_1}\left[\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]|\Upsilon(s,0)|ds\\
&+\frac{\sigma^*}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}|\Upsilon(s,w(s))-\Upsilon(s,0)|ds+\frac{\sigma^*}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}|\Upsilon(s,0)|ds\\
&\leq \frac{\ell\sigma^*}{\Gamma(\wp^*)}\int_0^{\varphi_1}\left[\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]s^{-\iota}|w(s)|ds\\
&+\frac{\sigma^*}{\Gamma(\wp^*)}\int_0^{\varphi_1}\left[\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]s^{-\iota}s^{\iota}|\Upsilon(s,0)|ds\\
&+\frac{\ell\sigma^*}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}|w(s)|ds+\frac{\sigma^*}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}\left(\frac{(\varphi_2-s)}{\sigma}\right)^{\wp_*-1}s^{-\iota}s^{\iota}|\Upsilon(s,0)|ds\\
&\leq \frac{\ell\sigma^*\|w\|_\Psi}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_0^{\varphi_1}\left[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\right]s^{-\iota}ds\\
&+\frac{\sigma^*\Upsilon^*}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_0^{\varphi_1}\left[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\right]s^{-\iota}ds\\
&+\frac{\ell\sigma^*\|w\|_\Psi}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds+\frac{\sigma^*\Upsilon^*}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds\\
&\leq \frac{\sigma^*(\ell\|w\|_\Psi+\Upsilon^*)}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_0^{\varphi_1}\left[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\right]s^{-\iota}ds\\
&+\frac{\sigma^*(\ell\|w\|_\Psi+\Upsilon^*)}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds
\end{aligned}
$$

$$\leq \frac{\sigma^*(\ell\|w\|_\Psi + \Upsilon^*)}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\Big[\int_0^{\varphi_1}(\varphi_2 - s)^{\wp_*-1}s^{-\iota}ds - \int_0^{\varphi_1}(\varphi_1 - s)^{\wp_*-1}s^{-\iota}ds\Big]$$

$$+ \frac{\sigma^*(\ell\|w\|_\Psi + \Upsilon^*)}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\varphi_1^{-\iota}(\varphi_2 - \varphi_1)^{\wp_*}$$

$$\leq \frac{\sigma^*(\ell\|w\|_\Psi + \Upsilon^*)}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\Big[\frac{\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\varphi_1^{\wp_*-\iota} - \frac{\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\varphi_2^{\wp_*-\iota}\Big]$$

$$+ \frac{\sigma^*(\ell\|w\|_\Psi + \Upsilon^*)}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\varphi_1^{-\iota}(\varphi_2 - \varphi_1)^{\wp_*}$$

Consequently, we have

$$\begin{aligned}|\Im(w)(\varphi_2) - \Im(w)(\varphi_1)| \;&\leq \frac{\sigma^*(\ell\|w\|_\Psi + \Upsilon^*)}{\Gamma(1-\iota+\wp^*)}\sigma^{1-\wp_*}\Big[\Gamma(1-\iota)\varphi_1^{\wp_*-\iota} - \Gamma(1-\iota)\varphi_2^{\wp_*-\iota}\Big]\\ &\quad + \frac{\sigma^*(\ell\|w\|_\Psi + \Upsilon^*)}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\varphi_1^{-\iota}(\varphi_2 - \varphi_1)^{\wp_*}.\end{aligned}$$

Hence $|(\Im w)(\varphi_2) - (\Im w)(\varphi_1)| \to 0$ *as* $|\varphi_2 - \varphi_1| \to 0$*. This implies that* $\Im(B_{R_\iota})$ *is* ***Equi-cont****. As a sequence of Steps 1 to 3 together with* ***Arzella Ascolli thm****, we conclude that* $\Im$ *is Compl cont. Then by SFPT the (2.1) has at least one solution.*

Remark 2.2.1 *The simplicity of condition (2.2) in comparison to the conditions of the earlier publications in the literature cannot be overlooked.*

Theorem 2.2.2 *Let* $(H1)$ *and* $(H2)$ *hold . If*

$$\frac{\ell\sigma^*\sigma^{1-\wp^*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota} < 1, \tag{2.4}$$

then the (2.1) has a unique solution in Ψ*.*

Proof 2.2.2 *We shall employ the* ***Banach contr princip*** *to prove that* $\Im$ *which is defined in (2.3) has* ***Fix pt****. To show that* $\Im$ *admits a unique* ***Fix pt*** *, it suffices to show that* $\Im$ *is a contraction.*

For $w_1(\varphi), w_2(\varphi) \in \Psi$*, we get*

$$\begin{aligned}\Big|\Im(w_1)(\varphi) - \Im(w_2)(\varphi)\Big| \;&=\; \Big|\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,w_1(s))ds - \int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,w_2(s))ds\Big|\\ &\leq\; \Big|\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Big|\Upsilon(s,w_1(s)) - \Upsilon(s,w_2(s))\Big|ds.\end{aligned}$$

It follows that

$$\begin{aligned}\left|\Im(w_1)(\varphi)-\Im(w_2)(\varphi)\right| &\leq \frac{\ell}{\Gamma(\wp^*)}\int_0^{\varphi}(\varphi-s)^{\wp_*-1}s^{-\iota}|w_1(s)-w_2(s)|ds\\ &\leq \frac{\ell}{\Gamma(\wp^*)}\int_0^{\varphi}\sigma^{\wp_*-1}\left(\frac{(\varphi-s)}{\sigma}\right)^{\wp_*-1}s^{-\iota}|w_1(s)-w_2(s)|ds\\ &\leq \frac{\ell\sigma^*\sigma^{1-\wp^*}}{\Gamma(\wp^*)}\|w_1-w_2\|_{\Psi}\int_0^{\varphi}(\varphi-s)^{\wp_*-1}s^{-\iota}ds\\ &\leq \frac{\ell\sigma^*\sigma^{1-\wp^*}\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)\Gamma(\wp^*)}\sigma^{\wp_*-\iota}\|w_1-w_2\|_{\Psi}\\ &\leq \frac{\ell\sigma^*\sigma^{1-\wp^*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|w_1-w_2\|_{\Psi}.\end{aligned}$$

By the boundedness of φ on D, we end up with

$$\|\Im(w_1)-\Im(w_2)\|\leq\frac{\ell\sigma^*\sigma^{1-\wp^*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|w_1-w_2\|_{\Psi}.$$

*Consequently by (2.4), the operator $\Im$ is a contraction. Thus, by **Banach contr princip**, $\Im$ has a unique **Fix pt** $w\in\Psi$, which is a unique solution of the (2.1).*

Remark 2.2.2 *Unlike the assumptions used in the literature , condition (2.4) does not involve the **piece-wise constant funct** $\wp$.*

2.3 Ulam Hyers stability

The stability of solutions to a specified problem is one of the key qualitative characteristics, and in the next section, we study the Ulam-Hyers stability for solutions to the alleged variable order (2.1).

Theorem 2.3.1 *Let the conditions $(H1)$ and $(H2)$ hold, then the (2.1) is **Ula Hyer stab**.*

Proof 2.3.1 *Let $\varepsilon>0$ an arbitrary number and the function $\chi(\varphi)$ from $\chi\in\Psi$ satisfy the following inequality*

$$|D^{\wp(\varphi)}\chi(\varphi)-\Upsilon(\varphi,\chi(\varphi))|<\varepsilon,\quad \varphi\in D$$

we have

$$D^{\wp(\varphi)}\chi(\varphi)=\int_0^{\varphi}\frac{(\varphi-s)^{1-\wp(s)}}{\Gamma(1-\wp(s))}\chi(s)ds,$$

we obtain

$$
\begin{aligned}
\left|\chi(\varphi)-\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,\chi(s))ds\right| &\leq \varepsilon\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}ds\\
&\leq \frac{\varepsilon}{\Gamma(\wp^*)}\int_0^{\varphi}\sigma^{\wp_*-1}\left(\frac{\varphi-s}{\sigma}\right)^{1-\wp_*}s^{-\iota}s^{\iota}ds\\
&\leq \frac{\varepsilon\sigma^*\sigma^{\wp_*-1}}{\Gamma(\wp^*)}\int_0^{\varphi}s^{-\iota}(\varphi-s)^{\wp_*-1}s^{\iota}ds\\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\int_0^{\varphi}s^{\iota}ds\\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\ell}\left[\frac{s^{\iota+1}}{\iota+1}\right]_0^{\varphi}\\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota}.
\end{aligned}
$$

Let $\varphi\in D$, however, we get

$$
\begin{aligned}
&|\chi(\varphi)-w(\varphi)|\\
&\leq\left|\varepsilon\frac{\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota}+\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,\chi(s))ds\right.\\
&\left.-\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,w(s))ds\right|\\
&\leq\varepsilon\frac{\sigma^*\sigma^{\wp_*-1}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota}+\int_0^{u}\frac{(u-s)^{\wp(s)-1}}{\Gamma(\wp(s))}|\Upsilon(s,\chi(s))-\Upsilon(s,w(s))|ds\\
&\leq\varepsilon\frac{\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota}+\frac{\ell}{\Gamma(\wp^*)}\int_0^{\varphi}\sigma^{\wp_*-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}|\chi(s)-w(s)|ds\\
&\leq\varepsilon\frac{\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota}+\frac{\ell\sigma^*\|\chi-w\|_{\Psi}}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_0^{\varphi}(\varphi-s)^{\wp_*-1}s^{-\iota}ds
\end{aligned}
$$

$$\leq \varepsilon \frac{\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota} + \frac{\ell\sigma^*\sigma^{1-\wp_*}\|\chi-w\|_{\Psi}\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}$$

$$\leq \varepsilon \frac{\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota} + \frac{\ell\sigma^*\sigma^{1-\wp_*}\|\chi-w\|_{\Psi}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}.$$

Thus, we get

$$\|\chi-w\|_{\Psi}\left(1-\frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\right) \leq \varepsilon \frac{\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota}.$$

For each $u \in M$, it follows that

$$\begin{aligned} |\chi(\varphi)-w(\varphi)| \leq \|\chi-w\|_{\Psi} &\leq \varepsilon \frac{\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota} \\ &\quad \times \frac{\Gamma(1-\iota+\wp^*)}{\Gamma(1-\iota+\wp^*)-\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}} \\ &\leq \varepsilon \frac{\sigma^*\sigma^{1-\wp_*}\sigma^{\wp_*-\iota}\varphi^{\iota}\Gamma(1-\iota)}{(\iota+1)\Big(\Gamma(1-\iota+\wp^*)-\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}\Big)} = c_{\Upsilon}\varepsilon. \end{aligned}$$

which implies that (2.1) is ***Ula Hyer stab****.*

Remark 2.3.1 *In light of conditions $(H1)$ and $(H2)$, one may readily draw the conclusion that the stability of (2.1) is concluded under less onerous assumptions.*

2.4 An application

Consider the following fractional problem

$$\begin{cases} D_{0^+}^{\wp(\varphi)}w(\varphi) &= \Upsilon(\varphi, w(\varphi)), \quad \varphi \in D = [0,1] \\ \\ w(0) &= 0, \end{cases} \tag{2.5}$$

where $\wp(\varphi) = \frac{1}{5}\varphi + \frac{3}{4}$ is a **Cont funct** on $[\frac{1}{2}, 1]$ and $\Upsilon(u,w) = \dfrac{\varphi^{-\frac{1}{5}}}{1+|\frac{w}{3}|}$ is a **Cont funct** on $(0,1] \times \mathbb{R}$. Clearly for $\varphi \in [0,1]$, we have

$$\frac{3}{4} \leq \wp(\varphi) \leq \frac{19}{20},$$

which implies that condition (H1) holds. Further, we have

$$\begin{aligned}
\varphi^{\frac{1}{5}}|\Upsilon(\varphi, w_1) - \Upsilon(\varphi, w_2)| &= \left|\varphi^{\frac{1}{5}}\left(\frac{\varphi^{-\frac{1}{5}}}{1+|\frac{w_1}{3}|} - \frac{\varphi^{-\frac{1}{5}}}{1+|\frac{w_2}{3}|}\right)\right| \\
&= \left|\frac{1+|\frac{w_2}{3}| - 1 - |\frac{w_1}{3}|}{(1+|\frac{w_1}{3}|)(1+|\frac{w_2}{3}|)}\right| \\
&= \left|\frac{|\frac{w_2}{3}| - |\frac{w_1}{3}|}{(1+|\frac{w_1}{3}|)(1+|\frac{w_2}{3}|)}\right| \\
&\leq \frac{1}{3}|w_1 - w_2|.
\end{aligned}$$

Hence condition (H2) holds with $\iota = \frac{1}{5}$ and $\ell = \frac{1}{3}$. For the purpose of verifying (2.4), it is clear that

$$\frac{\ell\sigma^* \sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota} = \frac{\frac{1}{3}\Gamma(1+\frac{1}{5})}{\Gamma(1-\frac{1}{5}+\frac{19}{20})} = \frac{1}{3}\frac{1}{5}\frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{27}{20})} = \frac{1}{15} \times \frac{4.5908}{0.8912} = 0.3434... < 1$$

By Theorem (2.2.2), problem (2.5) has a unique solution.

In the remaining part, we present the solution $w(\varphi)$ for $\wp(\varphi) = \frac{1}{5}\varphi + \frac{3}{4}$ with $\varphi \in [0,1]$ and $w_i(\varphi)$ for $\wp(\varphi_i) = \frac{1}{5}\varphi_i + \frac{3}{4}$ where φ_i is fixed. Figure (2.1) represents the plot of the solution w depending on φ. On other hand, Figure (2.2) presents a comparison between the solution w and some different solutions w_i with different $\wp$.

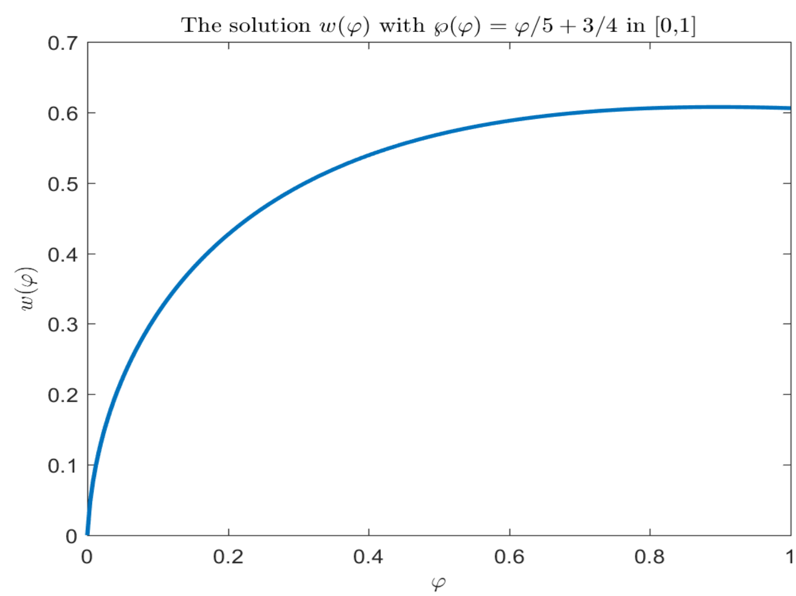
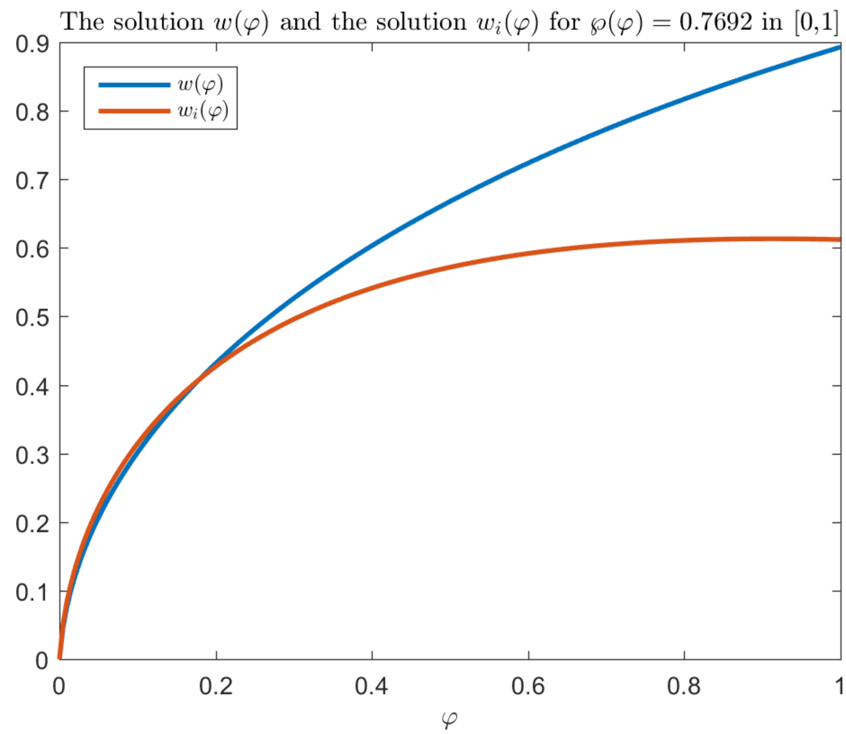
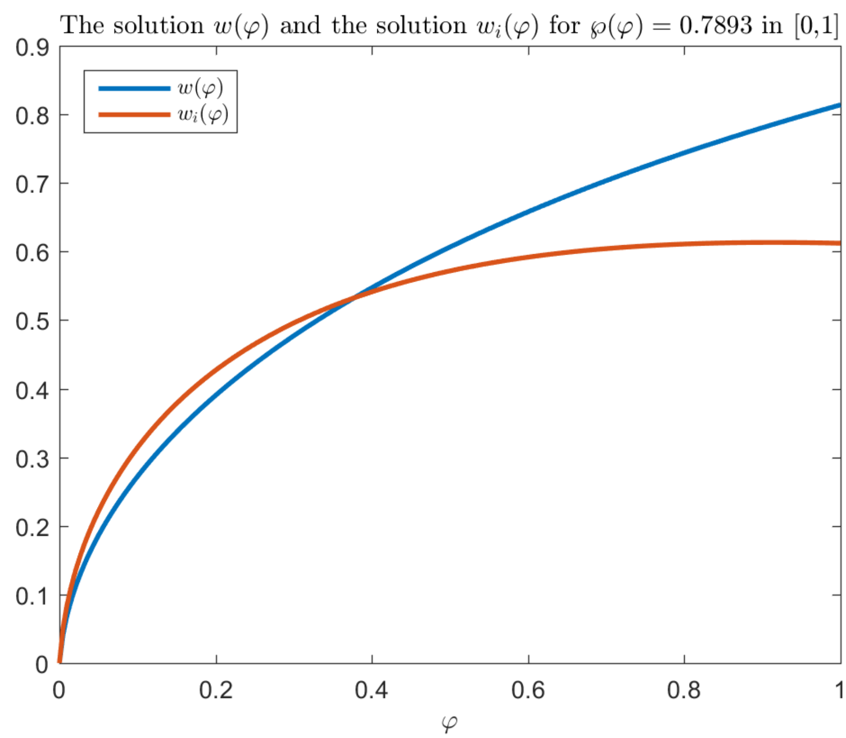
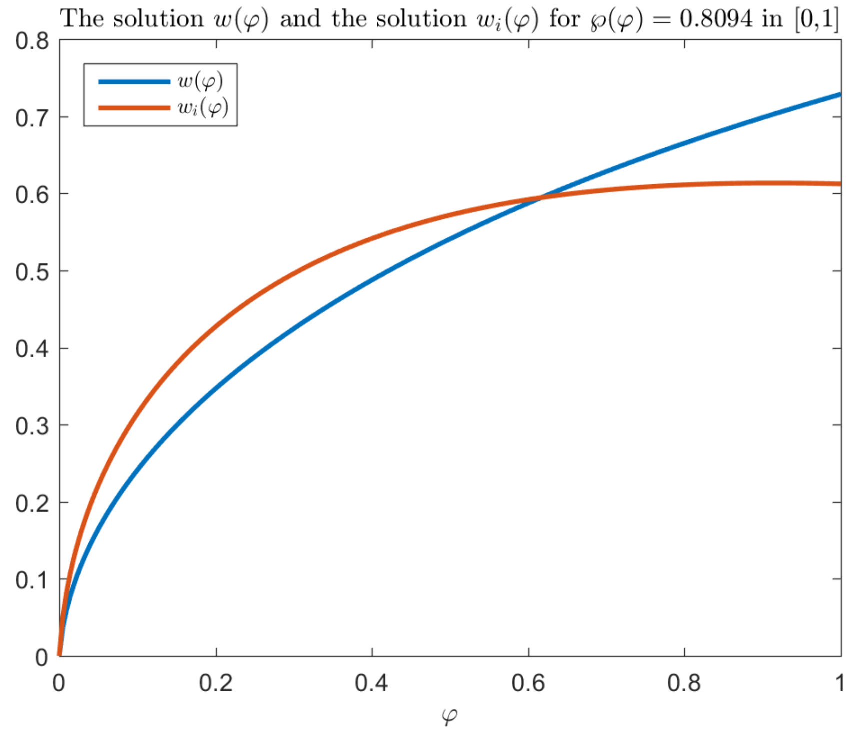
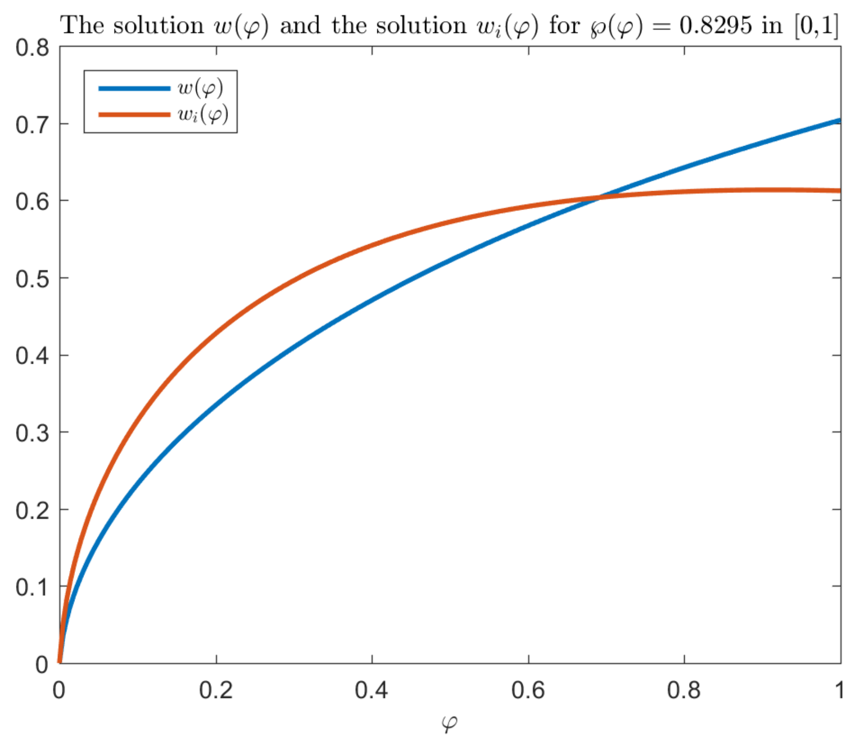


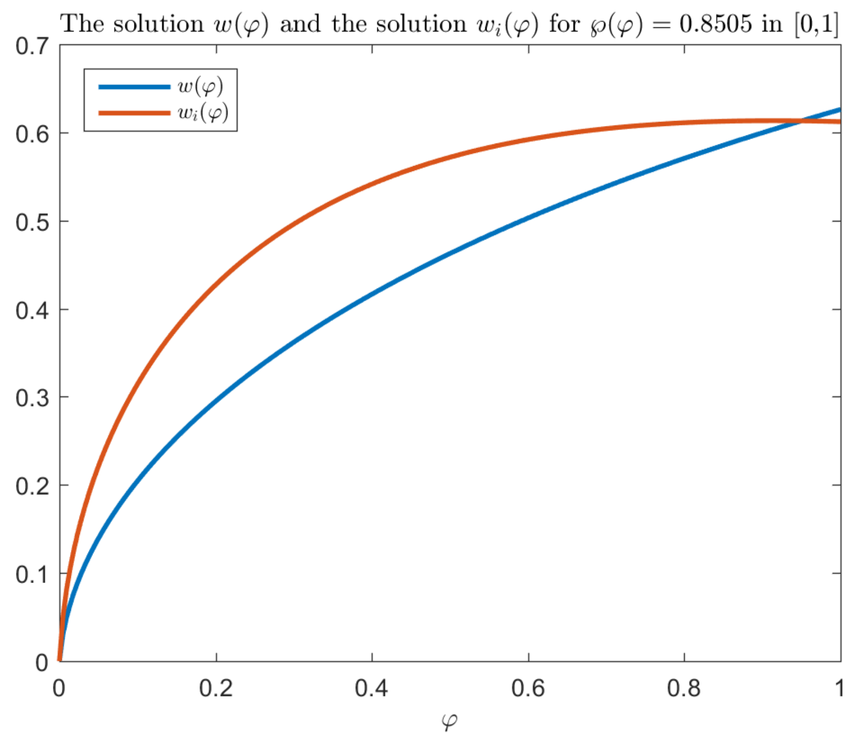
Figure 2.1: The solution $w(\varphi)$ in $[0,1]$ with $\wp(\varphi) = \frac{\varphi}{5} + \frac{3}{4}$


The solution $w(\varphi)$ and the solution $w_i(\varphi)$ for $\wp(\varphi) = 0.7692$ in [0,1]
$w(\varphi)$
$w_i(\varphi)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0
0.2
0.4
0.6
0.8
1
φ

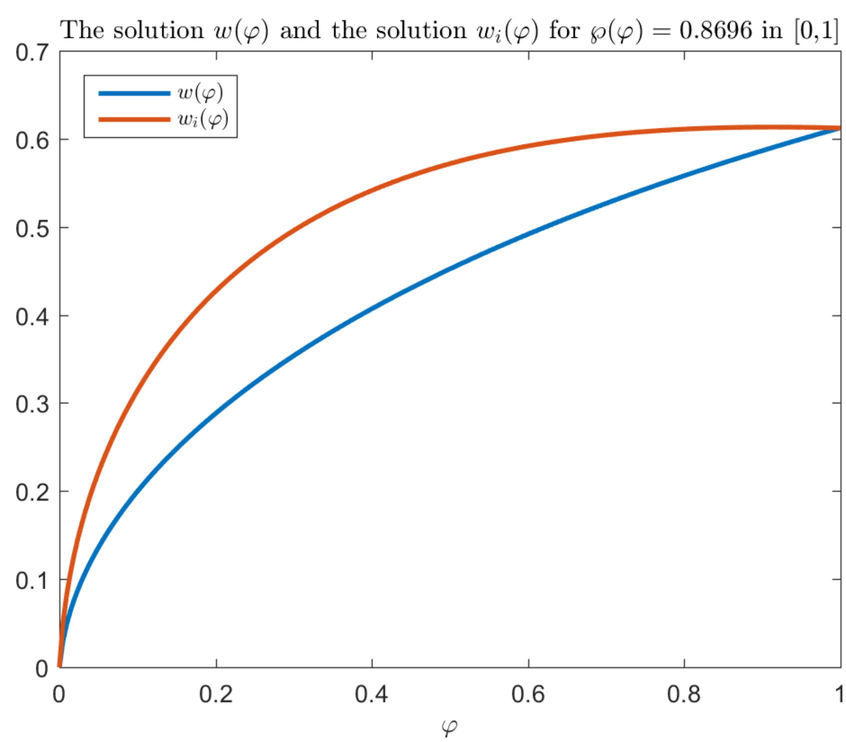

The solution $w(\varphi)$ and the solution $w_i(\varphi)$ for $\wp(\varphi) = 0.7893$ in [0,1]
$w(\varphi)$
$w_i(\varphi)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0
0.2
0.4
0.6
0.8
1
φ


The solution $w(\varphi)$ and the solution $w_i(\varphi)$ for $\wp(\varphi) = 0.8094$ in [0,1]
$w(\varphi)$
$w_i(\varphi)$
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
φ

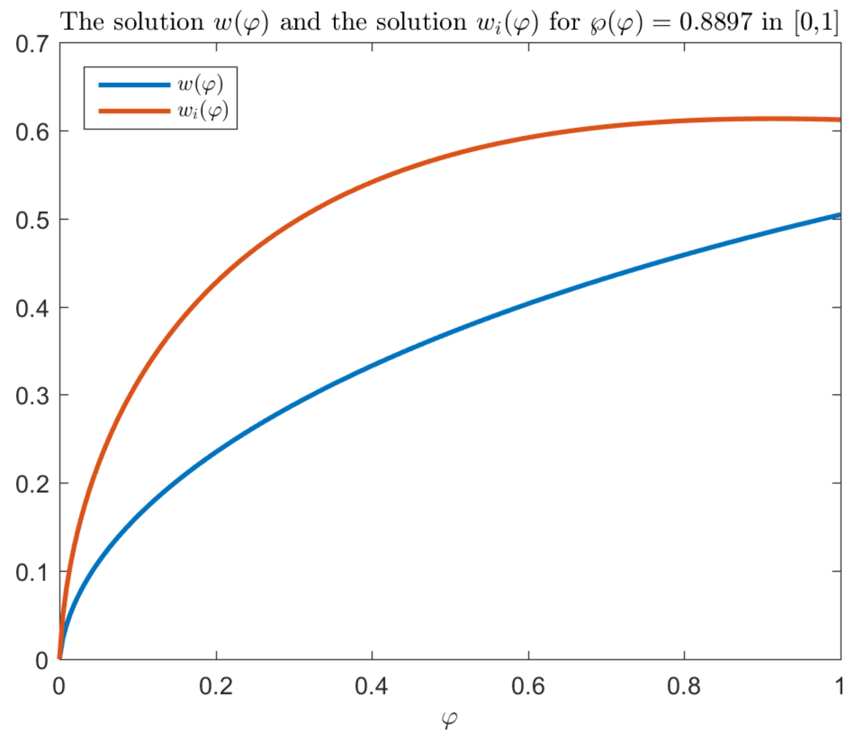

The solution $w(\varphi)$ and the solution $w_i(\varphi)$ for $\wp(\varphi) = 0.8295$ in [0,1]
$w(\varphi)$
$w_i(\varphi)$
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
φ

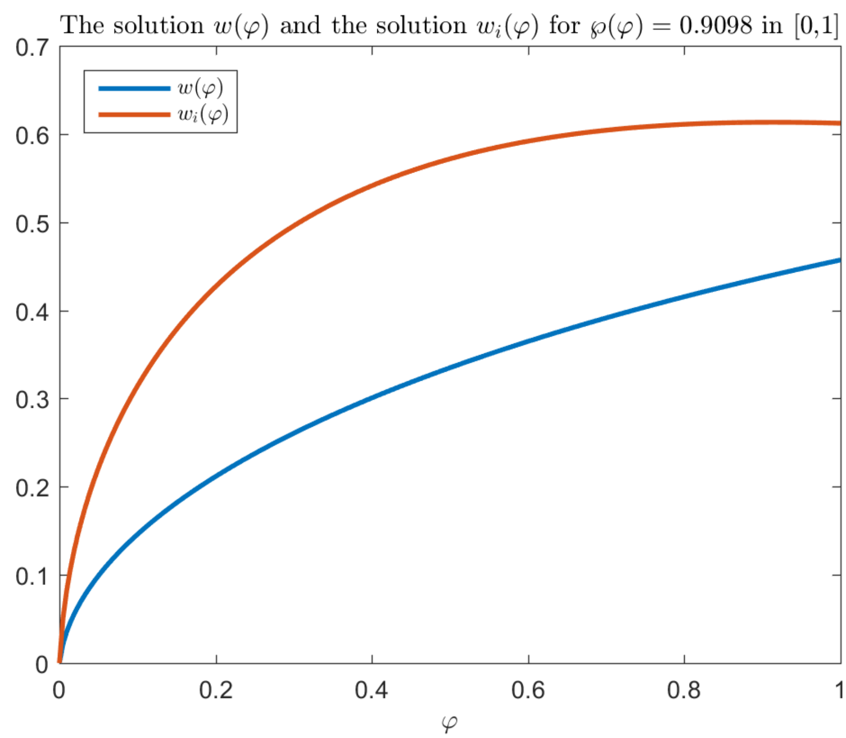


The solution $w(\varphi)$ and the solution $w_i(\varphi)$ for $\wp(\varphi) = 0.8505$ in [0,1]
$w(\varphi)$
$w_i(\varphi)$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
φ



The solution $w(\varphi)$ and the solution $w_i(\varphi)$ for $\wp(\varphi) = 0.8696$ in [0,1]
$w(\varphi)$
$w_i(\varphi)$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
φ


The solution $w(\varphi)$ and the solution $w_i(\varphi)$ for $\wp(\varphi) = 0.8897$ in [0,1]
$w(\varphi)$
$w_i(\varphi)$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
φ


The solution $w(\varphi)$ and the solution $w_i(\varphi)$ for $\wp(\varphi) = 0.9098$ in [0,1]
$w(\varphi)$
$w_i(\varphi)$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
φ

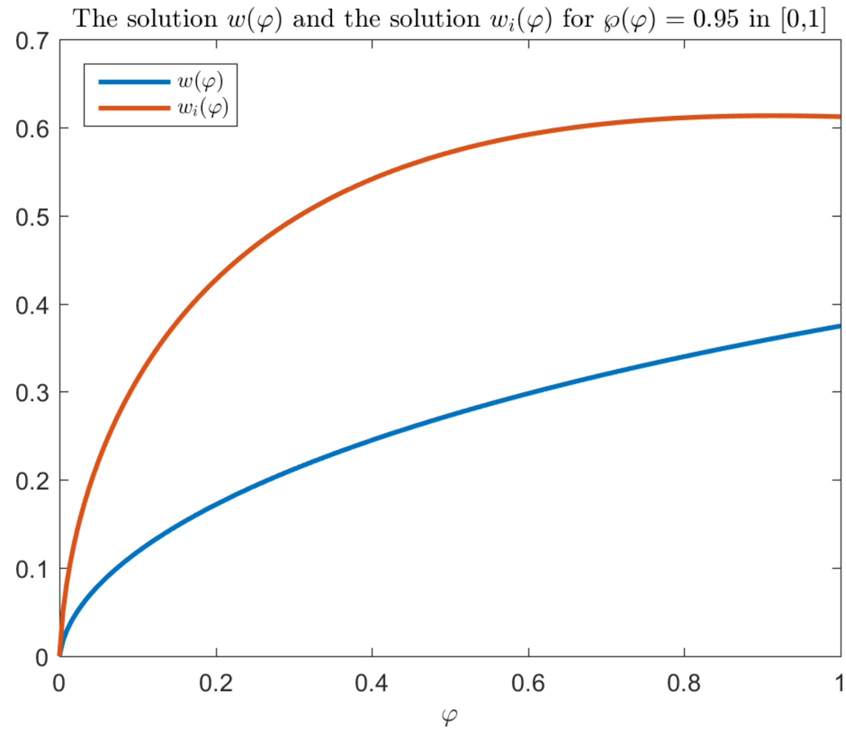


Figure 2.2: A plot of $w(\varphi)$, $w_i(\varphi)$ for different $\wp(\varphi)$

In this table, we present the $Norme_i = \max_{\varphi\in[0,1]} \Big| w(\varphi) - w_i(\varphi) \Big|$ for $\wp(\varphi) \in [0,1]$.

u_i	0.0959	0.1963	0.2968	0.3973	0.5023	0.5982	0.6986	0.7991	0.8995	1.0000
$\wp_i(u)$	0.7692	0.7893	0.8094	0.8295	0.8505	0.8696	0.8897	0.9098	0.9299	0.9500
$Norm_i$	0.2808	0.2017	0.1289	0.1084	0.1442	0.1764	0.2092	0.2406	0.2705	0.2987

We observe that the norm is small when $\wp_i(u)$ is close to 0.85. Almost the center of the interval.

Conclusion

This chapter introduces an innovative approach for solving fractional-order differential equations, specifically addressing the Cauchy-type problem with variable fractional order. By developing a new operator that eliminates the need for phase adjustments or interval splitting, we offer a more adaptable and efficient technique. The application of this method to the problem defined by the fractional-order Riemann-Liouville derivative shows promising results in terms of both stability and solvability. The originality of the approach is evident, as it has not been explored in existing literature. Overall, this work lays the groundwork for further research and potential applications in the field of fractional calculus and its numerous real-world applications.

Chapter 3

Cauchy-Type Problem With Finite Delay for Nonlinear Fractional Differential Equations of Variable Order

3.1 Introduction

[1] In this chapter we dispense with the use of the partial constant function , We discuss this work the **Exi,uniq and stab of sol** to the following (CFDPNFDEVO(2))

$$\begin{cases} D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(\varphi, w(\varphi)), \ \ \varphi \in D = [0,\sigma] \\ w(\varphi) = \eta(\varphi), \quad \varphi \in [-r, 0] \end{cases} \tag{CFDPNFDEVO(2)}$$

where $0 < \sigma < +\infty$, $0 < \wp(\varphi) \leq 1$, $\Upsilon : D \times C([-r,0], \mathbb{R}) \to \mathbb{R}$ is a continuous functions **Cont funct** and $D_{0^+}^{\wp(\varphi)}$, is the Riemann-Liouville fractional derivative of variable-order (R-LFDVO) $\wp(\varphi)$ and $\eta(\varphi) \in C([-r,0], \mathbb{R})$ with $\eta(0) = 0$.

For any function w defined on $[-r, \sigma]$ and any $\varphi \in D$, we denote by w_φ the element of $C([-r,0], \mathbb{R})$ defined by

$$w_\varphi(\jmath) = w(\varphi + \jmath), \quad \jmath \in [-r, 0]$$

[1]**M. Benaouda** , S. Sabit, , M. S. Souid, Hijaz Ahmad, Cauchy-Type Problem With Finite Delay for Nonlinear Fractional Differential Equations of Variable Order, (**submitted**).

3.2 Existence and uniqueness of solutions

We impose the following assumptions:

(A1) $\wp : [0, \sigma] \to (0, \wp^*]$ is Cont funct, such that $\frac{1}{2} < \wp(\varphi) \leq \wp_* \leq 1$.

(A2) Let $\varphi^\iota \Upsilon : D \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ is Cont funct $(0 \leq \iota \leq 1)$,exists a constants, $\ell > 0$, such that,

$$\varphi^\iota |\Upsilon(\varphi, u) - \Upsilon(\varphi, v)| \leq \ell |u - v| \quad for\ any\ u, v, \in \mathbb{R}\ and\ \varphi \in D. \tag{3.1}$$

(A3) Exists $\alpha, \beta \in C(D, \mathbb{R}^+)$ such that

$$\varphi^\iota \|\Upsilon(\varphi, u)\| \leq \alpha(\varphi) + \beta(\varphi)\|u\|_C\ for\ \varphi \in D\ and\ u \in \Psi$$

Definition 3.2.1 *A function $w \in C(D, \mathbb{R})$ is said to be a solution for (CFDPNFDEVO(2)) if and only if it verifies (CFDPNFDEVO(2))simultaneously.*

For the existence of solutions for the (CFDPNFDEVO(2)), an auxiliary lemma is needed as follows:

Lemma 3.2.1 *[72] Let $0 < \wp(\varphi) < 1$ and let $\psi : (0, \sigma] \to \mathbb{R}$ be continuous and $lim_{\varphi \to 0^+} \psi(\varphi) = \psi(0^+) \in \mathbb{R}$. Then w is a solution of the fractional integral equation*

$$w(\varphi) = \int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \psi(s) ds, \quad \tau \in D\ and\ \wp(s) > 0,$$

if and only if w is a solution of the initial value problem for the fractional differential equation

$$\begin{cases} D^{\wp(\varphi)} = \psi(\varphi), & \tau \in (0, \sigma] \\ \\ w(0) = 0. \end{cases} \tag{3.2}$$

The first result obtained by using the **Schauder fix pt thm** .

Theorem 3.2.1 *Assume that conditions $(A1), (A2), (A3)$ hold, if*

$$\frac{\sigma^*\ \sigma^{1-\wp_*} \|\alpha\|_\Psi\ \Gamma(1-\iota)\sigma^{\wp_* - \iota}}{\Gamma(1 - \iota + \wp^*) - \sigma^*\ \sigma^{1-\wp_*} \|\beta\|_\Psi\ \Gamma(1-\iota)\sigma^{\wp_* - \iota}} < 1.$$

Then the (CFDPNFDEVO(2)) has at least one solution on Ψ.

Proof

Consider the operator $\Im : \Psi \to \Psi$, defined by :

$$\Im(w)(\varphi) = \begin{cases} \eta(\varphi), & if\ \varphi \in [-r, 0] \\ \\ \displaystyle\int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w_s) ds, & if\ \varphi \in D \quad and \quad \wp(s) > 0 \end{cases} \tag{3.3}$$

We consider the set

$$B_{R_\iota} = \{w \in \Psi, \|w\|_\Psi \leq R_\iota\}. \tag{3.4}$$

Where

$$R_\iota = \frac{\sigma^* \ \sigma^{1-\wp_*} \|\alpha\|_\Psi \ \Gamma(1-\iota)\sigma^{\wp_*-\iota}}{\Gamma(1-\iota+\wp^*) - \sigma^* \ \sigma^{1-\wp_*} \|\beta\|_\Psi \ \Gamma(1-\iota)\sigma^{\wp_*-\iota}}. \tag{3.5}$$

Clearly B_{R_ι} is **Convex, closed bounded non-empty**.

The proof will be given in three steps.

`Step 1`: $\Im$ is Cont ope.

Let w_n be a sequence such that $w_n \to w$ in Ψ then

$$\|(\Im w_n) - (\Im w)\|_\Psi \to 0.$$

For $\varphi \in D$, we have

$$\begin{aligned}
|\Im(w_n)(\varphi) - \Im(w)(\varphi)| &= \left| \int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w_n(s)) ds - \int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s)) ds \right| \\
&\leq \int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} |\Upsilon(s, w_n(s)) - \Upsilon(s, w(s))| ds \\
&\leq \frac{\ell}{\Gamma(\wp^*)} \int_0^\varphi \sigma^{\wp^*-1} \left(\frac{\varphi - s}{\sigma} \right)^{\wp^*-1} s^{-\iota} |w_n(s) - w(s)| ds \\
&\leq \frac{\ell\sigma^* \ \sigma^{1-\wp_*}}{\Gamma(\wp^*)} \|w_n - w\|_\Psi \int_0^\varphi (\varphi - s)^{\wp_*-1} s^{-\iota} ds \\
&\leq \frac{\ell\sigma^* \ \sigma^{1-\wp_*} \ \Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*) \ \Gamma(1-\iota+\wp^*)} \sigma^{\wp_*-\iota} \|w_n - w\|_\Psi \\
\|\Im(w_n) - \Im(w)\|_\Psi &\leq \frac{\ell\sigma^* \ \sigma^{1-\wp_*} \ \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)} \sigma^{\wp_*-\iota} \|w_n - w\|_\Psi
\end{aligned}$$

$$\|(\Im w_n) - (\Im w)\|_\Psi \to 0 \quad as \quad n \to \infty.$$

Consequently, $\Im$ is Cont ope on Ψ.

`Step 2`: $\Im(B_{R_\iota}) \subseteq (B_{R_\iota})$.

For $w \in B_{R_\iota}$, and by $(A3)$, we get:

$$
\begin{aligned}
|\Im(w)(\varphi)| &= \Big|\int_0^\varphi \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,w_s)ds\Big| \\
&\le \int_0^\varphi \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}|\Upsilon(s,w_s)|ds \\
&\le \frac{1}{\Gamma(\wp^*)}\int_0^\varphi \sigma^{\wp_*-1}\Big(\frac{\varphi-s}{\sigma}\Big)^{\wp^*-1} s^{-\iota}\Big(|\alpha(s)|+|\beta(s)||w(s)|\Big)ds \\
&\le \frac{\sigma^*\ \sigma^{1-\wp_*}}{\Gamma(\wp^*)}\int_0^\varphi (\varphi-s)^{\wp_*-1}\Big(s^{-\iota}|\alpha(s)|+s^{-\iota}|\beta(s)||w(s)|\Big)ds \\
&\le \frac{\sigma^*\ \sigma^{1-\wp_*}}{\Gamma(\wp^*)}\int_0^\varphi (\varphi-s)^{\wp_*-1}s^{-\iota}|\alpha(s)|ds + \frac{\sigma^*\ \sigma^{1-\wp_*}}{\Gamma(\wp^*)}\int_0^\varphi (\varphi-s)^{\wp_*-1}s^{-\iota}|\beta(s)||w(s)|ds \\
&\le \frac{\sigma^*\ \sigma^{1-\wp_*}\|\alpha\|_\Psi}{\Gamma(\wp^*)}\int_0^\varphi (\varphi-s)^{\wp_*-1}s^{-\iota}ds + \frac{\sigma^*\ \sigma^{1-\wp_*}\|\beta\|_\Psi}{\Gamma(\wp^*)}\|w\|_\Psi\int_0^\varphi (\varphi-s)^{\wp_*-1}s^{-\iota}ds \\
&\le \frac{\sigma^*\ \sigma^{1-\wp_*}\|\alpha\|_\Psi\ \Gamma(\wp^*)\ \Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota} + \frac{\sigma^*\ \sigma^{1-\wp_*}\|\beta\|_\Psi\ \Gamma(\wp^*)\ \Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|w\|_\Psi \\
&\le \frac{\sigma^*\ \sigma^{1-\wp_*}\|\alpha\|_\Psi\ \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota} + \frac{\sigma^*\ \sigma^{1-\wp_*}\|\beta\|_\Psi\ \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|w\|_\Psi \\
&\le \frac{\sigma^*\ \sigma^{1-\wp_*}\|\alpha\|_\Psi\ \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota} \times \frac{\Gamma(1-\iota+\wp^*)}{\Gamma(1-\iota+\wp^*)-\sigma^*\ \sigma^{1-\wp_*}\|\beta\|_\Psi\ \Gamma(1-\iota)\sigma^{\wp_*-\iota}} \\
&\le \frac{\sigma^*\ \sigma^{1-\wp_*}\|\alpha\|_\Psi\ \Gamma(1-\iota)\sigma^{\wp_*-\iota}}{\Gamma(1-\iota+\wp^*)-\sigma^*\ \sigma^{1-\wp_*}\|\beta\|_\Psi\ \Gamma(1-\iota)\sigma^{\wp_*-\iota}} \\
&= R_\iota.
\end{aligned}
$$

Which means that $\Im(B_{R_\iota}) \subseteq (B_{R_\iota})$.

`Step 3` :$\Im$ is compact ope

Now, we will show that $\Im(B_{R_\iota})$ is **Relat comp**, meaning that $\Im$ is compact ope. Clearly $\Im(B_{R_\iota})$ is **Unif bnd** because by Step 2, we have $\Im(B_{R_\iota}) = \{\Im(w) : w \in B_{R_\iota}\} \subset B_{R_\iota}$ thus for each $w \in B_{R_\iota}$ we have $\|\Im(w)\|_\Psi \le R_\iota$ which means that $\Im(B_{R_\iota})$ is **Unif bnd**. It remains to show that $\Im(B_{R_\iota})$ is **Equi-cont**.

Firstly, we can remark that the function $\nu(\varphi) = \lambda^{\varphi} - \mu^{\varphi}$ is decreasing for $\varphi \in (0,1)$ and $0 < \lambda < \mu < 1$. Indeed, since $ln\lambda < ln\mu < 0, \lambda^{\varphi} > \mu^{\varphi} > 0$, we have that

$$\nu'(\varphi) = \lambda^{\varphi} ln\lambda - \mu^{\varphi} ln\mu < \mu^{\varphi} ln\lambda - \mu^{\varphi} ln\mu = \mu^{\varphi}(ln\lambda - ln\mu) < 0,$$

which implies that $\nu(\varphi)$ is decreasing function. Thus, for

$$\nu_{\wp}(s) = \left(\frac{\varphi_1 - s}{\sigma}\right)^{\wp(s)-1} - \left(\frac{\varphi_2 - s}{\sigma}\right)^{\wp(s)-1},$$

where $0 < \frac{\varphi_1 - s}{\sigma} < \frac{\varphi_2 - s}{\sigma} < 1$, we may look $\nu_{\wp}(s)$ as the same type as $\nu(s)$, then $\nu_{\wp}(s)$ is decreasing with respect to its exponent $\wp(s) - 1$.

For $\varphi_1, \varphi_2 \in D,\ \varphi_1 < \varphi_2$ and $w \in B_{R_\iota}$, we have:

$$|\wp(w)(\varphi_2) - \wp(w)(\varphi_1)|$$

$$= \Big| \int_0^{\varphi_2} \frac{(\varphi_2 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s)) ds - \int_0^{\varphi_1} \frac{(\varphi_1 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s)) ds \Big|$$

$$= \Big| \int_0^{\varphi_1} \frac{(\varphi_2 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s)) ds + \int_{\varphi_1}^{\varphi_2} \frac{(\varphi_2 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s)) ds$$
$$- \int_0^{\varphi_1} \frac{(\varphi_1 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w(s)) ds \Big|$$

$$\leq \Big| \int_0^{\varphi_1} \Big[(\varphi_2 - s)^{\wp(s)-1} - (\varphi_1 - s)^{\wp(s)-1} \Big] \Big| \frac{\Upsilon(s, w(s))}{\Gamma(\wp(s))} \Big| ds + \int_{\varphi_1}^{\varphi_2} (\varphi_2 - s)^{\wp(s)-1} \Big| \frac{\Upsilon(s, w(s))}{\Gamma(\wp(s))} \Big| ds,$$

This implies that

$$|\Im(w)(\varphi_2) - \Im(w)(\varphi_1)|$$

$$\leq \frac{1}{\Gamma(\wp^*)} \int_0^{\varphi_1} \left[\sigma^{\wp(s)-1} \left(\frac{\varphi_2 - s}{\sigma}\right)^{\wp(s)-1} - \sigma^{\wp(s)-1} \left(\frac{\varphi_1 - s}{\sigma}\right)^{\wp(s)-1} \right] s^{-\iota} \Big(\alpha(s) + \beta(s)|w(s)| \Big) ds$$

$$+ \frac{1}{\Gamma(\wp^*)} \int_{\varphi_1}^{\varphi_2} \sigma^{\wp(s)-1} \left(\frac{\varphi_2 - s}{\sigma}\right)^{\wp(s)-1} s^{-\iota} \Big(\alpha(s) + \beta(s)|w(s)| \Big) ds$$

$$\leq \frac{1}{\Gamma(\wp^*)}\int_0^{\varphi_1}\left[\sigma^{\wp_*-1}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\sigma^{\wp_*-1}\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]s^{-\iota}\Big(\alpha(s)+\beta(s)|w(s)|\Big)ds$$

$$+\frac{1}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}\sigma^{\wp_*-1}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}\Big(\alpha(s)+\beta(s)|w(s)|\Big)ds$$

$$\leq \frac{\sigma^*}{\Gamma(\wp^*)}\int_0^{\varphi_1}\left[\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]s^{-\iota}\alpha(s)ds$$

$$+\frac{\sigma^*}{\Gamma(\wp^*)}\int_0^{\varphi_1}\left[\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]s^{-\iota}\beta(s)|w(s)|ds$$

$$+\frac{\sigma^*}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}\alpha(s)ds+\frac{\sigma^*}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}\beta(s)|w(s)|ds$$

$$\leq \frac{\sigma^*\|\alpha\|_\Psi}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_0^{\varphi_1}\left[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\right]s^{-\iota}ds$$

$$+\frac{\sigma^*\|\beta\|_\Psi\|w\|_\Psi}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_0^{\varphi_1}\left[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\right]s^{-\iota}ds$$

$$+\frac{\sigma^*\|\alpha\|_\Psi}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds+\frac{\sigma^*\|\beta\|_\Psi\|w\|_\Psi}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds,$$

$$\leq \frac{\sigma^*\sigma^{1-\wp_*}(\|\alpha\|_\Psi+\|\beta\|_\Psi\|w\|_\Psi)}{\Gamma(\wp^*)}\int_0^{\varphi_1}\left[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\right]s^{-\iota}ds$$

$$+\frac{\sigma^*\sigma^{1-\wp_*}(\|\alpha\|_\Psi+\|\beta\|_\Psi\|w\|_\Psi)}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds$$

$$\leq \frac{\sigma^*\sigma^{1-\wp_*}(\|\alpha\|_\Psi+\|\beta\|_\Psi\|w\|_\Psi)}{\Gamma(\wp^*)}\Big[\int_0^{\varphi_1}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds-\int_0^{\varphi_1}(\varphi_1-s)^{\wp_*-1}s^{-\iota}ds\Big]$$

$$+\frac{\sigma^*\sigma^{1-\wp_*}(\|\alpha\|_\Psi+\|\beta\|_\Psi\|w\|_\Psi)}{\Gamma(\wp^*)}\varphi_1^{-\iota}(\varphi_2-\varphi_1)^{\wp_*}$$

$$\leq \frac{\sigma^*\sigma^{1-\wp_*}(\|\alpha\|_\Psi+\|\beta\|_\Psi\|w\|_\Psi)}{\Gamma(\wp^*)}\Big[\frac{\Gamma(1-\iota)\Gamma(\wp^*)}{\Gamma(1-\iota+\wp^*)}\varphi_1^{\wp_*-\iota}-\frac{\Gamma(1-\iota)\Gamma(\wp^*)}{\Gamma(1-\iota+\wp^*)}\varphi_2^{\wp_*-\iota}\Big]$$

$$+\frac{\sigma^*\sigma^{1-\wp_*}(\|\alpha\|_\Psi+\|\beta\|_\Psi\|w\|_\Psi)}{\Gamma(\wp^*)}\varphi_1^{-\iota}(\varphi_2-\varphi_1)^{\wp_*}$$

$$\leq \frac{\sigma^*\sigma^{1-\wp_*}(\|\alpha\|_\Psi+\|\beta\|_\Psi\|w\|_\Psi)}{\Gamma(1-\iota+\wp^*)}\Big[\Gamma(1-\iota)\varphi_1^{\wp_*-\iota}-\Gamma(1-\iota)\varphi_2^{\wp_*-\iota}\Big]$$

$$+\frac{\sigma^*\sigma^{1-\wp_*}(\|\alpha\|_\Psi+\|\beta\|_\Psi\|w\|_\Psi)}{\Gamma(\wp^*)}\varphi_1^{-\iota}(\varphi_2-\varphi_1)^{\wp_*},$$

Consequently, we have:

$$|\Im(w)(\varphi_2)-\Im(w)(\varphi_1)|\leq \frac{\sigma^*\sigma^{1-\wp_*}(\|\alpha\|_\Psi+\|\beta\|_\Psi\|w\|_\Psi)}{\Gamma(1-\iota+\wp^*)}\Big[\Gamma(1-\iota)\varphi_1^{\wp_*-\iota}-\Gamma(1-\iota)\varphi_2^{\wp_*-\iota}\Big]$$

$$+\frac{\sigma^*\sigma^{1-\wp_*}(\|\alpha\|_\Psi+\|\beta\|_\Psi\|w\|_\Psi)}{\Gamma(\wp^*)}\varphi_1^{-\iota}(\varphi_2-\varphi_1)^{\wp_*}.$$

Hence $|(\Im w)(\varphi_2)-(\Im w)(\varphi_1)|\to 0$ as $|\varphi_2-\varphi_1|\to 0$. It implies that $\Im(B_{R_\iota})$ is **Equi-cont**. As a sequence of Steps 1 to 3 together with **Arzella Ascolli thm**, we conclude that $\Im$ is **Compl cont**.
Then by **Schauder fix pt thm** the (CFDPNFDEVO(2)) has at least one solution.

Then, we give an existence the second result based on **Banach contr princip**.

Theorem 3.2.2 *Assume that conditions* $(A1),(A2)$ *hold, and if*

$$\frac{\ell\sigma^*\sigma^{1-\wp_*}\ \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}<1, \tag{3.6}$$

then the (CFDPNFDEVO(2)) has a unique solution in Ψ.

Proof We shall use the **Banach contr princip** to prove that $\Im$ be defined in (3.3) has **Fix pt**.

To show that $\Im$ admits a unique **Fix pt**, it suffices to show that $\Im$ is a contraction. For $w_1(\varphi), w_2(\varphi) \in \Psi$, we obtain that:

$$\begin{aligned}
\Big|\Im(w_1)(\varphi) - \Im(w_2)(\varphi)\Big| &= \Big|\int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s, w_1(s))ds - \int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s, w_2(s))ds\Big| \\
&\leq \int_0^\varphi \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))}|\Upsilon(s, w_1(s)) - \Upsilon(s, w_2(s))|ds \\
&\leq \frac{\ell}{\Gamma(\wp^*)}\int_0^\varphi \sigma^{\wp(s)-1}\left(\frac{\varphi - s}{\sigma}\right)^{\wp(s)-1} s^{-\iota}|w_1(s) - w_2(s)|ds \\
&\leq \frac{\ell}{\Gamma(\wp^*)}\int_0^\varphi \sigma^{\wp_*-1}\left(\frac{\varphi - s}{\sigma}\right)^{\wp_*-1} s^{-\iota}|w_1(s) - w_2(s)|ds \\
&\leq \frac{\ell\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)}\|w_1 - w_2\|_\Psi \int_0^\varphi (\varphi - s)^{\wp_*-1}s^{-\iota}ds \\
&\leq \frac{\ell\sigma^*\sigma^{1-\wp_*}\ \Gamma(\wp^*)\ \Gamma(1-\iota)}{\Gamma(\wp^*)\ \Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|w_1 - w_2\|_\Psi \\
&\leq \frac{\ell\sigma^*\sigma^{1-\wp_*}\ \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|w_1 - w_2\|_\Psi,
\end{aligned}$$

by bound φ on D we find

$$\|\Im(w_1) - \Im(w_2)\|_\Psi \leq \frac{\ell\sigma^*\sigma^{1-\wp_*}\ \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|w_1 - w_2\|_\Psi.$$

Consequently by (3.6), the operator $\Im$ is a contraction. Hence, by **Banach contr princip**, $\Im$ has a unique **Fix pt** $w \in \Psi$, which is a unique solution of the (CFDPNFDEVO(2)).

3.3 Ulam-Hyers stability

Theorem 3.3.1 *Let the conditions* $(A1)$ *and* $(A2)$,$(A3)$ *hold, then the (CFDPNFDEVO(2)) is* ***Ula Hyer stab****.*

Proof 3.3.1 *Let* $\varepsilon > 0$ *an arbitrary number and the function* $\chi(\varphi)$ *from* $\chi \in \Psi$ *satisfy the following inequality*

$$|D^{\wp(\varphi)}\chi(\varphi) - \Upsilon(\varphi, \chi(\varphi))| < \varepsilon, \quad \varphi \in D$$

we have

$$D^{\wp(\varphi)}\chi(\varphi)=\int_0^{\varphi}\frac{(\varphi-s)^{1-\wp(s)}}{\Gamma(1-\wp(s))}\chi(s)ds,$$

we obtain

$$\begin{aligned}
\left|\chi(\varphi)-\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,\chi(s))ds\right| &\leq \varepsilon\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}ds\\
&\leq \frac{\varepsilon}{\Gamma(\wp^*)}\int_0^{\varphi}\sigma^{\wp(s)-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp(s)-1}s^{-\iota}s^{\iota}ds\\
&\leq \frac{\varepsilon}{\Gamma(\wp^*)}\int_0^{\varphi}\sigma^{\wp_*-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}s^{\iota}ds\\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)}\int_0^{\varphi}(\varphi-s)^{\wp_*-1}s^{-\iota}s^{\iota}ds\\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\int_0^{\varphi}s^{\iota}ds\\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\left[\frac{s^{\iota+1}}{\iota+1}\right]_0^{\varphi}\\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1}.
\end{aligned}$$

Let $\varphi\in D$, *we get*

$$\begin{aligned}
|\chi(\varphi)-w(\varphi)| &\leq |\frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1}+\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,\chi(s))ds\\
&\quad-\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,w(s))ds|\\
&\leq |\frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1}|+\int_0^{\varphi}\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s)}|\Upsilon(s,\chi(s))-\Upsilon(s,w(s))|ds\\
&\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1}+\frac{\ell}{\Gamma(\wp^*)}\int_0^{\varphi}\sigma^{\wp_*-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}|\chi(s)-w(s)|ds
\end{aligned}$$

$$\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell\sigma^*\sigma^{1-\wp_*}\|\chi-w\|_\Psi}{\Gamma(\wp^*)}\int_0^\varphi(\varphi-s)^{\wp_*-1}s^{-\iota}ds$$

$$\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|\chi-w\|_\Psi$$

$$\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|\chi-w\|_\Psi.$$

Then

$$\|\chi-w\|_\Psi - \frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|\chi-w\|_\Psi \leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1},$$

$$\|\chi-w\|_\Psi(1-\frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}) \leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1}.$$

We obtain, for each $\varphi\in D$

$$|\chi(\varphi)-w(\varphi)| \leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} \times \frac{\Gamma(1-\iota+\wp^*)}{\Gamma(1-\iota+\wp^*)-\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota})}$$

$$\leq \frac{\varepsilon\sigma^*\sigma^{1-\wp_*}\sigma^{\wp_*-\iota}\varphi^{\iota+1}\Gamma(1-\iota)}{(\iota+1)\Big(\Gamma(1-\iota+\wp^*)-\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota})\Big)} = c_\Upsilon\varepsilon.$$

*then the (CFDPNFDEVO(2)) is **Ula Hyer stab**.*

3.4 Numerical example application

Consider the following fractional problem,

$$\begin{cases} D_{0^+}^{\wp(\varphi)}w(\varphi) &= \Upsilon(\varphi,w(\varphi)),\ \ \varphi\in D=[0,1] \\ \\ w(\varphi) &= \eta(\varphi)=sin(\varphi),\ \ \varphi\in[-1,0] \end{cases} \tag{3.7}$$

where $\wp(\varphi)=\frac{\varphi}{2}+\frac{1}{2}$ is a Cont funct on $[\frac{1}{2},1]$.
$\Upsilon(\varphi,w)=\dfrac{\frac{|w|}{2}+1}{\varphi^{\frac{1}{6}}}$ is a Cont funct on $(0,1]\times\mathbb{R}$.

For $\varphi\in[0,1]$, we have

$$\frac{1}{2}\leq\wp(\varphi)\leq 1,$$

Then, we have

$$\begin{aligned}
\varphi^{\frac{1}{6}}|\Upsilon(\varphi,w_1)-\Upsilon(\varphi,w_2)| &= \left|\varphi^{\frac{1}{6}}\left(\frac{\frac{|w_1|}{2}+1}{\varphi^{\frac{1}{6}}}-\frac{\frac{|w_2|}{2}+1}{\varphi^{\frac{1}{6}}}\right)\right| \\
&= \left|\frac{|w_1|}{2}-\frac{|w_2|}{2}\right| \\
&= \frac{1}{2}\Big||w_1|-|w_2|\Big| \\
&\leq \frac{1}{2}|w_1-w_2|.
\end{aligned}$$

Hence the condition (A2) holds with $\iota=\frac{1}{6}$ and $\ell=\frac{1}{2}$, $\sigma^*=1$.
Next, we prove that the condition (3.6)

$$\frac{\ell\sigma^*\sigma^{1-\wp_*}\,\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}=\frac{1}{2}\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{11}{6})}\simeq 0.600425....<1$$

Accordingly the condition (3.6) is achieved. By Theorem (3.2.2), the problem (3.7) has unique solution. In this section, we presente our solution $w(\varphi)$ for $\wp(\varphi)=\frac{1}{2}\varphi+\frac{1}{2}$ with $\varphi\in[0,1]$ and $w_i(\varphi)$ for $\wp(\varphi_i)=\frac{1}{2}\varphi_i+\frac{1}{2}$ where φ_i is fixed. In figure (3.1), we plot the solution z depending on u and the figure (3.2) present a comparison between the solution w and the different solutions w_i with a $\wp$ different.

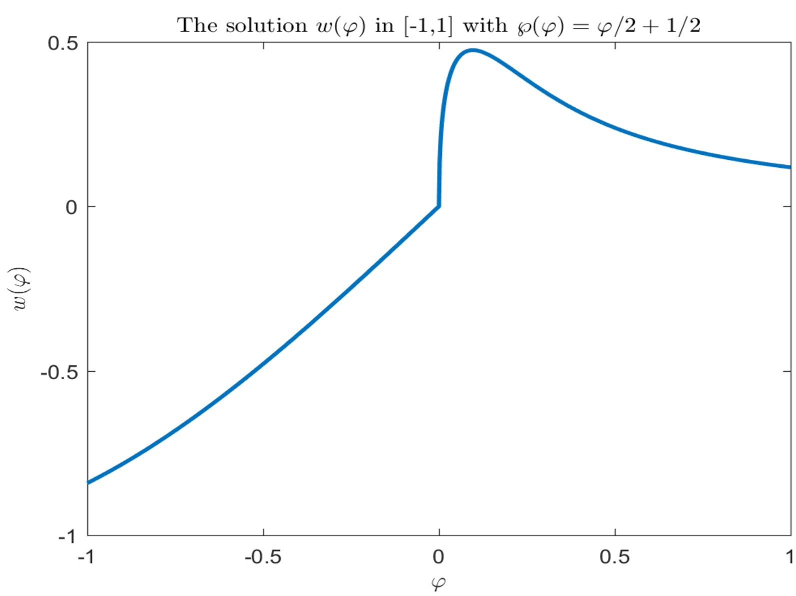
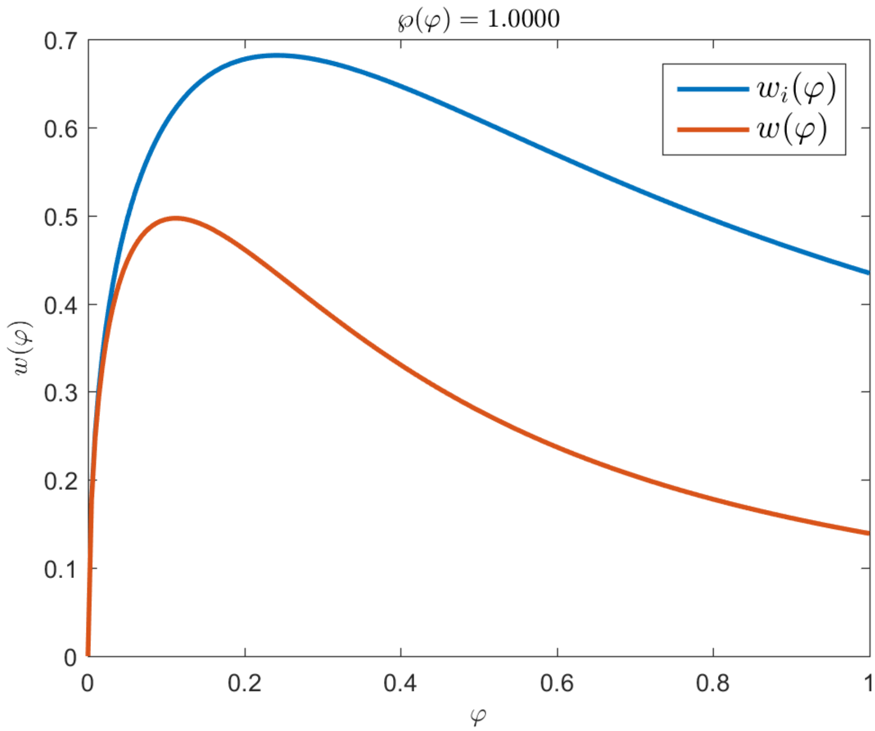
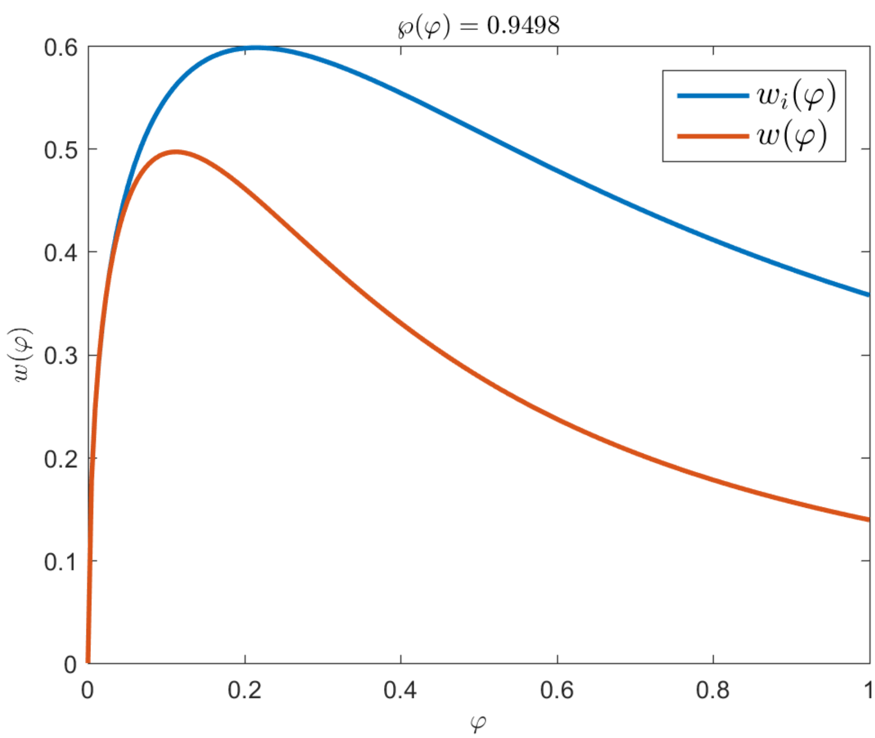
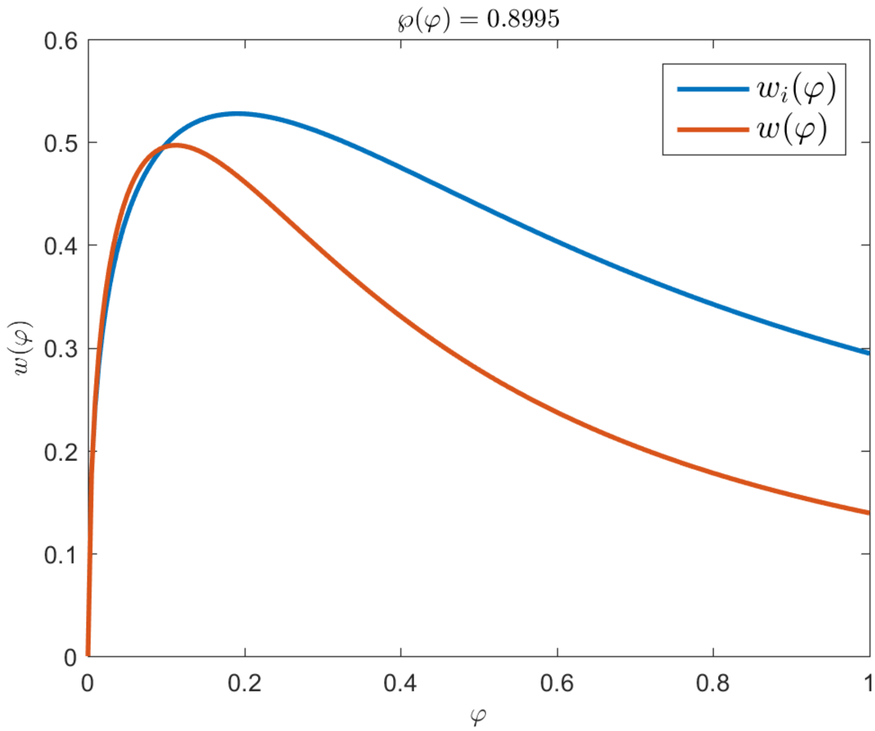


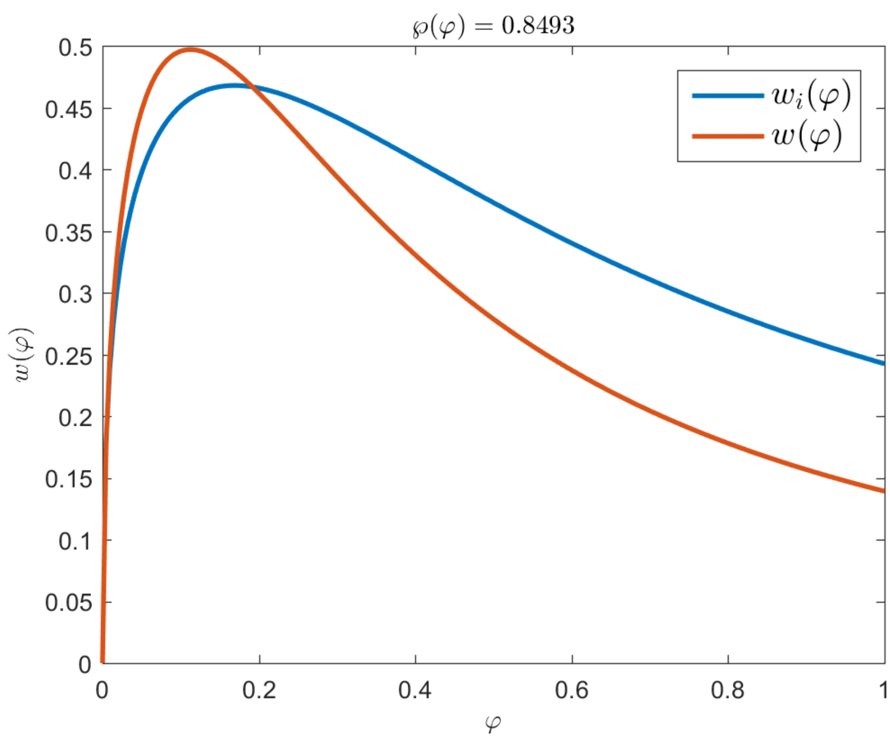
Figure 3.1: The solution $w(\varphi)$ in $[-1,1]$ with $\wp(\varphi)=\frac{\varphi}{2}+\frac{1}{2}$


℘(φ) = 1.0000
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
w(φ)
φ
w_i(φ)
w(φ)

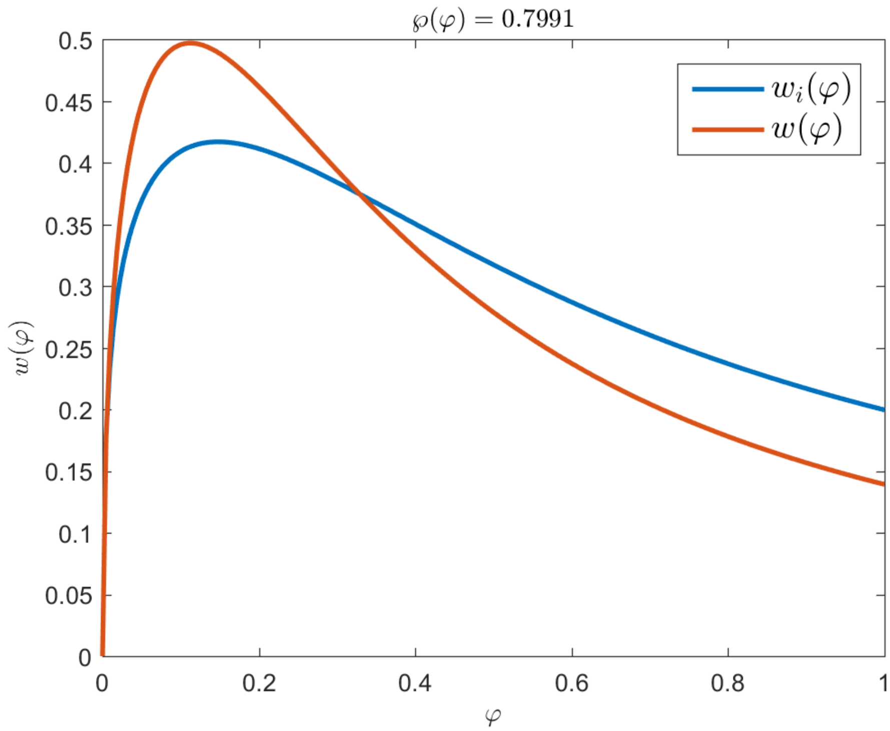

℘(φ) = 0.9498
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
w(φ)
φ
w_i(φ)
w(φ)



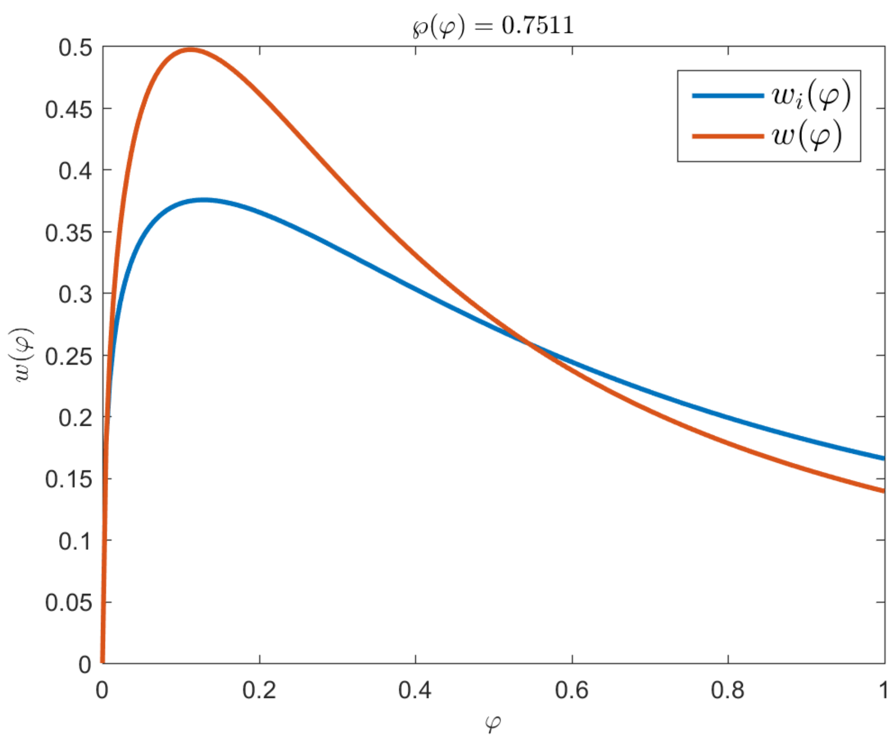
$\wp(\varphi) = 0.8995$
$w_i(\varphi)$
$w(\varphi)$
0.6
0.5
0.4
0.3
0.2
0.1
0
$w(\varphi)$
0
0.2
0.4
0.6
0.8
1
φ



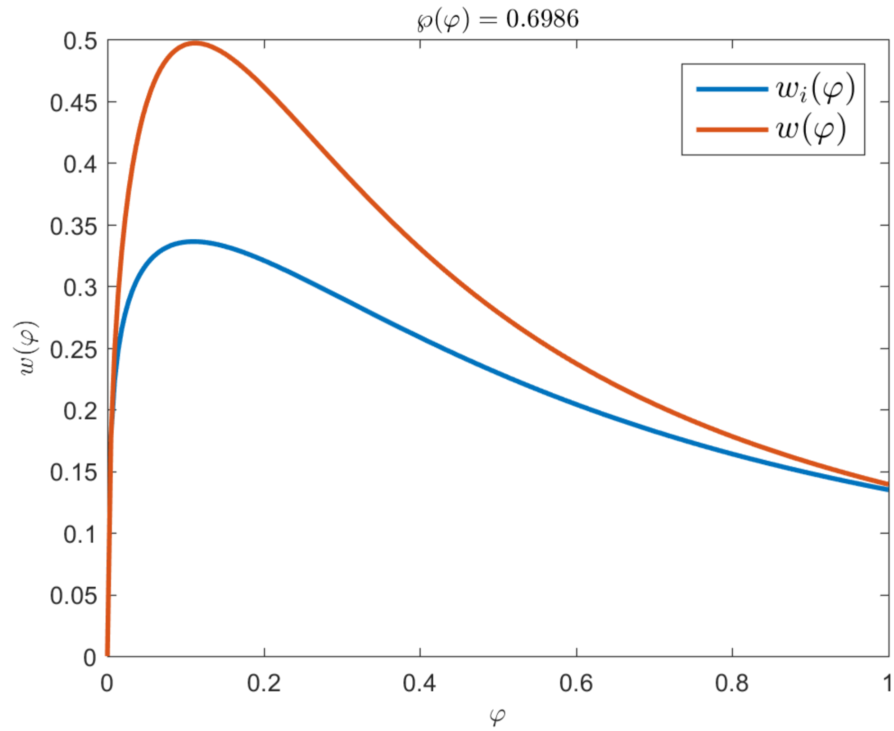
$\wp(\varphi) = 0.8493$
$w_i(\varphi)$
$w(\varphi)$
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
$w(\varphi)$
0
0.2
0.4
0.6
0.8
1
φ

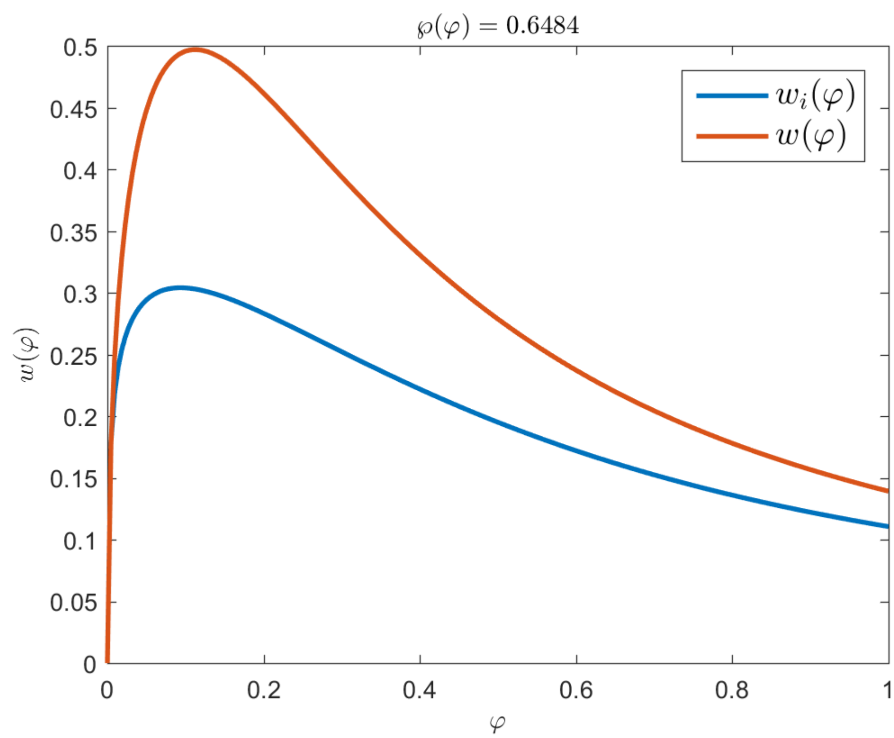


$\wp(\varphi) = 0.7991$
$w_i(\varphi)$
$w(\varphi)$
$w(\varphi)$
φ



$\wp(\varphi) = 0.7511$
$w_i(\varphi)$
$w(\varphi)$
$w(\varphi)$
φ


$\wp(\varphi) = 0.6986$
$w_i(\varphi)$
$w(\varphi)$
$w(\varphi)$
φ


$\wp(\varphi) = 0.6484$
$w_i(\varphi)$
$w(\varphi)$
$w(\varphi)$
φ

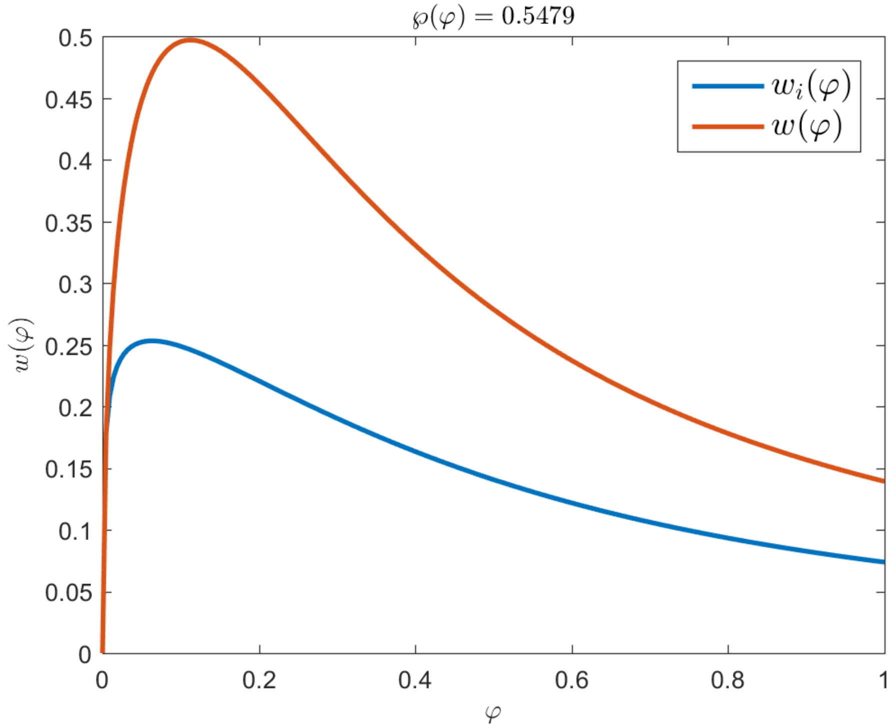


Figure 3.2: The plot of $w(\varphi)$, $w_i(\varphi)$ for different $\wp(\varphi)$

In this table, we present the $Norm_i = \max_{\varphi \in [0,1]} \Big| w(\varphi) - w_i(\varphi) \Big|$ for $\wp(\varphi) \in [0,1]$.

φ_i	0.0959	0.1963	0.2968	0.3973	0.5023	0.5982	0.6986	0.7991	0.8995	1
$\wp(\varphi_i)$	0.5479	0.5982	0.6484	0.6986	0.7511	0.7991	0.8493	0.8995	0.9498	1
$Norm_i$	0.2534	0.2254	0.1945	0.1609	0.1233	0.0879	0.1068	0.1670	0.2415	0.3320

We observe that the norm is small when $\wp(\varphi_i)$ is close to 0,8. Almost the center of the interval.

Chapter 4

Modern Technique to Study Cauchy-Type Problem of Fractional Variable Order

4.1 Introduction and motivations

[1] The Piece-Wise Constant Function will play a vital role in our study for converting the fractional problem of varaible order to an equivalent standard fractional problem of the constant order.

Benchohra et al [5] studied the existence of solutions for the following nonlinear fractional differential equations for constant order:

$$\begin{cases} D^{\varrho}_{0+}\xi(s) = \varphi(s, \xi_s); \quad s \in \mathcal{N} := [0, N] \quad (1) \\ \\ \xi(s) = \chi(s); \qquad\qquad s \in [-\gamma, 0], \gamma > 0 \quad (2) \end{cases} \tag{4.1}$$

Where D^{ϱ}_{0+} is standard Riemann-Liouville fractional derivative , $\varphi : \mathcal{N} \times C([-\gamma, 0], \mathbb{R}) \to \mathbb{R}$ is a given function $\chi \in C([-\gamma, 0], \mathbb{R})$ via $\chi(0) = 0$. For any function ξ defined on $[-\gamma, N]$ and any $s \in \mathcal{N}$, we denote by ξ_s the element of $C([-\gamma, 0], \mathbb{R})$ defined by

$$\xi_s(\tau) = \xi(s + \tau), \quad \tau \in [-\gamma, 0].$$

Since the authors in [5] consider an infinite delay, the obtained existence results can be examined as a generalization of several existence results for delayed fractional differential

[1] **M. Benaouda**, S. Sabit, H. Gunerhan, M. S. Souid, Modern Technique to Study Cauchy-Type Problem of Fractional Variable Order, *Journal of Modern Physics Letters B*, **2024**(2024).

equations with fractional constant order derivatives. In fact, there have been some important existence results for such equations where different techniques have been applied. [21, 25] How- ever, as stated above, the corresponding results for delayed fractional varaible order boundary-value problems are very few.

In this chapter we apply the new technique on the following fractional Cauchy-type problem (CPNFDEVOID(3))

$$\begin{cases} D_{0^+}^{\wp(\varphi)} w(\varphi) = \Upsilon(\varphi, w(\varphi)) \ \ \varphi \in D = [0, \sigma] \\ \\ w(\varphi) = \eta(\varphi), \quad \varphi \in]-\infty, 0] \end{cases} \tag{CPNFDEVOID(3)}$$

where $0 < \sigma < +\infty$, $0 < \wp(\varphi) \leq 1$, $\Upsilon : D \times \mathfrak{B} \to \mathbb{R}$ is a Continuous Functions **Cont funct** and $D_{0^+}^{\wp(\varphi)}$, is the Riemann-Liouville Fractional Derivative of Variable-Order $\wp(\varphi)$, $0 < \wp(\varphi) \leq \wp^* \leq 1$ and $\eta(\varphi) \in \mathfrak{B}$ with $\eta(0) = 0$ and $\mathfrak{B}$ is **Phase sp** .

For each function w defined on $]-\infty, \sigma]$ and each $\varphi \in D$, we note by w_φ the element of $\mathfrak{B}$ defined by

$$w_\varphi(\jmath) = w(\varphi + \jmath), \quad \jmath \in]-\infty, 0]$$

Here $w_\varphi(.)$ represents the history of the state from time ∞ up to the present time φ.

4.2 Study existence and uniqueness of solutions

Let us start by defininig what we mean by a solution of problem (CPNFDEVOID(3)). Let the space

$$\Omega = \{w : (\infty, \sigma] \to \mathbb{R} : w|_{(\infty,0]} \in \mathfrak{B} \text{ and } w|_{[0,\sigma]} \text{ is continuous } \}.$$

We impose the following assumptions:

(C1) $\wp : [\frac{1}{2}, \sigma] \to (\frac{1}{2}, \wp^*]$ is **Cont funct**, such that $\frac{1}{2} < \wp(\varphi) \leq \wp^* \leq 1$.

(C2) Let $\varphi^\iota \Upsilon : D \times \mathfrak{B} \to \mathbb{R}$ is **Cont funct** $(0 \leq \iota \leq 1)$, there exist a constants $\ell > 0$, such that,

$$\varphi^\iota |\Upsilon(\varphi, u) - \Upsilon(\varphi, v)| \leq \ell \|u - v\|_{\mathfrak{B}} \text{ for each } u, v, \in \mathfrak{B} \text{ and } \varphi \in D.$$

(C3) There existe $\alpha, \beta \in C(D, \mathbb{R}^+)$ such that

$$\varphi^{\iota}\|\Upsilon(\varphi, u)\| \leq \alpha(\varphi) + \beta(\varphi)\|u\|_{\mathfrak{B}},$$

for $\varphi \in D$ and each $u \in \mathfrak{B}$, and $\|I^{\wp^*}p\|_{\Psi} < \infty$.

Definition 4.2.1 *A function $w \in \Omega$ is said to be a solution for (CPNFDEVOID(3)) if w satisfies the equation $D^{\wp(\varphi)} = \Upsilon(\varphi, w(\varphi))$ on D and the condition $w(\varphi) = \eta(\varphi)$ on $(-\infty, 0]$.*

For the existence of solutions for the (CPNFDEVOID(3)), an auxiliary lemma is needed as follows:

Lemma 4.2.1 *[74] Let $0 < \wp(\varphi) < 1$ and let $F : (0, \sigma] \to \mathbb{R}$ be continuous and $lim_{\varphi \to 0^+} F(\varphi) = F(0^+) \in \mathbb{R}$. Then w is a solution of Fractional differential equations*

$$w(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} F(s)ds, \quad \varphi \in D \text{ and } \wp(s) > 0,$$

*if and only if w is a solution of the **IVP** for the Fractional differential equations*

$$\begin{cases} D^{\wp(\varphi)} = F(\varphi), & \varphi \in (0, \sigma] \\ w(0) = 0. \end{cases} \tag{4.2}$$

Lemma 4.2.2 *[74] Let $(C1)$ hold. And let $w_n, w \in \mathfrak{B}$ assume that*

$$w_n(\varphi) \to w(\varphi), \ \varphi \in D \text{ as } n \to \infty,$$

then

$$\int_0^{\varphi} \frac{(\varphi - s)^{\wp(\varphi)}}{\Gamma(1 - \wp(\varphi))} w_n(s)ds \to \int_0^{\varphi} \frac{(\varphi - s)^{\wp(\varphi)}}{\Gamma(1 - \wp(\varphi))} w(s)ds, t \in [0, \sigma] \text{ as } n \to \infty.$$

The first result obtained by using the **Schauder fix pt thm** .

Theorem 4.2.1 *Suppossing that conditions $(C1), (C2)$ and $(C3)$ are hold, if*

$$\frac{\sigma^* \ \sigma^{-\iota+1} \ (\|\alpha\|_{\Psi} + \|\beta\|_{\Psi} L_{\sigma}\|\eta\|_{\mathfrak{B}})}{(-\iota + 1)\Gamma(\wp^*) - \sigma^* \ \sigma^{-\iota+1}\|\beta\|_{\Psi}\kappa_{\sigma}} < 1$$

Where

$$\kappa_{\sigma} = sup\{|\kappa(\varphi)| : \varphi \in D\},$$

Then the (CPNFDEVOID(3)) has at least one solution on Ψ.

Proof We give the operator $\Phi : \Omega \to \Omega$ defined by :

$$\Phi(w)(\varphi) = \begin{cases} \eta(\varphi), & if\ \varphi \in (-\infty, 0] \\ \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w_s) ds, & if\ \varphi \in D \quad and \quad \wp(s) > 0 \end{cases} \tag{4.3}$$

Let $\varsigma(.) : (-\infty, \sigma] \to \mathrm{I\!R}$ be the function defined by

$$\varsigma(\varphi) = \begin{cases} 0, & if\ \varphi \in [0, \sigma] \\ \eta(\varphi), & if\ \varphi \in (-\infty, 0] \end{cases}$$

Then $\varsigma_0 = \eta$. For each $\upsilon \in C([0, \sigma], \mathrm{I\!R})$ with $\upsilon(0) = 0$, we denote υ^* the function defined by

$$\upsilon^*(\varphi) = \begin{cases} \upsilon(\varphi), & if\ \varphi \in [0, \sigma] \\ 0, & if\ \varphi \in (-\infty, 0] \end{cases}$$

If $w(.)$ satisfies the integral equation

$$w(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, w_s) ds,$$

we can decompose $w(.)$ as $w(\varphi) = \upsilon^*(\varphi) + \varsigma(\varphi), 0 \leq \varphi \leq \sigma$, and the function $\upsilon(.)$ satisfies

$$\upsilon(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, \upsilon^*(s) + \varsigma(s)) ds,$$

Set M_0 is **Banach sp** with norm $\|.\|_\sigma$ such that

$$M_0 = \{\upsilon \in C([0, \sigma], \mathrm{I\!R}) : \upsilon_0 = 0\},$$

and let $\|.\|_\sigma$ in M_0 defined by:

$$\|\upsilon\|_\sigma = \|\upsilon_0\|_{\mathfrak{B}} + sup\{|\upsilon(\varphi)| : 0 \leq \varphi \leq \sigma\} = sup\{|\upsilon(\varphi)| : 0 \leq \varphi \leq \sigma\}, \quad \upsilon \in M_0$$

We give the operator $\Im : M_0 \to M_0$ defined by:

$$(\Im \upsilon)(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, \upsilon^*(s) + \varsigma(s)) ds, \quad \varphi \in D. \tag{4.4}$$

We shall use the **Schauder fix pt thm** to prove that $\Im$ is **Fix pt**.
We consider the set

$$B_{R_\iota} = \{\upsilon \in M_0, \|\upsilon\|_\sigma \leq R_\iota\},$$

where

$$R_\iota = \frac{\sigma^* \ \sigma^{1-\wp_*} \ \Gamma(1-\iota)(\|\alpha\|_\Psi + \|\beta\|_\Psi L_\sigma \|\eta\|_{\mathfrak{B}}) \sigma^{\wp_* - \iota}}{\Gamma(1 - \iota + \wp^*) - \sigma^* \ \sigma^{1-\wp_*} \ \Gamma(1-\iota) \|\beta\|_\Psi \kappa_\sigma \sigma^{\wp_* - \iota}}.$$

Clearly $\mathfrak{B}_{R_\iota}$ is **Convex, closed bounded non-empty**.

We proved this in three steps.

`Step 1`: $\Im$ is **Cont ope**.

Let υ_n be a sequence such that $\upsilon_n \to \upsilon$ in M_0 then

$$\|(\Im\upsilon_n) - (\Im\upsilon)\|_\sigma \to 0.$$

For $\varphi \in [0,\sigma]$, we find

$$\begin{aligned}
|\Im(\upsilon_n)(\varphi) - \Im(\upsilon)(\varphi)| &= \Big| \int_0^\varphi \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))} \varUpsilon(s, \upsilon_n^*(s) + \varsigma(s))ds \\
&- \int_0^\varphi \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))} \varUpsilon(s, \upsilon^*(s) + \varsigma(s))ds \Big| \\
&\leq \int_0^\varphi \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))} |\varUpsilon(s, \upsilon_n^*(s) + \varsigma(s)) - \varUpsilon(s, \upsilon^*(s) + \varsigma(s))| ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\ell}{\Gamma(\wp^*)} \int_0^\varphi (\varphi-s)^{\wp(s)-1} s^{-\iota} \|\upsilon_n^*(s) - \upsilon^*(s)\|_{\mathfrak{B}} ds \\
&\leq \frac{\ell}{\Gamma(\wp^*)} \int_0^\varphi \sigma^{\wp(s)-1} \left(\frac{\varphi-s}{\sigma}\right)^{\wp(s)-1} s^{-\iota} \|\upsilon_n^*(s) - \upsilon^*(s)\|_{\mathfrak{B}} ds \\
&\leq \frac{\ell}{\Gamma(\wp^*)} \int_0^\varphi \sigma^{\wp_*-1} \left(\frac{\varphi-s}{\sigma}\right)^{\wp_*-1} s^{-\iota} \kappa(s) sup_{s\in[0,\varphi]} \|\upsilon_n(s) - \upsilon(s)\| ds \\
&\leq \frac{\ell\ \sigma^* \kappa_\sigma}{\Gamma(\wp^*)} \sigma^{1-\wp_*} \int_0^\varphi (\varphi-s)^{\wp_*-1} s^{-\iota}\ ds \|\upsilon_n - \upsilon\|_\sigma \\
&\leq \frac{\ell\sigma^*\ \sigma^{1-\wp_*}\ \kappa_\sigma\ \Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)} \sigma^{\wp_*-\iota} \|\upsilon_n - \upsilon\|_\sigma \\
\|\Im(\upsilon_n) - \Im(\upsilon)\|_\sigma &\leq \frac{\ell\sigma^*\ \sigma^{1-\wp_*}\ \kappa_\sigma\ \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)} \sigma^{\wp_*-\iota} \|\upsilon_n - \upsilon\|_\sigma,
\end{aligned}$$

$$\|(\Im\upsilon_n) - (\Im\upsilon)\|_\sigma \to 0 \quad as \quad n \to \infty$$

Consequently, $\Im$ is **Cont ope** on M_0.

`Step 2`: $\Im(\mathfrak{B}_{R_\iota}) \subseteq (\mathfrak{B}_{R_\iota})$. For $\upsilon \in \mathfrak{B}_{R_\iota}$, and by $(C3)$, we get:

$$|\Im(\upsilon)(\varphi)| = \Big| \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, \upsilon_s^* + \varsigma_s) ds \Big|$$

$$\leq \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))} |\Upsilon(s, \upsilon_s^* + \varsigma_s)| ds$$

$$\leq \frac{1}{\Gamma(\wp^*)} \int_0^{\varphi} \sigma^{\wp(s)-1} \left(\frac{\varphi - s}{\sigma}\right)^{\wp(s)-1} s^{-\iota} \Big(|\alpha(s)| + |\beta(s)| \|\upsilon_s^* + \varsigma_s\|_{\mathfrak{B}} \Big) ds$$

$$\leq \frac{1}{\Gamma(\wp^*)} \int_0^{\varphi} \sigma^{\wp_*-1} \left(\frac{\varphi - s}{\sigma}\right)^{\wp_*-1} \Big(s^{-\iota} |\alpha(s)| + s^{-\iota} |\beta(s)| \|\upsilon_s^* + \varsigma_s\|_{\mathfrak{B}} \Big) ds$$

$$\leq \frac{\sigma^*}{\Gamma(\wp^*)} \sigma^{1-\wp_*} \int_0^{\varphi} (\varphi - s)^{\wp_*-1} \Big(s^{-\iota} |\alpha(s)| + s^{-\iota} |\beta(s)| \|\upsilon_s^* + \varsigma_s\|_{\mathfrak{B}} \Big) ds$$

$$\leq \frac{\sigma^* \sigma^{1-\wp_*}}{\Gamma(\wp^*)} \int_0^{\varphi} (\varphi - s)^{\wp_*-1} s^{-\iota} |\alpha(s)| ds$$
$$+ \frac{\sigma^* \sigma^{1-\wp_*}}{\Gamma(\wp^*)} \int_0^{\varphi} (\varphi - s)^{\wp_*-1} s^{-\iota} |\beta(s)| \|\upsilon_s^* + \varsigma_s\|_{\mathfrak{B}} \Big) ds$$

$$\leq \frac{\sigma^* \sigma^{1-\wp_*} \|\alpha\|_{\Psi}}{\Gamma(\wp^*)} \int_0^{\varphi} (\varphi - s)^{\wp_*-1} s^{-\iota} ds$$
$$+ \frac{\sigma^* \sigma^{1-\wp_*} \|\beta\|_{\Psi}}{\Gamma(\wp^*)} \int_0^{\varphi} (\varphi - s)^{\wp_*-1} s^{-\iota} \Big(\|\upsilon_s^*\|_{\mathfrak{B}} + \|\varsigma_s\|_{\mathfrak{B}} \Big) ds$$

$$\leq \frac{\sigma^* \sigma^{1-\wp_*} \|\alpha\|_{\Psi}}{\Gamma(\wp^*)} \int_0^{\varphi} (\varphi - s)^{\wp_*-1} s^{-\iota} ds$$

$$+ \frac{\sigma^* \sigma^{1-\wp_*} \|\beta\|_{\Psi}}{\Gamma(\wp^*)} \Big(\kappa_\sigma \|\upsilon\|_\sigma + L_\sigma \|\eta\|_{\mathfrak{B}} \Big) \int_0^{\varphi} (\varphi - s)^{\wp_*-1} s^{-\iota} ds$$

$$\leq \frac{\sigma^* \sigma^{1-\wp_*} \|\alpha\|_{\Psi} \Gamma(\wp^*) \Gamma(1-\iota)}{\Gamma(\wp^*) \Gamma(1 - \iota + \wp^*)} \sigma^{\wp_* - \iota}$$

$$+\frac{\sigma^*\sigma^{1-\wp_*}\|\beta\|_\Psi\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)$$
$$\leq\frac{\sigma^*\sigma^{1-\wp_*}\|\alpha\|_\Psi\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}+\frac{\sigma^*\sigma^{1-\wp_*}\|\beta\|_\Psi\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)$$

$$\leq\frac{\sigma^*\ \sigma^{1-\wp_*}\ \Gamma(1-\iota)(\|\alpha\|_\Psi+\|\beta\|_\Psi L_\sigma\|\eta\|_{\mathfrak{B}})}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}+\frac{\sigma^*\ \sigma^{1-\wp_*}\ \Gamma(1-\iota)\|\beta\|_\Psi\kappa_\sigma}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|\upsilon\|_\sigma$$

$$\leq\frac{\sigma^*\ \sigma^{1-\wp_*}\ \Gamma(1-\iota)(\|\alpha\|_\Psi+\|\beta\|_\Psi L_\sigma\|\eta\|_{\mathfrak{B}})}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}$$

$$\times\frac{\Gamma(1-\iota+\wp^*)}{\Gamma(1-\iota+\wp^*)-\sigma^*\ \sigma^{1-\wp_*}\ \Gamma(1-\iota)\|\beta\|_\Psi\kappa_\sigma\sigma^{\wp_*-\iota}}$$
$$\leq\frac{\sigma^*\ \sigma^{1-\wp_*}\ \Gamma(1-\iota)(\|\alpha\|_\Psi+\|\beta\|_\Psi L_\sigma\|\eta\|_{\mathfrak{B}})\sigma^{\wp_*-\iota}}{\Gamma(1-\iota+\wp^*)-\sigma^*\ \sigma^{1-\wp_*}\ \Gamma(1-\iota)\|\beta\|_\Psi\kappa_\sigma\sigma^{\wp_*-\iota}}=R_\iota$$

where:

$$\|\upsilon_s^*+\varsigma_s\|_{\mathfrak{B}}\leq\|\upsilon_s^*\|_{\mathfrak{B}}+\|\varsigma_s\|_{\mathfrak{B}}\leq\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}},$$

and

$$L_\sigma=sup\{|L(\varphi)|:\varphi\in D\},$$

which means that $\Im(\mathfrak{B}_{R_\iota})\subseteq(\mathfrak{B}_{R_\iota})$.

`Step 3` :$\Im$ is **Compact ope**

Now, we will show that $\Im(\mathfrak{B}_{R_\iota})$ is **Rrlat comp**, meaning that $\Im$ is **Compact op**. Clearly $\Im(\mathfrak{B}_{R_\iota})$ is **Unif bnd** because by Step 2, we obtain $\Im(\mathfrak{B}_{R_\iota})=\{\Im(\upsilon):\upsilon\in\mathfrak{B}_{R_\iota}\}\subset\mathfrak{B}_{R_\iota}$ thus for each $\upsilon\in\mathfrak{B}_{R_\iota}$ we get $\|\Im(\upsilon)\|_{M_0}\leq R_\iota$ which means that $\Im(\mathfrak{B}_{R_\iota})$ is **Unif bnd**. It remains to show that $\Im(\mathfrak{B}_{R_\iota})$ is **Equi-cont**.

For $\varphi_1, \varphi_2 \in D,\ \varphi_1 < \varphi_2$ and $\varsigma \in \mathfrak{B}_{R_\iota}$, we have:

$|\Im(\upsilon)(\varphi_2) - \Im(\upsilon)(\varphi_1)|$

$$= \Big| \int_0^{\varphi_2} \frac{(\varphi_2 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, \upsilon_s^* + \varsigma_s) ds - \int_0^{\varphi_1} \frac{(\varphi_1 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, \upsilon_s^* + \varsigma_s) ds \Big|$$

$$= \Big| \int_0^{\varphi_1} \frac{(\varphi_2 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, \upsilon_s^* + \varsigma_s) ds + \int_{\varphi_1}^{\varphi_2} \frac{(\varphi_2 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, \upsilon_s^* + \varsigma_s) ds$$

$$- \int_0^{\varphi_1} \frac{(\varphi_1 - s)^{\wp(s)-1}}{\Gamma(\wp(s))} \Upsilon(s, \upsilon_s^* + \varsigma_s) ds \Big|$$

$$\leq \int_0^{\varphi_1} \Big[(\varphi_2 - s)^{\wp(s)-1} - (\varphi_1 - s)^{\wp(s)-1} \Big] \Big| \frac{\Upsilon(s, \upsilon_s^* + \varsigma_s)}{\Gamma(\wp(s))} \Big| ds$$

$$+ \int_{\varphi_1}^{\varphi_2} (\varphi_2 - s)^{\wp(s)-1} \Big| \frac{\Upsilon(s, \upsilon_s^* + \varsigma_s)}{\Gamma(\wp(s))} \Big| ds$$

$$\leq \frac{1}{\Gamma(\wp^*)} \int_0^{\varphi_1} \Big[\sigma^{\wp(s)-1} \Big(\frac{\varphi_2 - s}{\sigma} \Big)^{\wp(s)-1} - \sigma^{\wp(s)-1} \Big(\frac{\varphi_1 - s}{\sigma} \Big)^{\wp(s)-1} \Big] s^{-\iota} \Big(\alpha(s) + \beta(s) \|\upsilon_s^* + \varsigma_s\|_{\mathfrak{B}} \Big) ds$$

$$+ \frac{1}{\Gamma(\wp^*)} \int_{\varphi_1}^{\varphi_2} \sigma^{\wp(s)-1} \Big(\frac{\varphi_2 - s}{\sigma} \Big)^{\wp(s)-1} s^{-\iota} \Big(\alpha(s) + \beta(s) \|\upsilon_s^* + \varsigma_s\|_{\mathfrak{B}} \Big) ds$$

$$\leq \frac{1}{\Gamma(\wp^*)} \int_0^{\varphi_1} \Big[\sigma^{\wp_*-1} \Big(\frac{\varphi_2 - s}{\sigma} \Big)^{\wp_*-1} - \sigma^{\wp_*-1} \Big(\frac{\varphi_1 - s}{\sigma} \Big)^{\wp_*-1} \Big] s^{-\iota} \Big(\alpha(s) + \beta(s) \|\upsilon_s^* + \varsigma_s\|_{\mathfrak{B}} \Big) ds$$

$$+ \frac{1}{\Gamma(\wp^*)} \int_{\varphi_1}^{\varphi_2} \sigma^{\wp_*-1} \Big(\frac{\varphi_2 - s}{\sigma} \Big)^{\wp_*-1} s^{-\iota} \Big(\alpha(s) + \beta(s) \|\upsilon_s^* + \varsigma_s\|_{\mathfrak{B}} \Big) ds$$

$$\leq \frac{\sigma^*}{\Gamma(\wp^*)} \int_0^{\varphi_1} \Big[\Big(\frac{\varphi_2 - s}{\sigma} \Big)^{\wp_*-1} - \Big(\frac{\varphi_1 - s}{\sigma} \Big)^{\wp_*-1} \Big] s^{-\iota} \Big(\alpha(s) + \beta(s) \|\upsilon_s^* + \varsigma_s\|_{\mathfrak{B}} \Big) ds$$

$$+\frac{\sigma^*}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}\Big(\alpha(s)+\beta(s)\|\upsilon_s^*+\varsigma_s\|_{\mathfrak{B}}\Big)ds$$

$$\leq \frac{\sigma^*}{\Gamma(\wp^*)}\int_0^{\varphi_1}\left[\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]s^{-\iota}\alpha(s)ds$$

$$+\frac{\sigma^*}{\Gamma(\wp^*)}\int_0^{\varphi_1}\left[\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}-\left(\frac{\varphi_1-s}{\sigma}\right)^{\wp_*-1}\right]s^{-\iota}\beta(s)\|\upsilon_s^*+\varsigma_s\|_{\mathfrak{B}}ds$$
$$+\frac{\sigma^*}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}\alpha(s)ds+\frac{\sigma^*}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}\left(\frac{\varphi_2-s}{\sigma}\right)^{\wp_*-1}s^{-\iota}\beta(s)\|\upsilon_s^*+\varsigma_s\|_{\mathfrak{B}}ds$$

$$\leq \frac{\sigma^*\|\alpha\|_\Psi}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_0^{\varphi_1}\Big[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\Big]s^{-\iota}ds$$

$$+\frac{\sigma^*\|\beta\|_\Psi\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_0^{\varphi_1}\Big[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\Big]s^{-\iota}ds$$

$$+\frac{\sigma^*\|\alpha\|_\Psi}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds+\frac{\sigma^*\|\beta\|_\Psi\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds$$

$$\leq \frac{\sigma^*\sigma^{1-\wp_*}\Big(\|\alpha\|_\Psi+\|\beta\|_\Psi\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)\Big)}{\Gamma(\wp^*)}\int_0^{\varphi_1}\Big[(\varphi_2-s)^{\wp_*-1}-(\varphi_1-s)^{\wp_*-1}\Big]s^{-\iota}ds$$

$$+\frac{\sigma^*\sigma^{1-\wp_*}\Big(\|\alpha\|_\Psi+\|\beta\|_\Psi\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)\Big)}{\Gamma(\wp^*)}\int_{\varphi_1}^{\varphi_2}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds$$

$$\leq \frac{\sigma^*\sigma^{1-\wp_*}\Big(\|\alpha\|_\Psi+\|\beta\|_\Psi\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)\Big)}{\Gamma(\wp^*)}\Big[\int_0^{\varphi_1}(\varphi_2-s)^{\wp_*-1}s^{-\iota}ds-\int_0^{\varphi_1}(\varphi_1-s)^{\wp_*-1}s^{-\iota}ds\Big]$$

$$+\frac{\sigma^*\sigma^{1-\wp_*}\Big(\|\alpha\|_\Psi+\|\beta\|_\Psi\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)\Big)}{\Gamma(\wp^*)}\varphi_1^{-\iota}(\varphi_2-\varphi_1)^{\wp_*}$$

$$\leq \frac{\sigma^*\sigma^{1-\wp_*}\Big(\|\alpha\|_\Psi+\|\beta\|_\Psi\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)\Big)}{\Gamma(\wp^*)}\Big[\frac{\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\varphi_1^{\wp_*-\iota}-\frac{\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\varphi_2^{\wp_*-\iota}\Big]$$

$$+\frac{\sigma^*\sigma^{1-\wp_*}\Big(\|\alpha\|_\Psi+\|\beta\|_\Psi\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)\Big)}{\Gamma(\wp^*)}\varphi_1^{-\iota}(\varphi_2-\varphi_1)^{\wp_*}$$

Consequently, we obtain

$$|\Im(\upsilon)(\varphi_2)-\Im(\upsilon)(\varphi_1)|\leq\frac{\sigma^*\sigma^{1-\wp_*}\Big(\|\alpha\|_\Psi+\|\beta\|_\Psi\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)\Big)}{\Gamma(1-\iota+\wp^*)}\Big[\Gamma(1-\iota)\varphi_1^{\wp_*-\iota}-\Gamma(1-\iota)\varphi_2^{\wp_*-\iota}\Big]$$

$$+\frac{\sigma^*\sigma^{1-\wp_*}\Big(\|\alpha\|_\Psi+\|\beta\|_\Psi\Big(\kappa_\sigma\|\upsilon\|_\sigma+L_\sigma\|\eta\|_{\mathfrak{B}}\Big)\Big)}{\Gamma(\wp^*)}\varphi_1^{-\iota}(\varphi_2-\varphi_1)^{\wp_*}$$

Hence $|(\Im\upsilon)(\varphi_2)-(\Im\upsilon)(\varphi_1)|\to 0$ as $|\varphi_2-\varphi_1|\to 0$. It implies that $\Im(\mathfrak{B}_{R_\iota})$ is **Equi-cont**. As a sequence of Steps 1 to 3 together with **Arzella Ascolli thm**, we conclude that $\Im$ is **Compl cont**.

`Step 4` (**A priori bounds**): Now we can expose there exists an open set $V\subseteq M_0$ with $\upsilon\neq\lambda\Im(\upsilon)$ for some $0<\lambda<1$. Then for each $\varphi\in[0,\sigma]$ we have

$$\upsilon(\varphi)=\lambda\Big[\int_0^\varphi\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,\upsilon_s^*+\varsigma_s)ds\Big].$$

This implies by $(C3)$

$$\begin{aligned}
|\upsilon(\varphi)| &\leq \int_0^\varphi\frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}|\Upsilon(s,\upsilon_s^*+\varsigma_s)|ds\\
&\leq \frac{1}{\Gamma(\wp^*)}\int_0^\varphi(\varphi-s)^{\wp(s)-1}s^{-\iota}\Big(\alpha(s)+\beta(s)\|\upsilon_s^*+\varsigma_s\|_{\mathfrak{B}}\Big)ds\\
&\leq \frac{1}{\Gamma(\wp^*)}\int_0^\varphi\sigma^{\wp(s)-1}\bigg(\frac{\varphi-s}{\sigma}\bigg)^{\wp(s)-1}\Big(s^{-\iota}\alpha(s)+s^{-\iota}\beta(s)\|\upsilon_s^*+\varsigma_s\|_{\mathfrak{B}}\Big)ds\\
&\leq \frac{1}{\Gamma(\wp^*)}\int_0^\varphi\sigma^{\wp_*-1}\bigg(\frac{\varphi-s}{\sigma}\bigg)^{\wp_*-1}\Big(s^{-\iota}\alpha(s)+s^{-\iota}\beta(s)\|\upsilon_s^*+\varsigma_s\|_{\mathfrak{B}}\Big)ds
\end{aligned}$$

$$\leq \frac{\sigma^*}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_0^{\varphi}(\varphi-s)^{\wp_*-1}s^{-\iota}\beta(s)\|\upsilon_s^*$$

$$+ \varsigma_s\|_{\mathfrak{B}}ds + \frac{\sigma^*\sigma^{1-\wp_*}\|\alpha\|_{\Psi}\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}$$

$$\leq \frac{\sigma^*}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_0^{\varphi}(\varphi-s)^{\wp_*-1}s^{-\iota}\beta(s)\|\upsilon_s^*+\varsigma_s\|_{\mathfrak{B}}ds + \frac{\sigma^*\sigma^{1-\wp_*}\|\alpha\|_{\Psi}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota},$$

but

$$\begin{aligned}\|\upsilon_s^*+\varsigma_s\|_{\mathfrak{B}} &\leq \|\upsilon_s^*\|_{\mathfrak{B}}+\|\varsigma_s\|_{\mathfrak{B}}\\ &\leq \kappa(\varphi)sup\{|\upsilon(s)|:0\leq s\leq 1\}+L(\varphi)\|\upsilon_0\|_{\mathfrak{B}}+\kappa(\varphi)sup\{|\varsigma(s)|:0\leq s\leq 1\}+L(\varphi)\|\varsigma_0\|_{\mathfrak{B}}\\ &\leq \kappa_\sigma sup\{|\upsilon(s)|:0\leq s\leq 1\}+L_\sigma\|\eta\|_{\mathfrak{B}}.\end{aligned}$$

If we name $\psi(\varphi)$ the right-hand side of (4.5), then we get

$$\|\upsilon_s^*+\varsigma_s\|_{\mathfrak{B}}\leq\psi(t),$$

and therefor

$$|\upsilon(\varphi)|\leq\frac{\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)}\int_0^{\varphi}(\varphi-s)^{\wp_*-1}s^{-\iota}\beta(s)\psi(s)ds+\frac{\sigma^*\sigma^{1-\wp_*}\|\alpha\|_{\Psi}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota},\qquad \varphi\in[0,\sigma].$$

Using the above inequality and the definition of ψ we have that

$$\psi(\varphi)\leq L_\sigma\|\eta\|_{\mathfrak{B}}+\frac{\kappa_\sigma\sigma^*\sigma^{1-\wp_*}\|\alpha\|_{\Psi}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}+\frac{\kappa_\sigma\sigma^*\sigma^{1-\wp_*}\|\beta\|_{\Psi}}{\Gamma(\wp^*)}\int_0^{\varphi}(\varphi-s)^{\wp_*-1}s^{-\iota}\psi(s)ds,\varphi\in[0,\sigma].$$

Then from Lemma (1.4.1) , there exists $\kappa=\kappa(\wp^*)$ in this manner we have

$$|\psi(\varphi)|\leq L_\sigma\|\eta\|_{\mathfrak{B}}+\frac{\kappa_\sigma\sigma^*\sigma^{1-\wp_*}\|\alpha\|_{\Psi}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}+\kappa(\wp^*)\frac{\kappa_\sigma\sigma^*\sigma^{1-\wp_*}\|\beta\|_{\Psi}}{\Gamma(\wp^*)}\int_0^{\varphi}(\varphi-s)^{\wp_*-1}s^{-\iota}\delta ds,\varphi\in[0,\sigma].$$

where

$$\delta=L_\sigma\|\eta\|_{\mathfrak{B}}+\frac{\kappa_\sigma\sigma^*\sigma^{1-\wp_*}\|\alpha\|_{\Psi}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota},$$

hence

$$\|\psi\|_{\Psi}\leq\delta+\frac{\delta\kappa(\wp^*)\kappa_\sigma\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}}{\Gamma(1-\iota+\wp^*)}=\widehat{L},$$

then

$$\|\upsilon\|_{\Psi}\leq\widehat{L}\|I^{\wp^*}\beta\|_{\Psi}+\frac{\sigma^*\sigma^{1-\wp_*}\|\alpha\|_{\Psi}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}=L^*.$$

Set

$$V = \{\upsilon \in M_0 : \|\upsilon\|_\sigma < L^* + 1\}.$$

$\Im : \overline{V} \to M_0$ is continuous and **Compl cont**. From the choice of V , there is no $\upsilon \in \partial V$ such that $\upsilon = \lambda \Im(\upsilon)$ for $\lambda \in (0,1)$. As a consequence of **Alt nonlinear L-S thm** ([32]) , we deduce that $\Im$ has a fixed point υ in V.

The second result obtained by using **Banach contr princip**.

Theorem 4.2.2 *Suppossing that conditions $(C1),(C2)$ are hold, and if*

$$\frac{\sigma^* \sigma^{1-\wp_*}\ \ell\ \kappa_\sigma\ \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota} < 1, \tag{4.5}$$

then the (CPNFDEVOID(3)) has a unique solution in Ψ.

Proof:

Let $\Im : M_0 \to M_0$ be defined as in (4.4). That the operator Φ admits a unique **Fix pt** is equivalent to $\Im$ has a **Fix pt**, and so we turn to proving that $\Im$ has a **Fix pt**. We shall that $\Im : M_0 \to M_0$ is a contraction map.

For $\upsilon_1(\varphi), \upsilon_2(\varphi) \in M_0$, we obtain that:

$$\begin{aligned}
|\Im(\upsilon_1)(\varphi) - \Im(\upsilon_2)(\varphi)| &\leq \int_0^\varphi \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}|\Upsilon(s,\upsilon_1^*(s)+\varsigma(s)) - \Upsilon(s,\upsilon_2^*(s)+\varsigma(s))|ds \\
&\leq \frac{1}{\Gamma(\wp^*)}\int_0^\varphi \sigma^{\wp(s)-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp(s)-1} \ell\ s^{-\iota}\|\upsilon_1^*(s)-\upsilon_2^*(s)\|_{\mathfrak{B}}ds \\
&\leq \frac{\ell}{\Gamma(\wp^*)}\int_0^\varphi \sigma^{\wp_*-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp_*-1} s^{-\iota}\kappa(s) sup_{s\in[0,\varphi]}\|\upsilon_1(s)-\upsilon_2(s)\|ds \\
&\leq \frac{\sigma^*\ \ell\ \kappa_\sigma}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\int_0^\varphi (\varphi-s)^{\wp_*-1}s^{-\iota}\|\upsilon_1-\upsilon_2\|_\sigma ds \\
&\leq \frac{\sigma^*\ \ell\ \kappa_\sigma}{\Gamma(\wp^*)}\sigma^{1-\wp_*}\|\upsilon_1-\upsilon_2\|_\sigma\int_0^\varphi (\varphi-s)^{\wp_*-1}s^{-\iota}ds \\
&\leq \frac{\sigma^*\sigma^{1-\wp_*}\ \ell\ \kappa_\sigma\ \Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|\upsilon_1-\upsilon_2\|_\sigma
\end{aligned}$$

by bound φ on D we find

$$\|\Im(\upsilon_1) - \Im(\upsilon_2)\|_\sigma \leq \frac{\sigma^*\sigma^{1-\wp_*}\ \ell\ \kappa_\sigma\ \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|\upsilon_1-\upsilon_2\|_\sigma,$$

Consequently by (4.5), the operator $\Im$ is a contraction. Hence, by **Banach contr principle**, $\Im$ has a unique **Fix pt** $\upsilon \in M_0$, which is a unique solution of the (CPNFDEVOID(3)).

4.3 Ulam Hyers stability

Theorem 4.3.1 *Let the conditions* $(C1)$ *and* $(C2)$,$(C3)$ *hold, then the (CPNFDEVOID(3)) is* ***Ula Hyer stab****.*

Proof 4.3.1 *Let* $\varepsilon > 0$ *an arbitrary number and the function* $w(\varphi)$ *from* $\upsilon \in \mathfrak{B}$ *satisfy the following inequality*

$$|D^{\wp(\varphi)}\chi(\varphi) - \Upsilon(\varphi, \chi(\varphi))| < \varepsilon, \quad \varphi \in D$$

we have

$$D^{\wp(\varphi)}\chi(\varphi) = \int_0^{\varphi} \frac{(\varphi - s)^{1-\wp(s)}}{\Gamma(1 - \wp(s))}\chi(s)ds,$$

we obtain

$$\begin{aligned}
\left|\chi(\varphi) - \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s, \chi(s))ds\right| &\leq \varepsilon \int_0^{\varphi} \frac{(\varphi - s)^{\wp(s)-1}}{\Gamma(\wp(s))}ds \\
&\leq \frac{\varepsilon}{\Gamma(\wp^*)} \int_0^{\varphi} \sigma^{\wp(s)-1}\left(\frac{\varphi - s}{\sigma}\right)^{\wp(s)-1} ds \\
&\leq \frac{\varepsilon}{\Gamma(\wp^*)} \int_0^{\varphi} \sigma^{\wp_*-1}\left(\frac{\varphi - s}{\sigma}\right)^{\wp_*-1} ds \\
&\leq \frac{\varepsilon\sigma^*}{\Gamma(\wp^*)}\sigma^{1-\wp_*} \int_0^{\varphi} (\varphi - s)^{\wp_*-1}s^{-\iota}s^{\iota}ds \\
&\leq \frac{\varepsilon\sigma^*\Gamma(\wp^*)\Gamma(1 - \iota)}{\Gamma(\wp^*)\Gamma(1 - \iota + \wp^*)}\sigma^{1-\wp_*}\sigma^{\wp_*-\iota} \int_0^{\varphi} s^{\iota}ds \\
&\leq \frac{\varepsilon\sigma^*\ \sigma^{1-\wp_*}\Gamma(1 - \iota)}{\Gamma(1 - \iota + \wp^*)}\sigma^{\wp_*-\iota}\left[\frac{s^{\iota+1}}{\iota + 1}\right]_0^{\varphi} \\
&\leq \frac{\varepsilon\sigma^*\ \sigma^{1-\wp_*}\Gamma(1 - \iota)}{(\iota + 1)\Gamma(1 - \iota + \wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1}.
\end{aligned}$$

Let $\varphi \in D$, we get

$$|\chi(\varphi) - w(\varphi)| = |\frac{\varepsilon\sigma^* \ \sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \int_0^\varphi \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,\chi(s))ds$$

$$- \int_0^\varphi \frac{(t-s)^{\wp(s)-1}}{\Gamma(\wp(s))}\Upsilon(s,w(s))ds|$$

$$\leq \frac{\varepsilon\sigma^* \ \sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \int_0^\varphi \frac{(\varphi-s)^{\wp(s)-1}}{\Gamma(\wp(s))}|\Upsilon(s,\chi(s)) - \Upsilon(s,w(s))|ds$$
$$\leq \frac{\varepsilon\sigma^* \ \sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell}{\Gamma(\wp^*)}\int_0^\varphi \sigma^{\wp(s)-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp(s)-1} s^{-\iota}|\chi(s) - w(s)|ds$$

$$\leq \frac{\varepsilon\sigma^* \ \sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell}{\Gamma(\wp^*)}\int_0^\varphi \sigma^{\wp_*-1}\left(\frac{\varphi-s}{\sigma}\right)^{\wp_*-1} s^{-\iota}|\chi(s) - w(s)|ds$$

$$\leq \frac{\varepsilon\sigma^* \ \sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell\sigma^*\sigma^{1-\wp_*}}{\Gamma(\wp^*)}\|\chi - w\|_\Psi \int_0^\varphi (\varphi-s)^{\wp_*-1}s^{-\iota}ds$$

$$\leq \frac{\varepsilon\sigma^* \ \sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(\wp^*)\Gamma(1-\iota)}{\Gamma(\wp^*)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|\chi - w\|_\Psi$$

$$\leq \frac{\varepsilon\sigma^* \ \sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} + \frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\|\chi - w\|_\Psi.$$

Then

$$\|\chi - w\|_\Psi(1 - \frac{\ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}) \leq \frac{\varepsilon\sigma^* \ \sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1}$$

We obtain, for each $\varphi \in D$

$$\begin{aligned}
|\chi(\varphi) - w(\varphi)| \leq \|\chi - w\|_\Psi \quad &\leq \quad \frac{\varepsilon\sigma^* \ \sigma^{1-\wp_*}\Gamma(1-\iota)}{(\iota+1)\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota}\varphi^{\iota+1} \\
&\times \quad \frac{\Gamma(1-\iota+\wp^*)}{\Gamma(1-\iota+\wp^*) - \ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}} \\
&\leq \quad \varepsilon\frac{\sigma^* \ \sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}\varphi^{\iota+1}}{(\iota+1)\Big(\Gamma(1-\iota+\wp^*) - \ell\sigma^*\sigma^{1-\wp_*}\Gamma(1-\iota)\sigma^{\wp_*-\iota}\Big)} = c_\Upsilon\varepsilon.
\end{aligned}$$

then the (CPNFDEVOID(3)) is ***Ula Hyer stab****.*

4.4 Approximate numerical example

we define this **(Fractional problem)**,

$$\begin{cases} D^{\wp(\varphi)}w(\varphi) &= \dfrac{re^{-\theta\varphi+\varphi}|w_\varphi|}{(\varphi+1)^{\frac{1}{6}}(e^{\varphi}+e^{-\varphi})(1+|w_\varphi|)}, \ \ \varphi\in D=[0,1],\ 0\le\wp(\varphi)\le 1,\ and\ \theta>0. \\ \\ w(\varphi) &= \eta(\varphi),\ \ \varphi\in(-\infty,0] \end{cases} \tag{4.6}$$

where $\wp(\varphi)=\frac{1}{2}\varphi+\frac{1}{2}$ is a **Cont funct** on $[0,1]$.
Clearly for $\varphi\in[0,1]$ we have

$$\frac{1}{2}\le\wp(\varphi)\le 1.$$

where $r\Gamma(\wp^*)=r_0\int_0^\sigma s^{\wp_*-1}e^{-s}ds$, and $r_0>1$ fixed. Let $\theta>0$ and

$$\mathfrak{B}_\emptyset=\{w\in C\Big((-\infty,0],\mathbb{R}\Big): lim_{\jmath\to-\infty}e^{\theta\ \jmath}w(\jmath), exists\ in\ \mathbb{R}\}.$$

The norme of $\mathfrak{B}_\theta$ is given by

$$\|w\|_\theta=sup_{-\infty<\jmath\le 0}e^{\theta\jmath}\ |w(\jmath)|,$$

Let $w:(-\infty,\sigma]\to\mathbb{R}$ be such that $w_0\in\mathfrak{B}_\theta$. Then

$$lim_{\jmath\to-\infty}e^{\theta\jmath}w_\varphi(\jmath)=lim_{\jmath\to-\infty}e^{\theta\jmath}w(\varphi+\jmath)=lim_{\jmath\to-\infty}e^{\theta(\jmath-\varphi)}w(\jmath)=e^{-\theta\varphi}lim_{\jmath\to-\infty}e^{\theta\jmath}w_0(\jmath)<\infty.$$

Hence $w_\varphi\in\mathfrak{B}_\theta$. Finally we prove that

$$\|w_\varphi\|_\theta\le\kappa(\varphi)sup\{|w(s)|:0\le s\le\varphi\}+L(\varphi)\|w_0\|_\theta,$$

where $\kappa=L=1$ and $T=1$ we have $|w_\varphi(\jmath)|=|w(\varphi+\jmath)|$.

If $\jmath+\varphi\le 0$, we get

$$|w_\varphi(\jmath)|\le sup\{|w(s)|:-\infty<s\le 0\}.$$

For $\jmath+\varphi\ge 0$, then we have

$$|w_\varphi(\jmath)|\le sup\{|w(s)|:0\le s\le\varphi\}.$$

Thus for all $\jmath+\varphi\in[0,\sigma]$, we get

$$|w_\varphi(\jmath)|\le sup\{|w(s)|:-\infty<s\le 0\}+sup\{|w(s)|:0\le s\le\varphi\}.$$

Then

$$\|w_\varphi\|_\theta \leq \|w_0\|_\theta + sup\{|w(s)| : 0 \leq s \leq \varphi\}.$$

Then $(\mathfrak{B}_\theta, \|.\|_\theta)$ is a **Banach sp**. We deduce that $\mathfrak{B}$ is a **Phase sp** .
Where $r = \frac{1}{2}$, set

$$\Upsilon(\varphi, w) = \frac{e^{-\theta\varphi+\varphi}w}{2\ (\varphi+1)^{\frac{1}{6}}(e^{\varphi}+e^{-\varphi})(1+w)}, \quad \varphi \in [0,\sigma] \times \mathfrak{B}_\theta.$$

Let $w_1, w_2 \in \mathfrak{B}_\theta$, then we have

$$\begin{aligned}
\varphi^{\frac{1}{6}}|\Upsilon(\varphi, w_1) - \Upsilon(\varphi, w_2)| &= \varphi^{\frac{1}{6}} \frac{e^{-\theta\varphi+\varphi}}{2\ (\varphi+1)^{\frac{1}{6}}\ (e^{\varphi}+e^{-\varphi})} \left| \frac{w_1}{1+w_1} - \frac{w_2}{1+w_2} \right| \\
&\leq \varphi^{\frac{1}{6}} \frac{e^{-\theta\varphi+\varphi}}{2\ \varphi^{\frac{1}{6}}\ (e^{\varphi}+e^{-\varphi})} \left| \frac{w_1}{1+w_1} - \frac{w_2}{1+w_2} \right| \\
&= \frac{e^{\varphi-\theta\varphi}|w_1 - w_2|}{2\ (e^{\varphi}+e^{-\varphi})(1+w_1)(1+w_2)} \\
&\leq \frac{e^{\varphi}e^{-\theta\varphi}|w_1 - w_2|}{2\ (e^{\varphi}+e^{-\varphi})(1+w_1)(1+w_2)} \\
&\leq \frac{e^{\varphi}|w_1 - w_2|_{\mathfrak{B}_\theta}}{2\ (e^{\varphi}+e^{-\varphi})} \\
&\leq \frac{1}{2}\|w_1 - w_2\|_{\mathfrak{B}_\theta}.
\end{aligned}$$

Hence the condition (C2) holds with $\iota = \frac{1}{6}$ and $\ell = \frac{1}{2}$, $\kappa_\sigma = 1$, $\wp_* = 1$ and
$\sigma^* = max\{1, \sigma^{\wp_*-1}\} = 1$.
Next, we prove that the condition (4.5)

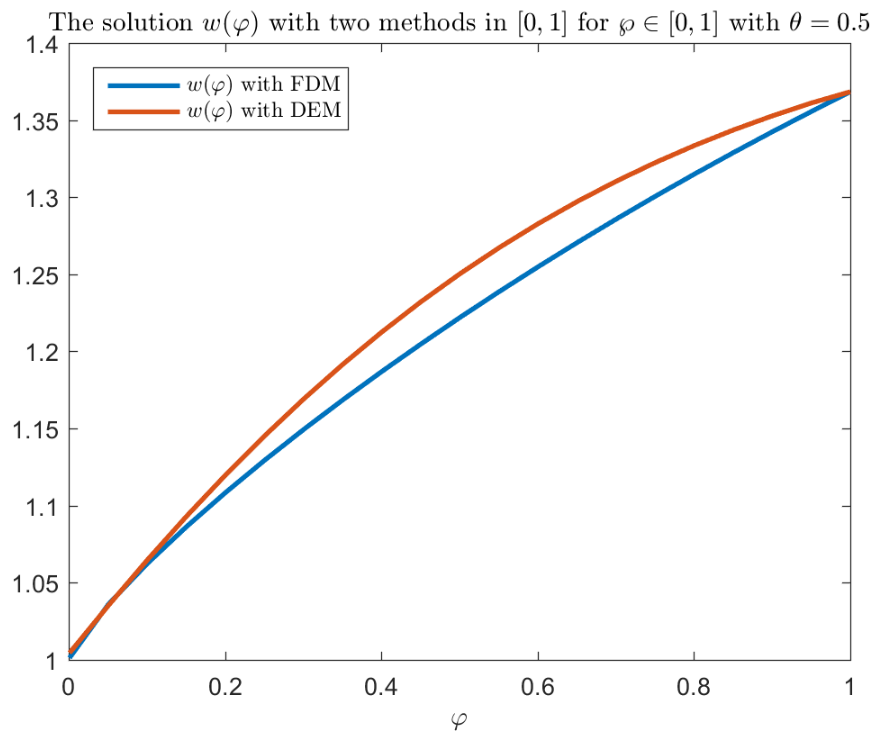
$$\frac{\sigma^*\sigma^{1-\wp_*}\ \ell\ \kappa_\sigma\ \Gamma(1-\iota)}{\Gamma(1-\iota+\wp^*)}\sigma^{\wp_*-\iota} = \frac{\frac{1}{2}\Gamma(\frac{5}{6})}{\frac{5}{6}\Gamma(\frac{5}{6})} = \frac{\frac{1}{2}}{\frac{5}{6}} = \frac{1}{2} \times \frac{6}{5} = 0.6 < 1.$$

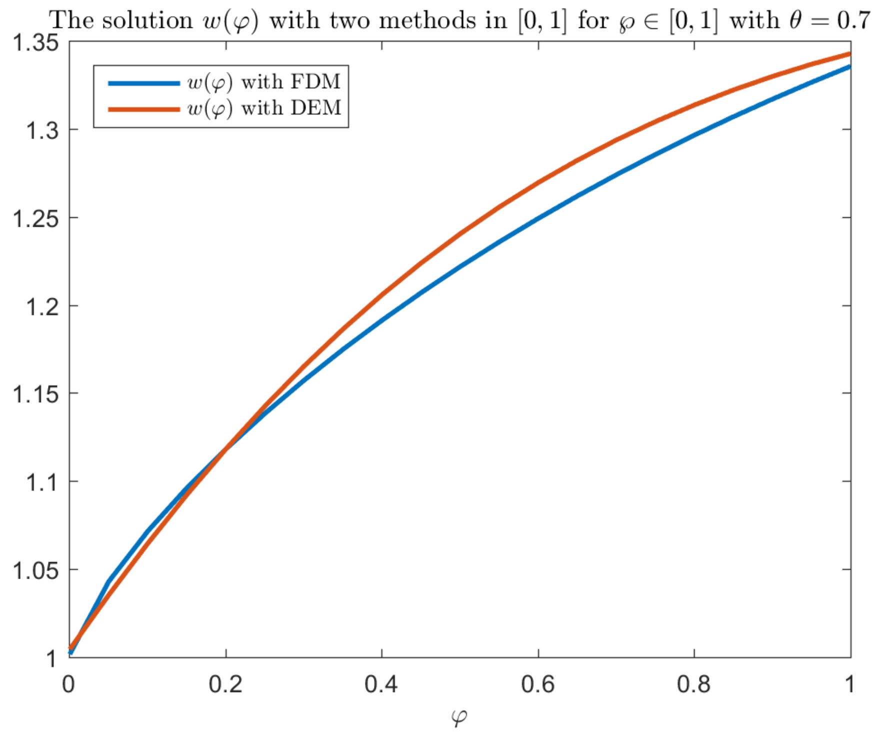
Accordingly the condition (4.5) is achieved. By Theorem (4.2.2), the problem (4.6) has unique solution.

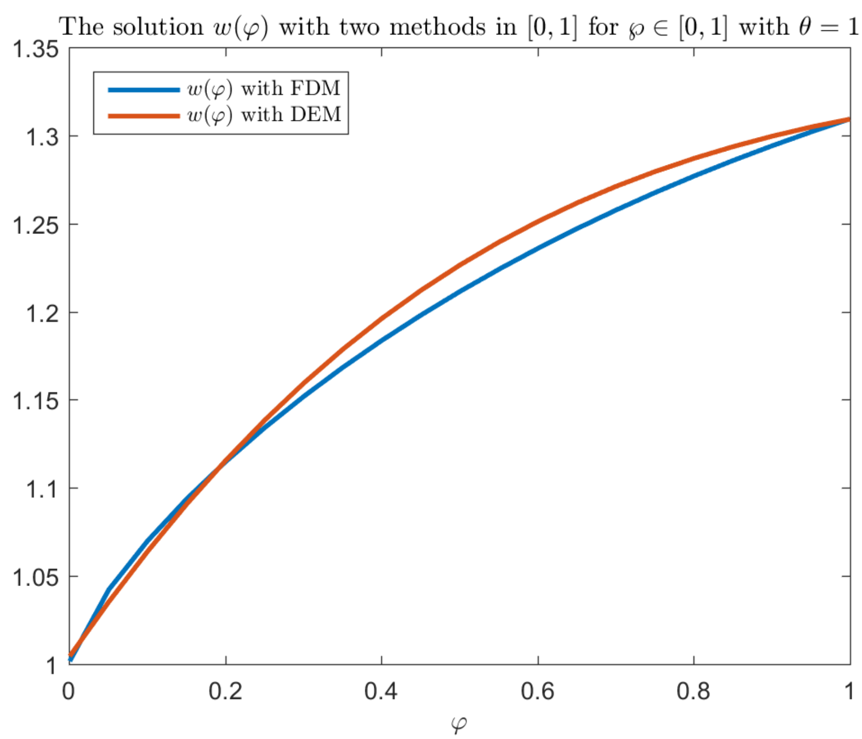
4.4.1 Numerical results

In this section, we use two numerical methods. The first is the finite difference method (**FDM**)[50], the second method is the Euler's discretization method (**DEM**)[63]. Both methods are based on the subdivision of the interval, we took 1000 points. We calculated the solution $w_{FDM}(\varphi)$ with the (**FDM**) method and the solution $w_{DEM}(\varphi)$ with the (**DEM**) method for $\wp(\varphi) = \frac{1}{2}\varphi + \frac{1}{2}$ with $\varphi \in [0, 1]$.

In figure (4.1), we plot the solutions $w_{FDM}(\varphi)$ and $w_{DEM}(\varphi)$ depending on φ and different θ.




The solution $w(\varphi)$ with two methods in $[0,1]$ for $\wp \in [0,1]$ with $\theta = 0.7$
$w(\varphi)$ with FDM
$w(\varphi)$ with DEM
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ


The solution $w(\varphi)$ with two methods in $[0,1]$ for $\wp \in [0,1]$ with $\theta = 1$
$w(\varphi)$ with FDM
$w(\varphi)$ with DEM
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ

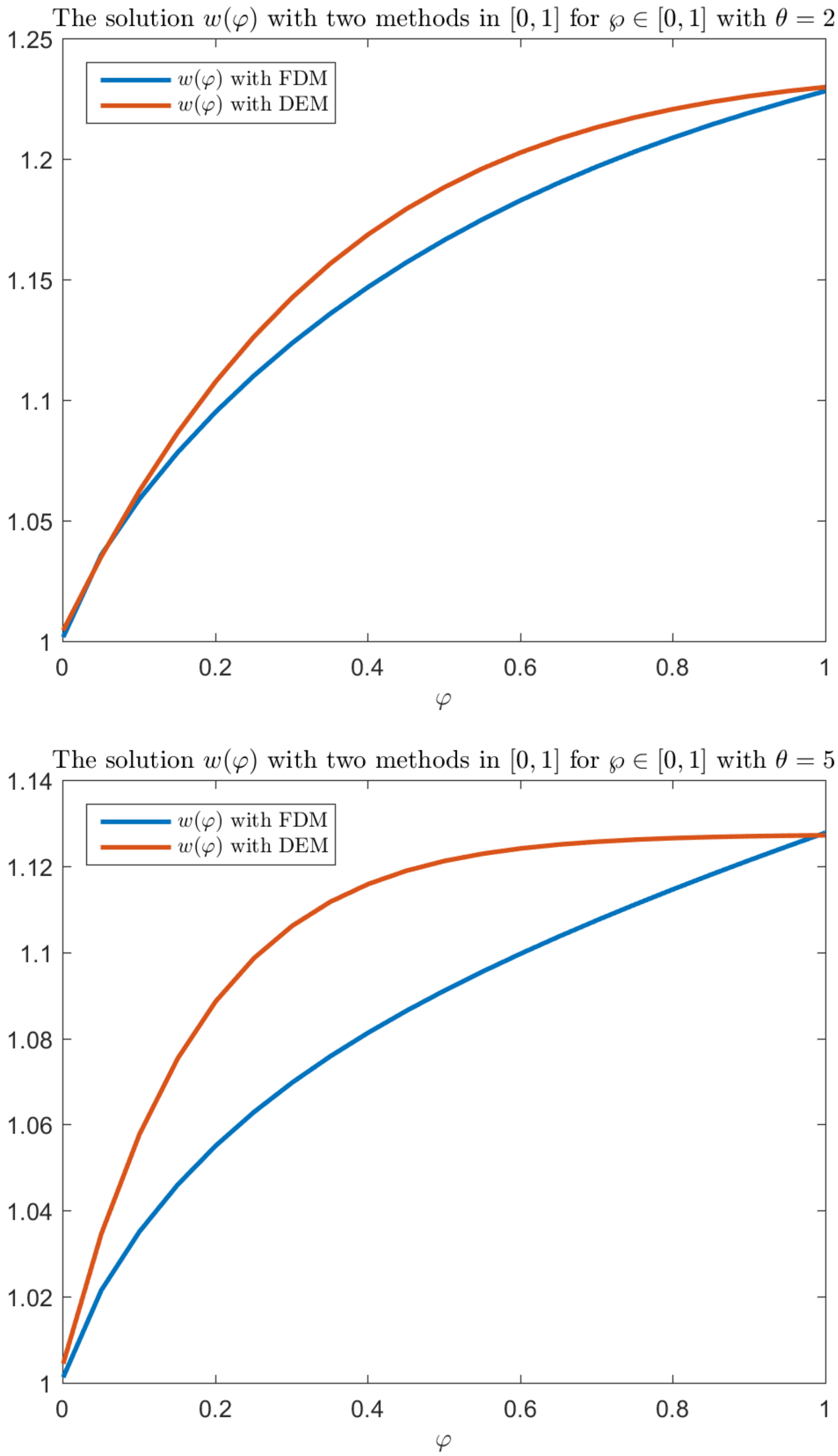


Figure 4.1: The solutions $w_{FDM}(\varphi)$ and $w_{DEM}(\varphi)$ in $[0,1]$ with $\wp(\varphi) = \frac{1}{2}\varphi + \frac{1}{2}$ and different θ.

In this table, we present the $Norm = \max_{\varphi\in[0,1]} \Big| w_{FDM}(\varphi) - w_{DEM}(\varphi) \Big|$ for $\wp(\varphi) \in [0,1]$.

θ	$\theta = 0.5$	$\theta = 0.7$	$\theta = 1$	$\theta = 2$	$\theta = 5$
$Norm$	0.02665	0.02454	0.01541	0.02008	0.02940

We notice that the error between the two methods is of order 2.
We observe that for $\theta \le 1$, the approximate solution w_{FDM} is close to the approximate solution w_{DEM} when θ is proche to 1. For $\theta > 1$, the two approximate solution is the same when θ is proche to 1(see figure(4.1)).
Now,we calculated the solution $w_{i,FDM}(\varphi)$ with the (**FDM**) method and the solution $w_{i,DEM}(\varphi)$ with the (**DEM**) method for $\wp(\varphi_i) = \frac{1}{2}\varphi_i + \frac{1}{2}$ where φ_i is fixed and for different θ.

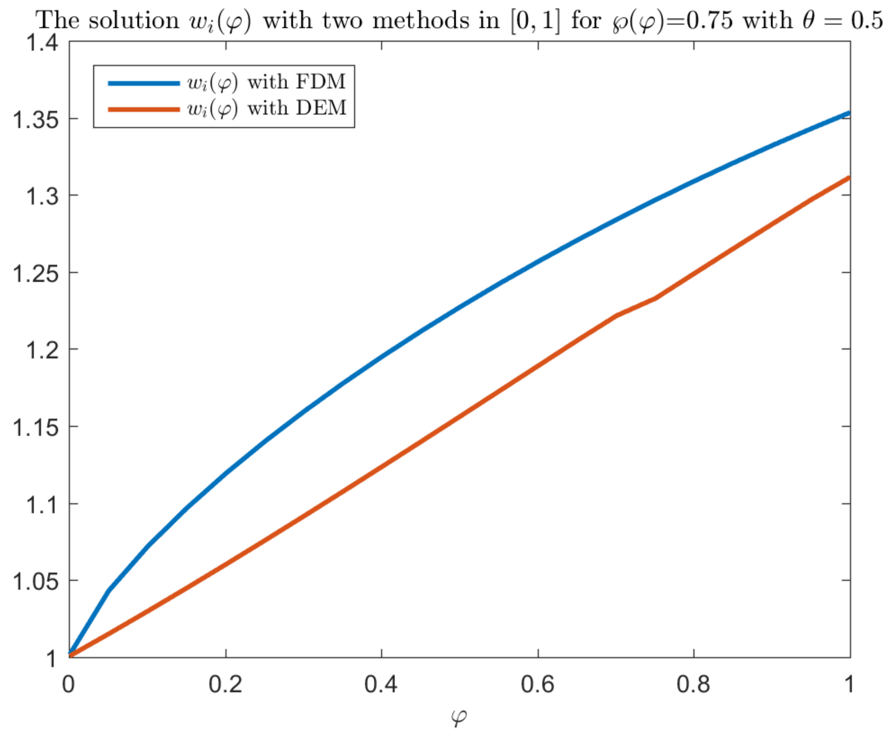
In this table, we present the $Norm_i = \max_{\varphi\in[0,1]} \Big| w_{i,FDM}(\varphi) - w_{i,DEM}(\varphi) \Big|$ for $\wp(\varphi)$ fixed in $[0,1]$.

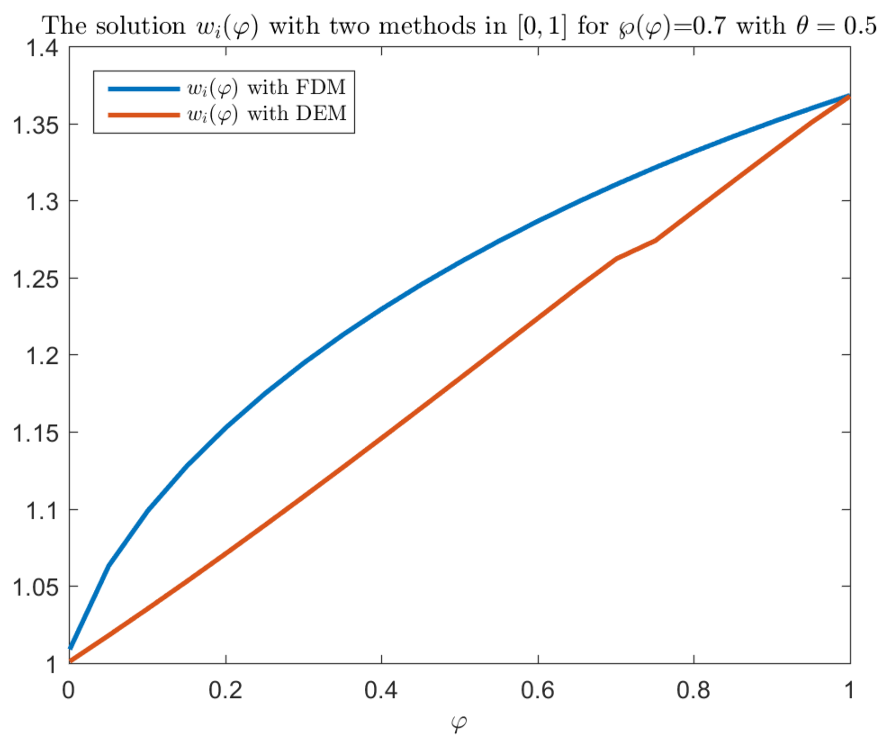
φ	0.2	0.4	0.5	0.6	0.8	1.0
$\wp(\varphi)$	0.6	0.7	0.75	0.8	0.9	1
$Norm_i, \theta = 0.5$	0.13623	0.03688	0.02726	0.02962	0.05435	0.07210
$Norm_i, \theta = 0.7$	0.13608	0.03703	0.02102	0.02450	0.04922	0.06663
$Norm_i, \theta = 1$	0.12351	0.02798	0.01608	0.03119	0.05407	0.07028
$Norm_i, \theta = 2$	0.02132	0.02152	0.02279	0.03419	0.05312	0.06625
$Norm_i, \theta = 5$	0.02124	0.02177	0.02989	0.03681	0.04750	0.05489

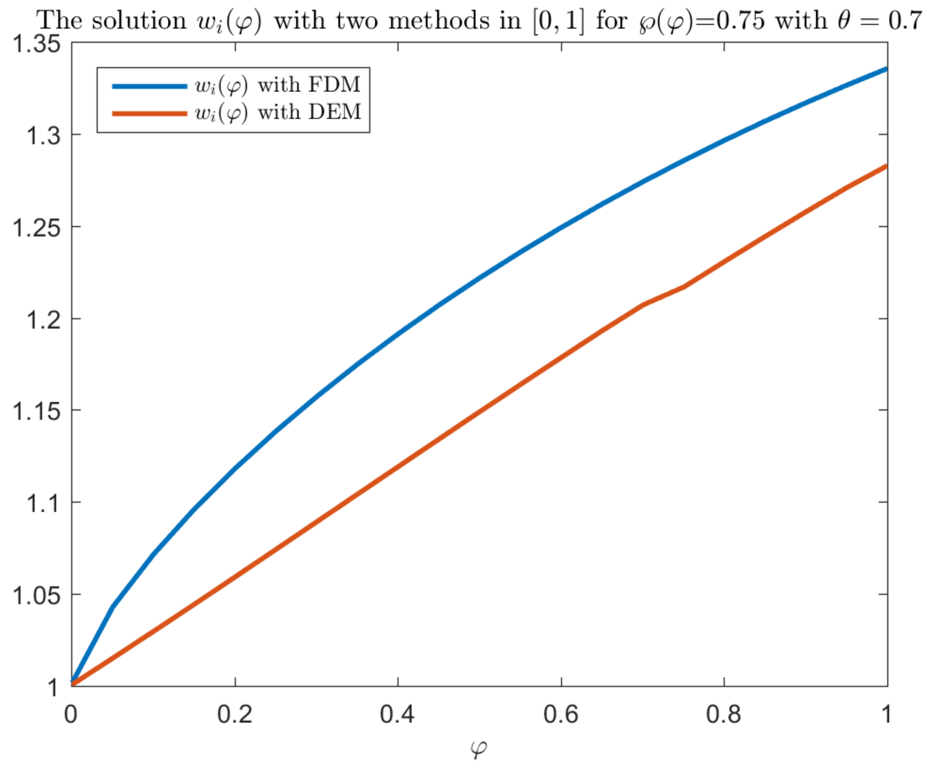
We observe that $\theta \le 1$ the $Norm_i$ is small at $\varphi = 0.5, \wp = 0.75$, it decreasing in $\varphi \in [0, 0.5]$ and crescent in $\varphi \in [0.5, 1]$ but when $\theta > 1$ is creasing in $\varphi \in [0, 1]$. Then for $\theta \le 1$, the approximate solution $w_{i,FDM}$ is close to the approximate solution $w_{i,DEM}$ when φ is proche to 0.5($\wp = 0.75$). For $\theta > 1$, the two approximate solution is the same when φ is proche to

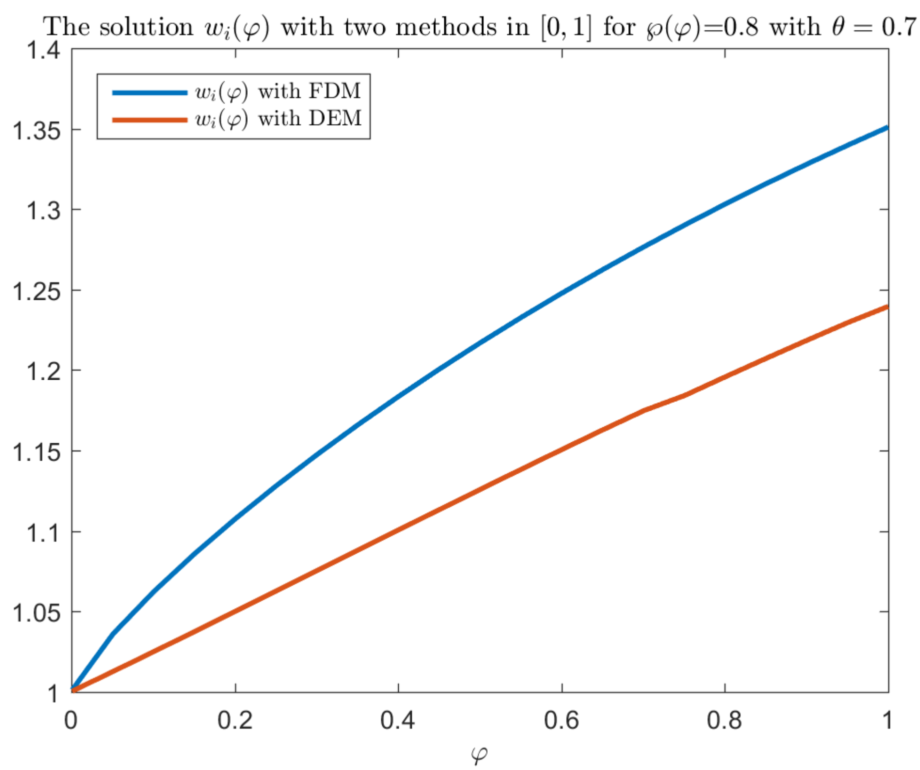
$0(\wp = 0.5)$(see figure(4.2)).

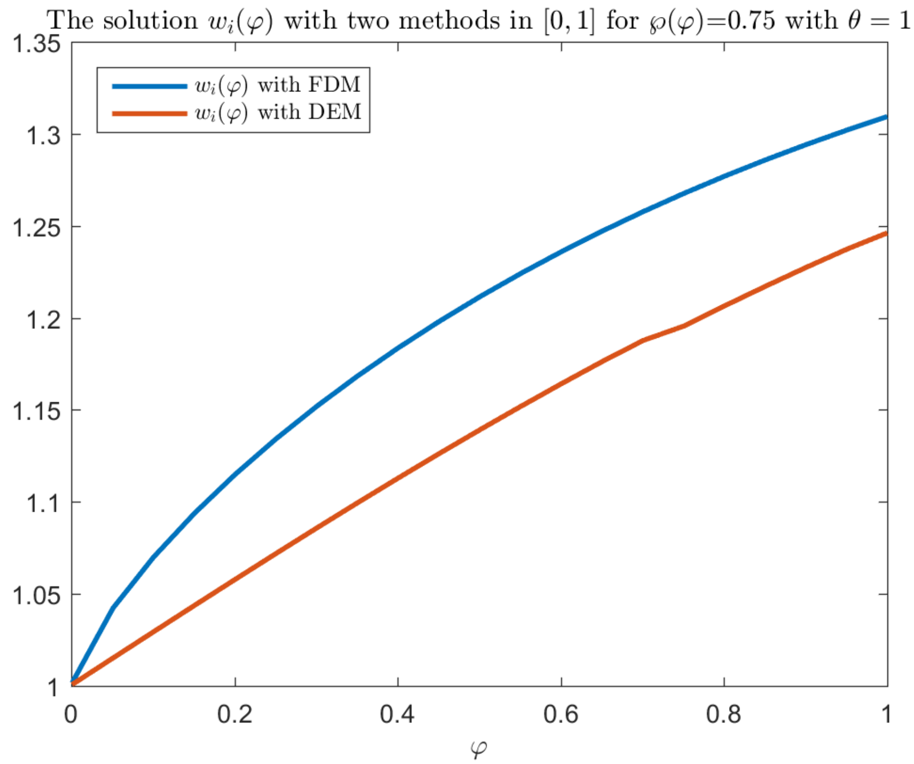
In the figure (4.2), we present a comparison between the solution $w_{i,FDM}(\varphi)$ and the solutions $w_{i,DEM}(\varphi)$ with a $\wp$ different and different θ.






The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.75 with $\theta = 0.7$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ


The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.8 with $\theta = 0.7$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.4
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ


The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.75 with $\theta=1$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.35
1.3
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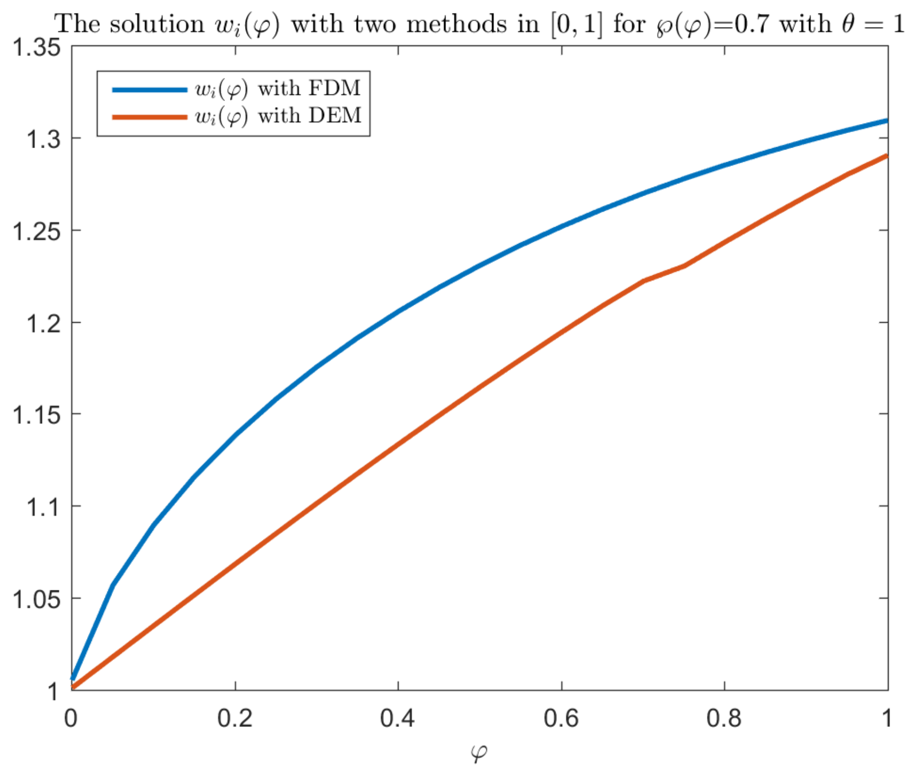
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
φ


The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.7 with $\theta=1$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.35
1.3
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0
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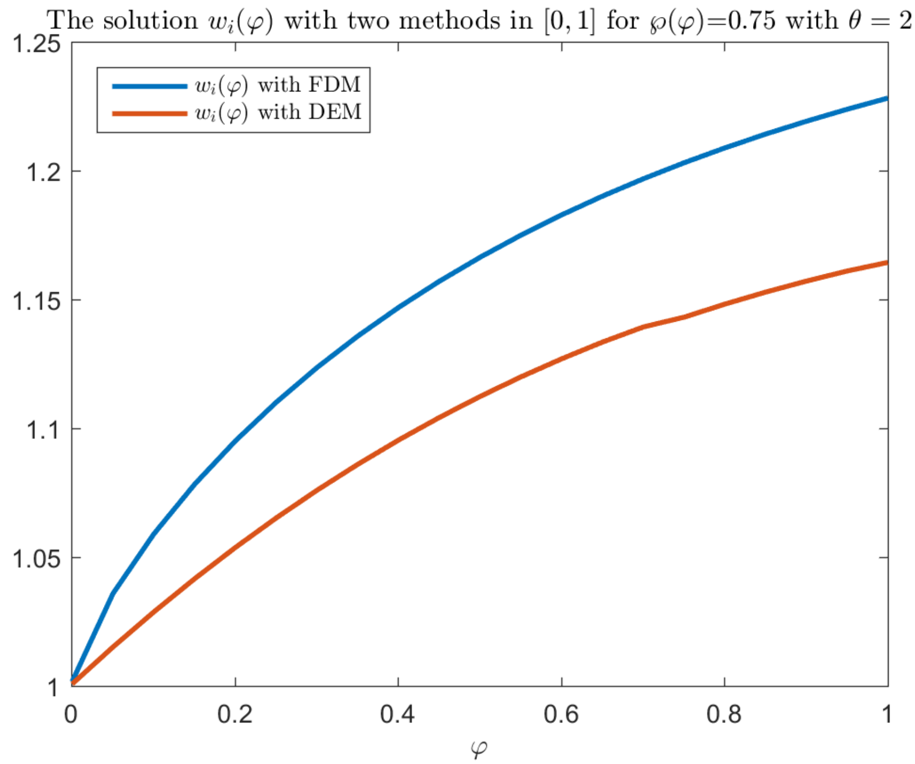
0.8
1
φ


The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.75 with $\theta = 2$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.25
1.2
1.15
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1.05
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0
0.2
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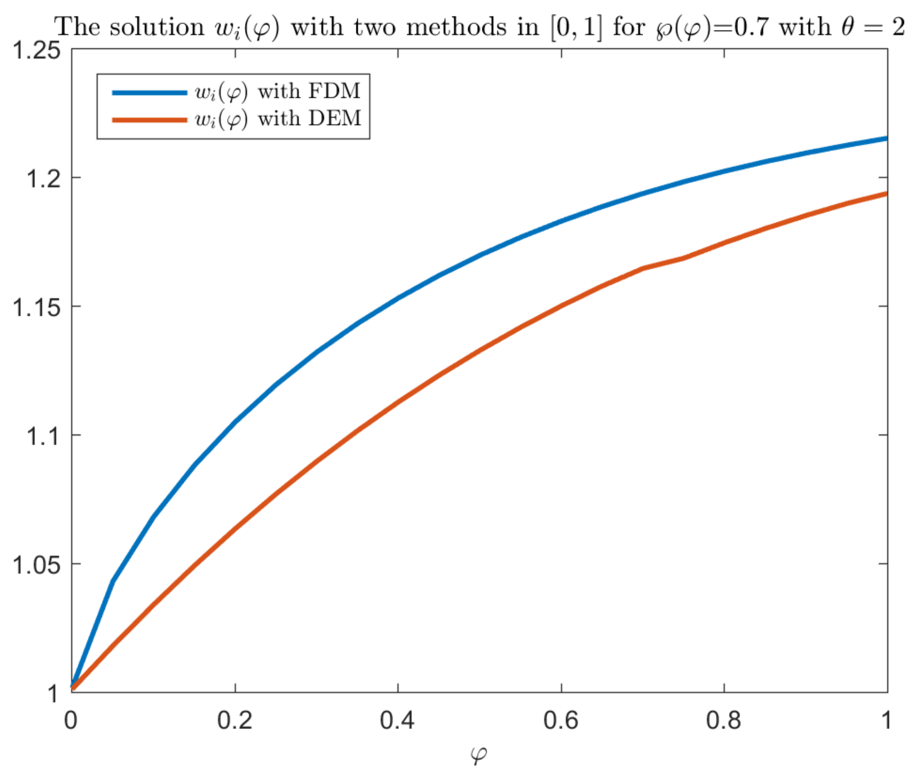
φ


The solution $w_i(\varphi)$ with two methods in $[0,1]$ for $\wp(\varphi)$=0.7 with $\theta = 2$
$w_i(\varphi)$ with FDM
$w_i(\varphi)$ with DEM
1.25
1.2
1.15
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1.05
1
0
0.2
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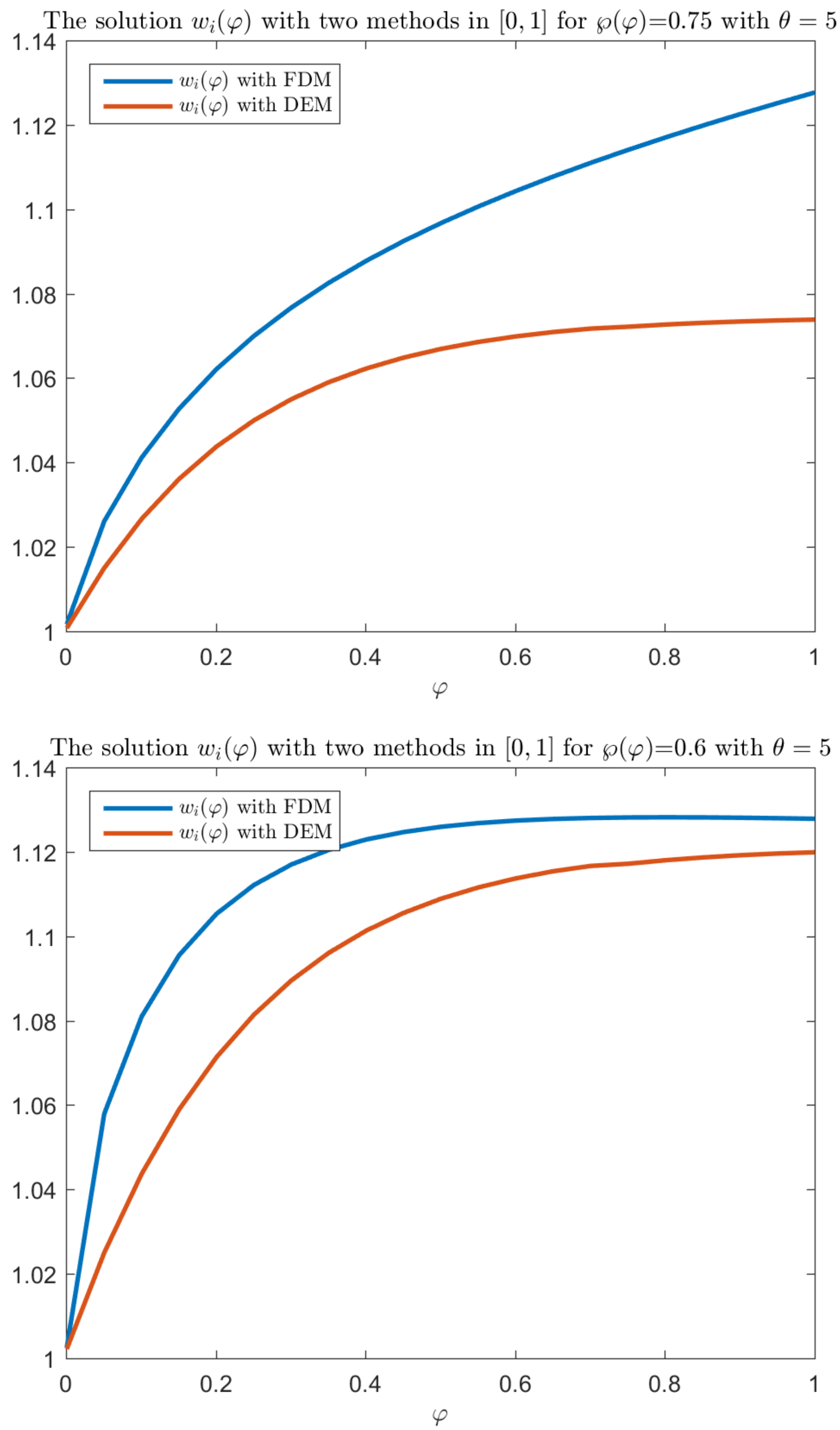

Figure 4.2: A plot of $w_i(\varphi)$ with two methods for different $\wp(\varphi)$ and different θ.

Conclusion

In this study, we investigate the existence, uniqueness and Ulam-Hyers stability of solutions for fractional diffrentail equations of varaible order with infinite delay. The achieved results are based on the Banach contraction principle and Alternative nonlinear Leray-Schauder theorem with some properties of phase space. Our method is completely new and simple to use, in contrast to earlier findings that were attained by utilizing the ideas of the generalized interval and the piecewise function. In fact, the findings might be seen as a successful attempt to sidestep dificult calculations and demonstrate the major findings under less stringent presumptions. We concluded the paper the with a concrete example that exposes their appropriateness. In the future, we investigate into a variety of problems such as hybrid fractional Cauchy-type problems of varaible order or impulsive fractional Cauchy-type problems of varaible order, which can be conducted using our methodology.

General Conclusions and Perspectives

In this study, we investigate the existence, uniqueness, and stability of solutions for the Cauchy-type problem of fractional variable order differential equations. Our method is completely new and simple to use, in contrast to earlier findings that were attained by utilizing the ideas of the generalized interval and the piecewise function. In fact, the findings might be seen as a successful attempt to sidestep difficult calculations and demonstrate the major findings under less stringent presumptions.

At the end of each chapter of this thesis, we presented numerical applications in which we explained how to obtain the approximate solution to our problems by using accurate and different methods. This study is considered a continuation of the theoretical study. The numerical study is very important practically and scientifically for researchers in order to use the exact approximate solution in various sciences, whether physical, biological or economic.

In the near future, we want to study these value problems with different conditions by using complex order.

Finally, we hope that this thesis will be an excellent addition to the field of scientific research and a good reference for readers, authors, and researchers in this specialty.

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